7

Internal and External Descriptions: Relations and Properties

7.1 Introduction

In this chapter it is shown how external descriptions of a system, such as the transfer function and the impulse response, depend only on the controllable and observable parts of internal state-space descriptions (Section 7.2). Based on these results, the exact relation between internal (Lyapunov) stability and input–output stability is established in Section 7.3. In Section 7.4 the poles of the transfer function matrix, the poles of the system (eigenvalues), the zeros of the transfer function, the invariant zeros, the decoupling zeros, and their relation to uncontrollable or unobservable eigenvalues are addressed. In the final Section 7.5, polynomial matrix and matrix fractional descriptions are introduced. Polynomial matrix descriptions are generalizations of state-space internal descriptions. The matrix fractional descriptions of transfer function matrices offer a convenient way to work with transfer functions in control design and to establish the relations between internal and external descriptions of systems.

7.2 Relations Between State-Space and Input–Output Descriptions

In this section it is shown that the input–output description, namely the transfer function or the impulse response of a system, depends only on the part of the state-space representation that is both controllable and observable. The uncontrollable and/or unobservable parts of the system "cancel out" and play no role in the input–output system descriptions.

Consider the system

$$\dot{x} = Ax + Bu, \quad y = Cx + Du, \tag{7.1}$$

where $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{p\times n}$, $D \in R^{p\times m}$ has $p \times m$. The transfer function matrix

$$H(s) = C(sI - A)^{-1}B + D = \widehat{C}(sI - \widehat{A})^{-1}\widehat{B} + \widehat{D}, \tag{7.2}$$

where $\{\widehat{A}, \widehat{B}, \widehat{C}, \widehat{D}\}$ is an equivalent representation given in (6.9) with $\widehat{A} = Q^{-1}AQ, \widehat{B} = Q^{-1}B, \widehat{C} = CQ$, and $\widehat{D} = D$. Consider now the Kalman Decomposition Theorem in Section 6.2.3 and the representation (6.22). We wish to investigate which of the submatrices A_{ij}, B_i, C_j determine $H(s)$ and which do not. The inverse of $sI - \widehat{A}$ can be determined by repeated application of the formulas

$$\begin{bmatrix} \alpha & \beta \\ 0 & \delta \end{bmatrix}^{-1} = \begin{bmatrix} \alpha^{-1} & -\alpha^{-1}\beta\delta^{-1} \\ 0 & \delta^{-1} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \alpha & 0 \\ \gamma & \delta \end{bmatrix}^{-1} = \begin{bmatrix} \alpha^{-1} & 0 \\ -\delta^{-1}\gamma\alpha^{-1} & \delta^{-1} \end{bmatrix}, \tag{7.3}$$

where $\alpha, \beta, \gamma, \delta$ are matrices with α and δ square and nonsingular. It turns out that

$$H(s) = C_1(sI - A_{11})^{-1}B_1 + D, \tag{7.4}$$

that is, the only part of the system that determines the external description is $\{A_{11}, B_1, C_1, D\}$, the subsystem that is both controllable and observable [see Theorem 6.6(iii)]. Analogous results exist in the time domain. Specifically, taking the inverse Laplace transform of both sides in (7.4), the impulse response of the system for $t \geq 0$ is derived as

$$H(t, 0) = C_1 e^{A_{11}t} B_1 + D\delta(t), \tag{7.5}$$

which depends only on the controllable and observable parts of the system, as expected.

Similar results exist for discrete-time systems described by (6.4). For such systems, the transfer function matrix $H(z)$ and the pulse response $H(k, 0)$ are given by

$$H(z) = C_1(zI - A_{11})^{-1}B_1 + D \tag{7.6}$$

and

$$H(k, 0) = \begin{cases} C_1 A_{11}^{k-1} B_1, & k > 0, \\ D, & k = 0. \end{cases} \tag{7.7}$$

Again, these depend only on the part of the system that is both controllable and observable, as in the continuous-time case.

Example 7.1. For the system $\dot{x} = Ax + Bu$, $y = Cx$, where A, B, C are as in Examples 6.7 and 6.10, we have $H(s) = C(sI - A)^{-1}B = C_1(sI - A_{11})^{-1}B_1 = (1)(1/s)[1, 1] = [1/s,\ 1/s]$. Notice that only the controllable and observable eigenvalue of A, $\lambda_1 = 0$ (in A_{11}), appears in the transfer function as a pole. All other eigenvalues ($\lambda_2 = -1, \lambda_3 = -2$) cancel out.

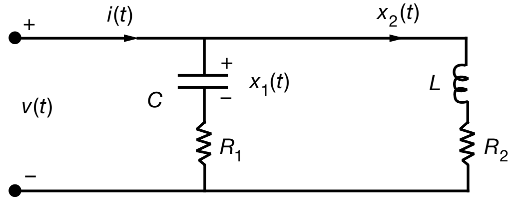

Figure 7.1. An RL/RC circuit

Example 7.2. The circuit depicted in Figure 7.1 is described by the state-space equations

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -1/(R_1C) & 0 \\ 0 & -R_2/L \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1/(R_1C) \\ 1/L \end{bmatrix} v(t)$$

$$i(t) = [-1/R_1, 1] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + (1/R_1)v(t),$$

where the voltage $v(t)$ and current $i(t)$ are the input and output variables of the system, $x_1(t)$ is the voltage across the capacitor, and $x_2(t)$ is the current through the inductor. We have $\hat{i}(s) = H(s)\hat{v}(s)$ with the transfer function given by

$$H(s) = C(sI - A)^{-1}B + D = \frac{(R_1^2C - L)s + (R_1 - R_2)}{(Ls + R_2)(R_1^2Cs + R_1)} + \frac{1}{R_1}.$$

The eigenvalues of A are $\lambda_1 = -1/(R_1C)$ and $\lambda_2 = -R_2/L$. Note that in general $\text{rank}[\lambda_i I - A, B] = \text{rank}\begin{bmatrix} \lambda_i I - A \\ C \end{bmatrix} = 2 = n$; i.e., the system is controllable and observable, unless the relation $R_1R_2C = L$ is satisfied. In this case, $\lambda_1 = \lambda_2 = -R_2/L$ and the system matrix $P(s)$ assumes the form

$$P(s) = \begin{bmatrix} sI - A, & B \\ -C, & D \end{bmatrix} = \begin{bmatrix} s + R_2/L & 0 & R_2/L \\ 0 & s + R_2/L & 1/L \\ 1/R_1 & -1 & 1/R_1 \end{bmatrix}.$$

In the following discussion, assume that $R_1R_2C = L$ is satisfied.

(i) Let $R_1 \neq R_2$ and take

$$[v_1, v_2] = \begin{bmatrix} R_2 & R_1 \\ 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} \hat{v}_1 \\ \hat{v}_2 \end{bmatrix} = [v_1, v_2]^{-1} = \frac{1}{R_2 - R_1} \begin{bmatrix} 1 & -R_1 \\ -1 & R_2 \end{bmatrix}$$

to be the linearly independent right and left eigenvectors corresponding to the eigenvalues $\lambda_1 = \lambda_2 = -R_2/L$. The eigenvectors could have been any two linearly independent vectors since $\lambda_i I - A = 0$. They were chosen as above because they also have the property that $\hat{v}_2 B = 0$ and $Cv_2 = 0$, which in view of Corollary 6.9, implies that $\lambda_2 = -R_2/L$ is both uncontrollable and unobservable. The eigenvalue $\lambda_1 = -R_2/L$ is both controllable and observable, as it can be seen using $Q = \begin{bmatrix} R_2 & R_1 \\ 1 & 1 \end{bmatrix}$ to reduce the representation to the canonical structure form (Kalman Decomposition Theorem). The transfer function is in this case given by

$$H(s) = \frac{(s + R_1/L)(s + R_2/L)}{R_1(s + R_2/L)(s + R_2/L)} = \frac{s + R_1/L}{R_1(s + R_2/L)};$$

that is, only the controllable and observable eigenvalue appears as a pole in $H(s)$, as expected.

(ii) Let $R_1 = R_2 = R$ and take

$$[v_1, v_2] = \begin{bmatrix} 1 & R \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \hat{v}_1 \\ \hat{v}_2 \end{bmatrix} = [v_1, v_2]^{-1} = \begin{bmatrix} 1 & -R \\ 0 & 1 \end{bmatrix}.$$

In this case $\hat{v}_1 B = 0$ and $Cv_2 = 0$. Thus, one of the eigenvalues, $\lambda_1 = -R/L$, is uncontrollable (but can be shown to be observable) and the other eigenvalue, $\lambda_2 = -R/L$, is unobservable (but can be shown to be controllable). In this case, none of the eigenvalues appear in the transfer function. In fact,

$$H(s) = 1/R,$$

as can readily be verified. Thus, in this case, the network behaves as a constant resistance network.

At this point it should be made clear that the modes that are uncontrollable and/or unobservable from certain inputs and outputs do not actually disappear; they are simply invisible from certain vantage points under certain conditions. (The voltages and currents of this network in the case of constant resistance $[H(s) = 1/R]$ *are studied in Exercise 7.2.)*

Example 7.3. Consider the system $\dot{x} = Ax + Bu, y = Cx$, where $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, and $C = [1,\ 1,\ 0]$. Using the eigenvalue/eigenvector test, it can be shown that the three eigenvalues of A (resp., the three modes of A) are $\lambda_1 = 1$ (resp., e^t), which is controllable and observable; $\lambda_2 = -2$ (resp., e^{-2t}), which is uncontrollable and observable; and $\lambda_3 = -1$ (resp., e^{-t}), which is controllable and unobservable.

The response due to the initial condition $x(0)$ and the input $u(t)$ is

$$
\begin{aligned}
x(t) &= e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \\
&= \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix} x(0) + \int_0^t \begin{bmatrix} e^{(t-\tau)} \\ 0 \\ e^{-(t-\tau)} \end{bmatrix} u(\tau)d\tau
\end{aligned}
$$

and

$$
\begin{aligned}
y(t) &= Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau \\
&= [e^t, e^{-2t}, 0]x(0) + \int_0^t e^{(t-\tau)}u(\tau)d\tau.
\end{aligned}
$$

Notice that only controllable modes appear in $e^{At}B$ [resp., only controllable eigenvalues appear in $(sI - A)^{-1}B$], only observable modes appear in Ce^{At} [resp., only observable eigenvalues appear in $C(sI - A)^{-1}$], and only modes that are both controllable and observable appear in $Ce^{At}B$ [resp., only eigenvalues that are both controllable and observable appear in $C(sI - A)^{-1}\ B = H(s)$]. For the discrete-time case, refer to Exercise 7.1d.

7.3 Relations Between Lyapunov and Input–Output Stability

In view of the relation between eigenvalues of A and poles of $H(s)$ developed above [see also (7.20) and (7.22)] we are now in a position to provide complete insight into the relation between exponential stability or Lyapunov stability and BIBO (Bounded Input Bounded Output) stability of a system.

Consider the system $\dot{x} = Ax + Bu$, $y = Cx + Du$, and recall the following results:

(i) The system is asymptotically stable (internally stable, stable in the sense of Lyapunov) if and only if the real parts of all the eigenvalues of A, $\mathcal{R}e\lambda_i(A)$ $i = 1, \ldots, n$, are negative. Recall also that asymptotic stability is equivalent to exponential stability in the case of linear time-invariant systems.

(ii) Let the transfer function be $H(s) = C(sI - A)^{-1}B + D$. The system is BIBO stable if and only if the real parts of all the poles of $H(s), \mathcal{R}ep_i(H(s))$ $i = 1, \ldots, r$, are negative [see Section 4.7].

The relation between the eigenvalues of A and the poles of $H(s)$ is

$$
\{\text{eigenvalues of } A\} \supset \{\text{poles of } H(s)\} \tag{7.8}
$$

with equality holding when all eigenvalues are controllable and observable [see (7.20), (7.22) and Chapter 8, Theorems 8.9 and 8.12]. Specifically, the eigenvalues of A may be controllable and observable, uncontrollable and/or unobservable, and the poles of $H(s)$ are exactly the eigenvalues of A that are both controllable and observable. The remaining eigenvalues of A, the uncontrollable and/or unobservable ones, cancel out when $H(s) = C(sI - A)^{-1}B + D$ is determined. Note also that the uncontrollable/unobservable eigenvalues that cancel correspond to input and output decoupling zeros (see Section 7.4). So the cancellations that take place in forming $H(s)$ are really pole/zero cancellations, i.e., cancellations between poles of the system (uncontrollable and unobservable eigenvalues of A) and zeros of the system (input and output decoupling zeros).

It is now straightforward to see that

$$\{\text{Internal stability}\} \overset{\Rightarrow}{\not\Leftarrow} \{\text{BIBO stability}\};$$

that is, internal stability implies, but is not necessarily implied by, BIBO stability. BIBO stability implies internal stability only when the system is completely controllable and observable ([1, p. 487, Theorem 9.4]).

Example 7.4. Consider the system $\dot{x} = Ax + Bu, y = Cx$, where $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $C = [-2,\ 1]$. The eigenvalues of A are the roots of $|sI - A| = s^2 - s - 2 = (s+1)(s-2)$ at $\{-1, 2\}$, and so the system is not internally stable (it is not stable in the sense of Lyapunov). The transfer function is

$$H(s) = C(sI - A)^{-1}B = \frac{s - 2}{(s + 1)(s - 2)} = \frac{1}{s + 1}.$$

Since there is one pole of $H(s)$ at $\{-1\}$, the system is BIBO stable, which verifies that BIBO stability does not necessarily imply internal stability. As it can be easily verified, the -1 eigenvalue of A is controllable and observable and it is the eigenvalue that appears as a pole of $H(s)$ at -1. The other eigenvalue at $+2$ that is unobservable, which is also the output decoupling zero of the system, cancels in a pole/zero cancellation in $H(s)$ as expected.

7.4 Poles and Zeros

In this section the poles and zeros of a time-invariant system are defined and discussed. The poles and zeros are related to the (controllable and observable, resp., uncontrollable and unobservable) eigenvalues of A. These relationships shed light on the eigenvalue cancellation mechanisms encountered when input–output relations, such as transfer functions, are formed.

In the following development, the finite *poles of a transfer function matrix* $H(s)$ [or $H(z)$] are defined first (for the definition of poles at infinity, refer to the Exercise 7.9). It should be noted here that the eigenvalues of A are sometimes called *poles of the system* $\{A, B, C, D\}$. To avoid confusion, we shall use the complete term *poles of* $H(s)$, when necessary. The zeros of a system are defined using internal descriptions (state-space representations).

7.4.1 Smith and Smith–McMillan Forms

To define the poles of $H(s)$, we shall first introduce the Smith form of a polynomial matrix $P(s)$ and the Smith–McMillan form of a rational matrix $H(s)$.

The *Smith form* $S_P(s)$ of a $p \times m$ polynomial matrix $P(s)$ (in which the entries are polynomials in s) is defined as

$$S_P(s) = \begin{bmatrix} \Lambda(s) & 0 \\ 0 & 0 \end{bmatrix} \tag{7.9}$$

with $\Lambda(s) \triangleq \mathrm{diag}[\epsilon_1(s), \ldots, \epsilon_r(s)]$, where $r = \mathrm{rank}\, P(s)$. The unique *monic* polynomials $\epsilon_i(s)$ (polynomials with leading coefficient equal to one) are the *invariant factors* of $P(s)$. It can be shown that $\epsilon_i(s)$ divides $\epsilon_{i+1}(s)$, $i = 1, \ldots, r-1$. Note that $\epsilon_i(s)$ can be determined by

$$\epsilon_i(s) = D_i(s)/D_{i-1}(s), \quad i = 1, \ldots, r,$$

where $D_i(s)$ is the monic greatest common divisor of all the nonzero ith-order minors of $P(s)$ with $D_0(s) = 1$. The $D_i(s)$ are the *determinantal divisors* of $P(s)$. A matrix $P(s)$ can be reduced to Smith form by elementary row and column operations or by a pre- and post-multiplication by unimodular matrices, namely

$$U_L(s)P(s)U_R(s) = S_p(s). \tag{7.10}$$

Unimodular Matrices. Let $R[s]^{p\times m}$ denote the set of $p \times m$ matrices with entries that are polynomials in s with real coefficients. A polynomial matrix $U(s) \in R[s]^{p\times p}$ is called *unimodular* (or $R[s]$-unimodular) if there exists a $\hat{U}(s) \in R[s]^{p\times p}$ such that $U(s)\hat{U}(s) = I_p$. This is the same as saying that $U^{-1}(s) = \hat{U}(s)$ exists and is a polynomial matrix. Equivalently, $U(s)$ is unimodular if $\det U(s) = \alpha \in R, \alpha \neq 0$. It can be shown that every unimodular matrix is a matrix representation of a finite number of successive elementary row and column operations. See [1, p. 526].

Consider now a $p \times m$ rational matrix $H(s)$. Let $d(s)$ be the monic least common denominator of all nonzero entries, and write

$$H(s) = \frac{1}{d(s)} N(s), \tag{7.11}$$

with $N(s)$ a polynomial matrix. Let $S_N(s) = \mathrm{diag}[n_1(s), \ldots, n_r(s), 0_{p-r,m-r}]$ be the Smith form of $N(s)$, where $r = \mathrm{rank}\, N(s) = \mathrm{rank}\, H(s)$. Divide each

$n_i(s)$ of $S_N(s)$ by $d(s)$, canceling all common factors to obtain the *Smith–McMillan* form of $H(s)$,

$$SM_H(s) = \begin{bmatrix} \tilde{\Lambda}(s) & 0 \\ 0 & 0 \end{bmatrix}, \tag{7.12}$$

with $\tilde{\Lambda}(s) \triangleq \text{diag}\left[\frac{\epsilon_1(s)}{\psi_1(s)}, \ldots, \frac{\epsilon_r(s)}{\psi_r(s)}\right]$, where $r = \text{rank}\, H(s)$. Note that $\epsilon_i(s)$ divides $\epsilon_{i+1}(s)$, $i = 1, 2, \ldots, r-1$, and $\psi_{i+1}(s)$ divides $\psi_i(s)$, $i = 1, 2, \ldots, r-1$.

7.4.2 Poles

Pole Polynomial of $H(s)$. Given a $p \times m$ rational matrix $H(s)$, its *characteristic polynomial* or *pole polynomial*, $p_H(s)$, is defined as

$$p_H(s) = \psi_1(s) \cdots \psi_r(s), \tag{7.13}$$

where the ψ_i, $i = 1, \cdots, r$, are the denominators of the Smith–McMillan form, $SM_H(s)$, of $H(s)$. It can be shown that $p_H(s)$ is the monic least common denominator of all nonzero minors of $H(s)$.

Definition 7.5. *The* poles of $H(s)$ *are the roots of the pole polynomial* $p_H(s)$. ■

Note that the monic least common denominator of all nonzero first-order minors (entries) of $H(s)$ is called the *minimal polynomial* of $H(s)$ and is denoted by $m_H(s)$. The $m_H(s)$ divides $p_H(s)$ and when the roots of $p_H(s)$ [poles of $H(s)$] are distinct, $m_H(s) = p_H(s)$, since the additional roots in $p_H(s)$ are repeated roots of $m_H(s)$.

It is important to note that when the minors of $H(s)$ [of order $1, 2, \ldots, \min(p, m)$] are formed by taking the determinants of all square submatrices of dimension $1 \times 1, 2 \times 2$, etc., all cancellations of common factors between numerator and denominator polynomials should be carried out.

In the scalar case, $p = m = 1$, Definition 7.5 reduces to the well-known definition of poles of a transfer function $H(s)$, since in this case there is only one minor (of order 1), $H(s)$, and the poles are the roots of the denominator polynomial of $H(s)$. Notice that in this case, it is assumed that all the possible cancellations have taken place in the transfer function of a system. Here $p_H(s) = m_H(s)$, that is, the pole or characteristic polynomial equals the minimal polynomial of $H(s)$. Thus, $p_H(s) = m_H(s)$ are equal to the (monic) denominator of $H(s)$.

Example 7.6. Let $H(s) = \begin{bmatrix} 1/[s(s+1)] & 1/s & 1 \\ 0 & 0 & 1/s^2 \end{bmatrix}$. The nonzero minors of order 1 are the nonzero entries. The least common denominator is $s^2(s+1) = m_H(s)$, the minimal polynomial of $H(s)$. The nonzero minors of order 2 are

$1/[s^3(s+1)]$ and $1/s^3$ (taking columns 1 and 3, and 2 and 3, respectively). The least common denominator of all minors (of order 1 and 2) is $s^3(s+1) = p_H(s)$, the characteristic polynomial of $H(s)$. The poles are $\{0, 0, 0, -1\}$. Note that $m_H(s)$ is a factor of $p_H(s)$, and the additional root at $s = 0$ in $p_H(s)$ is a repeated pole. To obtain the Smith–McMillan form of $H(s)$, write $H(s) = \frac{1}{s^2(s+1)} \begin{bmatrix} s & s(s+1) & s^2(s+1) \\ 0 & 0 & (s+1) \end{bmatrix} = \frac{1}{d(s)} N(s)$, where $d(s) = s^2(s+1) = m_H(s)$ [see (7.11)]. The Smith form of $N(s)$ is

$$S_N(s) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & s(s+1) & 0 \end{bmatrix}$$

since $D_0 = 1, D_1 = 1, D_2 = s(s+1)$ [the determinantal divisors of $N(s)$], and $n_1 = D_1/D_0 = 1, n_2 = D_2/D_1 = s(s+1)$, the invariant factors of $N(s)$. Dividing by $d(s)$, we obtain the Smith–McMillan form of $H(s)$,

$$SM_H(s) = \begin{bmatrix} \epsilon_1/\psi_1 & 0 & 0 \\ 0 & \epsilon_2/\psi_2 & 0 \end{bmatrix} = \begin{bmatrix} 1/[s^2(s+1)] & 0 & 0 \\ 0 & 1/s & 0 \end{bmatrix}.$$

Note that ψ_2 divides ψ_1 and ϵ_1 divides ϵ_2. Now the characteristic or pole polynomial of $H(s)$ is $p_H(s) = \psi_1\psi_2 = s^3(s+1)$ and the poles are $\{0, 0, 0, -1\}$, as expected.

Example 7.7. Let $H(s) = \frac{1}{s+2} \begin{bmatrix} 1 & \alpha \\ 1 & 1 \end{bmatrix}$. If $\alpha \neq 1$, then the second-order minor is $|H(s)| = \frac{1-\alpha}{(s+2)^2}$. The least common denominator of this nonzero second-order minor $|H(s)|$ and of all the entries of $H(s)$ (the first-order minors) is $(s+2)^2 = p_H(s)$; i.e., the poles are at $\{-2, -2\}$. Also, $m_H(s) = s + 2$.

Now if $\alpha = 1$, then there are only first-order nonzero minors ($|H(s)| = 0$). In this case $p_H(s) = m_H(s) = s + 2$, which is quite different from the case when $\alpha \neq 1$. Presently, there is only one pole at -2.

As will be shown in Chapter 8 via Theorems 8.9 and 8.12, the poles of $H(s)$ are exactly the controllable and observable eigenvalues of the system (in A_{11}) and no factors of $|sI - A_{11}|$ in $H(s)$ cancel [see (7.52)].

In general, for the set of poles of $H(s)$ and the eigenvalues of A, we have

$$\{\text{Poles of } H(s)\} \subset \{\text{eigenvalues of } A\}, \tag{7.14}$$

with equality holding when all the eigenvalues of A are controllable and observable eigenvalues of the system. Similar results hold for discrete-time systems and $H(z)$.

Example 7.8. Consider $A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$, and $C = [0, 1, 0]$. Then the transfer function $H(s) = [1/s, 1/s]$. $H(s)$ has only one pole, $s_1 = 0$ ($p_H(s) = s$), and $\lambda_1 = 0$, is the only controllable and observable eigenvalue. The other two eigenvalues of A, $\lambda_2 = -1$, $\lambda_3 = -2$, which are not both controllable and observable, do not appear as poles of $H(s)$.

Example 7.9. Recall the circuit in Example 7.2 in Section 7.2. If $R_1R_2C \neq L$, then {poles of $H(s)$} = {eigenvalues of A at $\lambda_1 = -1/(R_1C)$ and $\lambda_2 = -R_2/L$}. In this case, both eigenvalues are controllable and observable. Now if $R_1R_2C = L$ with $R_1 \neq R_2$, then $H(s)$ has only one pole, $s_1 = -R_2/L$, since in this case only one eigenvalue $\lambda_1 = -R_2/L$ is controllable and observable. The other eigenvalue λ_2 at the same location $-R_2/L$ is uncontrollable and unobservable. Now if $R_1R_2C = L$ with $R_1 = R_2 = R$, then one of the eigenvalues becomes uncontrollable and the other (also at $-R/L$) becomes unobservable. In this case $H(s)$ has no finite poles ($H(s) = 1/R$).

7.4.3 Zeros

In a scalar transfer function $H(s)$, the roots of the denominator polynomial are the poles, and the roots of its numerator polynomial are the zeros of $H(s)$. As was discussed, the *poles of* $H(s)$ are some or all of the eigenvalues of A (the eigenvalues of A are sometimes also called *poles of the system* $\{A, B, C, D\}$). In particular, the uncontrollable and/or unobservable eigenvalues of A can never be poles of $H(s)$. In Chapter 8 (Theorems 8.9 and 8.12), it is shown that only those eigenvalues of A that are both controllable and observable appear as poles of the transfer function $H(s)$. Along similar lines, the *zeros of* $H(s)$ (to be defined later) are some or all of the characteristic values of another matrix, the system matrix $P(s)$. These characteristic values are called the *zeros of the system* $\{A, B, C, D\}$.

The *zeros of a system* for both the continuous- and the discrete-time cases are defined and discussed next. We consider now only finite zeros. For the case of zeros at infinity, refer to the exercises.

Let the *system matrix* (also called *Rosenbrock's system matrix*) of $\{A, B, C, D\}$ be

$$P(s) \triangleq \begin{bmatrix} sI - A & B \\ -C & D \end{bmatrix}. \tag{7.15}$$

Note that in view of the system equations $\dot{x} = Ax + Bu, y = Cx + Du$, we have

$$P(s) \begin{bmatrix} -\hat{x}(s) \\ \hat{u}(s) \end{bmatrix} = \begin{bmatrix} 0 \\ \hat{y}(s) \end{bmatrix},$$

where $\hat{x}(s)$ denotes the Laplace transform of $x(t)$.

Zero Polynomial of (A, B, C, D). Let $r = \operatorname{rank} P(s)$ [note that $n \le r \le \min(p+n, m+n)$], and consider all those rth order nonzero minors of $P(s)$ that are formed by taking the first n rows and n columns of $P(s)$, i.e., all rows and columns of $sI - A$, and then adding appropriate $r - n$ rows (of $[-C, D]$) and columns (of $[B^T, D^T]^T$). The *zero polynomial of the system* $\{A, B, C, D\}$, $z_p(s)$, is defined as the monic greatest common divisor of all these minors.

Definition 7.10. *The* zeros of the system $\{A, B, C, D\}$ *or the* system zeros *are the roots of the zero polynomial of the system,* $z_P(s)$. ■

In addition, we define the *invariant zeros of the system* as the roots of the invariant polynomials of $P(s)$.

In particular, consider the $(p+n) \times (m+n)$ system matrix $P(s)$ and let

$$S_P(s) = \begin{bmatrix} \Lambda(s) & 0 \\ 0 & 0 \end{bmatrix}, \quad \Lambda(s) = \operatorname{diag}[\epsilon_1(s), \ldots, \epsilon_r(s), 0] \tag{7.16}$$

be its Smith form. The *invariant zero polynomial of the system* $\{A, B, C, D\}$ is defined as

$$z_P^I(s) = \epsilon_1(s)\epsilon_2(s) \cdots \epsilon_r(s), \tag{7.17}$$

and its roots are the *invariant zeros of the system*. It can be shown that the monic greatest common divisor of all the highest order nonzero minors of $P(s)$ equals $z_P^I(s)$.

In general,

$$\{\text{zeros of the system}\} \supset \{\text{invariant zeros of the system}\}.$$

When $p = m$ with $\det P(s) \neq 0$, then the zeros of the system coincide with the invariant zeros.

Now consider the $n \times (m+n)$ matrix $[sI - A, B]$ and determine its n invariant factors $\epsilon_i(s)$ and its Smith form. The product of its invariant factors is a polynomial, the roots of which are the *input-decoupling zeros of the system* $\{A, B, C, D\}$. Note that this polynomial equals the monic greatest common divisor of all the highest order nonzero minors (of order n) of $[sI - A, B]$. Similarly, consider the $(p+n) \times n$ matrix $\begin{bmatrix} sI - A \\ -C \end{bmatrix}$ and its invariant polynomials, the roots of which define the *output-decoupling zeros of the system* $\{A, B, C, D\}$.

Using the above definitions, it is not difficult to show that the input-decoupling zeros of the system are eigenvalues of A and also zeros of the system $\{A, B, C, D\}$. In addition note that if λ_i is such an input-decoupling zero, then $\operatorname{rank}[\lambda_i I - A, B] < n$, and therefore, there exists a $1 \times n$ vector $\hat{v}_i \neq 0$ such that $\hat{v}_i[\lambda_i I - A, B] = 0$. This, however, implies that λ_i is an uncontrollable eigenvalue of A (and $\hat{v}_i$ is the corresponding left eigenvector), in view of Section 6.3. Conversely, it can be shown that an uncontrollable eigenvalue is an input-decoupling zero. Therefore, *the input-decoupling zeros* *of the system* $\{A, B, C, D\}$ *are the uncontrollable eigenvalues of* A. Similarly, it can be shown that the *output-decoupling zeros of the system* $\{A, B, C, D\}$ *are the unobservable eigenvalues of* A. They are also zeros of the system, as can easily be seen from the definitions.

There are eigenvalues of A that are both uncontrollable and unobservable. These can be determined using the left and right corresponding eigenvector test or by the Canonical Structure Theorem (Kalman Decomposition Theorem) (see Sections 6.2 and 6.3). These uncontrollable and unobservable eigenvalues of A are zeros of the system that are both input- and output-decoupling zeros and are called *input–output decoupling zeros*. These input–output decoupling zeros can also be defined directly from $P(s)$ given in (7.15); however, care should be taken in the case of repeated zeros.

If the zeros of a system are determined and the zeros that are input- and/or output-decoupling zeros are removed, then the zeros that remain are the *zeros of* $H(s)$ and can be found directly from the transfer function $H(s)$.

Zero Polynomial of $H(s)$. In particular, if the Smith–McMillan form of $H(s)$ is given by (7.12), then

$$z_H(s) = \epsilon_1(s)\epsilon_2(s)\cdots\epsilon_r(s) \tag{7.18}$$

is the *zero polynomial of* $H(s)$ and its roots are the *zeros of* $H(s)$. These are also called the *transmission zeros of the system.*

Definition 7.11. *The* zeros of $H(s)$ *or the* transmission zeros of the system *are the roots of the zero polynomial of* $H(s), z_H(s)$. ∎

When $P(s)$ is square and nonsingular, the relationship between the zeros of the system and the zeros of $H(s)$ can easily be determined. Consider the identity

$$P(s) = \begin{bmatrix} sI - A & B \\ -C & D \end{bmatrix} = \begin{bmatrix} sI - A & 0 \\ -C & I \end{bmatrix} \begin{bmatrix} I & (sI - A)^{-1}B \\ 0 & H(s) \end{bmatrix}$$

and note that $|P(s)| = |sI - A|\ |H(s)|$. In this case, the invariant zeros of the system [the roots of $|P(s)|$], which are equal here to the zeros of the system, are the zeros of $H(s)$ [the roots of $|H(s)|$] *and* those eigenvalues of A that are not both controllable and observable [the ones that do not cancel in $|sI - A||H(s)|$].

Note that the zero polynomial of $H(s), z_H(s)$, equals the monic greatest common divisor of the numerators of all the highest order nonzero minors in $H(s)$ after all their denominators have been set equal to $p_H(s)$, the characteristic polynomial of $H(s)$. In the scalar case ($p = m = 1$), our definition of the zeros of $H(s)$ reduces to the well-known definition of zeros, namely, the roots of the numerator polynomial of $H(s)$.

Example 7.12. Consider $H(s)$ of Example 7.6. From the Smith–McMillan form of $H(s)$, we obtain the zero polynomial $z_H(s) = 1$, and $H(s)$ has no (finite) zeros. Alternatively, the highest order nonzero minors are $1/[s^3(s+1)]$ and $1/s^3 = (s+1)/[s^3(s+1)]$ and the greatest common divisor of the numerators is $z_H(s) = 1$.

Example 7.13. We wish to determine the zeros of $H(s) = \begin{bmatrix} \frac{s}{s+1} & 0 \\ \frac{1}{s+1} & \frac{s+1}{s^2} \end{bmatrix}$. The first-order minors are the entries of $H(s)$, namely $\frac{s}{s+1}, \frac{1}{s+1}, \frac{s+1}{s^2}$, and there is only one second-order minor $\frac{s}{s+1} \cdot \frac{s+1}{s^2} = \frac{1}{s}$. Then $p_H(s) = s^2(s+1)$, the least common denominator, is the characteristic polynomial. Next, write the highest (second-) order minor as $\frac{1}{s} = \frac{s(s+1)}{s^2(s+1)} = \frac{s(s+1)}{p_H(s)}$ and note that $s(s+1)$ is the zero polynomial of $H(s), z_H(s)$, and the zeros of $H(s)$ are $\{0, -1\}$. It is worth noting that the poles and zeros of $H(s)$ are at the same locations. This may happen only when $H(s)$ is a matrix.

If the Smith–McMillan form of $H(s)$ is to be used, write $H(s) = \frac{1}{s^2(s+1)} \begin{bmatrix} s^3 & 0 \\ s^2 & (s+1)^2 \end{bmatrix} = \frac{1}{d(s)} N(s)$. The Smith form of $N(s)$ is now $\begin{bmatrix} 1 & 0 \\ 0 & s^3(s+1)^2 \end{bmatrix}$ since $D_0 = 1, D_1 = 1, D_2 = s^3(s+1)^2$ with invariant factors of $N(s)$ given by $n_1 = D_1/D_0 = 1$ and $n_2 = D_2/D_1 = s^3(s+1)^2$. Therefore, the Smith–McMillan form (7.12) of $H(s)$ is

$$SM_H(s) = \begin{bmatrix} \frac{1}{s^2(s+1)} & 0 \\ 0 & \frac{s(s+1)}{1} \end{bmatrix} = \begin{bmatrix} \epsilon_1/\psi_1 & 0 \\ 0 & \epsilon_2/\psi_2 \end{bmatrix}.$$

The zero polynomial is then $z_H(s) = \epsilon_1 \epsilon_2 = s(s+1)$, and the zeros of $H(s)$ are $\{0, -1\}$, as expected. Also, the pole polynomial is $p_H(s) = \psi_1 \psi_2 = s^2(s+1)$, and the poles are $\{0, 0, -1\}$.

Example 7.14. We wish to determine the zeros of $H(s) = \begin{bmatrix} \frac{s}{s+1} & 0 \\ \frac{1}{s+1} & \frac{s+1}{s^2} \\ 0 & \frac{1}{s} \end{bmatrix}$.

The second-order minors are $\frac{1}{s}, \frac{1}{s+1}, \frac{1}{s(s+1)}$, and the characteristic polynomial is $p_H(s) = s^2(s+1)$. Rewriting the highest (second-) order minors as $s(s+1)/p_H(s), s^2/p_H(s)$, and $s/p_H(s)$, the greatest common divisor of the numerators is s; i.e., the zero polynomial of $H(s)$ is $z_H(s) = s$. Thus, there is only one zero of $H(s)$ located at 0. Alternatively, note that the Smith–McMillan form is

$$SM_H(s) = \begin{bmatrix} 1/[s^2(s+1)] & 0 \\ 0 & s/1 \\ 0 & 0 \end{bmatrix}.$$

7.4.4 Relations Between Poles, Zeros, and Eigenvalues of A

Consider the system $\dot{x} = Ax + Bu, y = Cx + Du$ and its transfer function matrix $H(s) = C(sI - A)^{-1}B + D$. Summarizing the above discussion, the following relations can be shown to be true.

1. We have the set relationship

$$\begin{aligned}\{\text{zeros of the system}\} = \{&\text{zeros of } H(s)\}\\ &\cup \{\text{input-decoupling zeros}\} \cup \{\text{output-decoupling zeros}\}\\ &- \{\text{input–output decoupling zeros}\}.\end{aligned} \tag{7.19}$$

   Note that the invariant zeros of the system contain all the zeros of $H(s)$ (transmission zeros), but not all the decoupling zeros (see Example 7.15). When $P(s)$ is square and nonsingular, the zeros of the system are exactly the invariant zeros of the system. Also, in the case when $\{A, B, C, D\}$ is controllable and observable, the zeros of the system, the invariant zeros, and the transmission zeros [zeros of $H(s)$] all coincide.
2. We have the set relationship

$$\begin{aligned}&\{\text{eigenvalues of } A \text{ (or poles of the system)}\} = \{\text{poles of } H(s)\}\\ &\cup \{\text{uncontrollable eigenvalues of } A\} \cup \{\text{unobservable eigenvalues of } A\}\\ &- \{\text{both uncontrollable and unobservable eigenvalues of } A\}.\end{aligned} \tag{7.20}$$

3. We have the set relationships

$$\{\text{input-decoupling zeros}\} = \{\text{uncontrollable eigenvalues of } A\},$$
$$\{\text{output-decoupling zeros}\} = \{\text{unobservable eigenvalue of } A\},$$

   and

$$\begin{aligned}&\{\text{input–output decoupling zeros}\} =\\ &\quad\{\text{eigenvalues of } A \text{ that are both uncontrollable and unobservable}\}.\end{aligned} \tag{7.21}$$

4. When the system $\{A, B, C, D\}$ is controllable and observable, then

$$\begin{aligned}\{\text{zeros of the system}\} &= \{\text{zeros of} H(s)\}\\ \text{and } \{\text{eigenvalues of } A \text{ (or poles of the system)}\} &= \{\text{poles of } H(s)\}.\end{aligned} \tag{7.22}$$

   Note that *the eigenvalues of A (the poles of the system)* can be defined as the roots of the invariant factors of $sI - A$ in $P(s)$ given in (7.15).

Example 7.15. Consider the system $\{A, B, C\}$ of Example 7.8. Let

$$P(s) = \begin{bmatrix} sI - A & B \\ -C & D \end{bmatrix} = \left[\begin{array}{ccc|cc} s & 1 & -1 & 1 & 0 \\ -1 & s+2 & -1 & 1 & 1 \\ 0 & -1 & s+1 & 1 & 2 \\ \hline 0 & -1 & 0 & 0 & 0 \end{array}\right].$$

There are two fourth-order minors that include all columns of $sI - A$ obtained by taking columns 1, 2, 3, 4 and columns 1, 2, 3, 5 of $P(s)$; they are $(s+1)(s+2)$ and $(s+1)(s+2)$. The zero polynomial of the system is $z_P = (s+1)(s+2)$, and the zeros of the system are $\{-1, -2\}$. To determine the input-decoupling zeros, consider all the third-order minors of $[sI - A, B]$. The greatest common divisor is $s + 2$, which implies that the input-decoupling zeros are $\{-2\}$. Similarly, consider $\begin{bmatrix} sI - A \\ -C \end{bmatrix}$ and show that $s + 1$ is the greatest common divisor of all the third-order minors and that the output-decoupling zeros are $\{-1\}$. The transfer function for this example was found in Example 7.8 to be $H(s) = [1/s, 1/s]$. The zero polynomial of $H(s)$ is $z_H(s) = 1$, and there are no zeros of $H(s)$. Notice that there are no input–output decoupling zeros. It is now clear that relation (7.19) holds.

The controllable (resp., uncontrollable) and the observable (resp., unobservable) eigenvalues of A (poles of the system) have been found in Example 6.10. Compare these results to show that (7.21) holds. The poles of $H(s)$ are $\{0\}$. Verify that (7.20) holds.

One could work with the Smith form of the matrices of interest and the Smith–McMillan form of $H(s)$. In particular, it can be shown that the Smith form of $P(s)$ is $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & (s+2) & 0 \end{bmatrix}$, of $[sI - A, B]$ is $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & s+2 & 0 & 0 \end{bmatrix}$, of $\begin{bmatrix} sI - A \\ -C \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s+1 \\ 0 & 0 & 0 \end{bmatrix}$, and of $[sI - A]$ is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s(s+1)(s+2) \end{bmatrix}$. Also, it can be shown that the Smith–McMillan form of $H(s)$ is

$$SM_H(s) = [1/s, 0].$$

It is straightforward to verify the above results. Note that in the present case the invariant zero polynomial is $z_P^I(s) = s + 2$ and there is only one invariant zero at -2.

Example 7.16. Consider the circuit of Example 7.9 and of Example 7.2 and the system matrix $P(s)$ for the case when $R_1 R_2 C = L$ given by

$$P(s) = \begin{bmatrix} sI - A & B \\ -C & D \end{bmatrix} = \left[\begin{array}{cc|c} s + R_2/L & 0 & R_2/L \\ 0 & s + R_2/L & 1/L \\ \hline 1/R_1 & -1 & 1/R_1 \end{array}\right].$$

(i) First, let $R_1 \neq R_2$. To determine the zeros of the system, consider $|P(s)| = (1/R_1)(s+R_1/L)(s+R_2/L)$, which implies that the zeros of the system are $\{-R_1/L, -R_2/L\}$. Consider now all second-order (nonzero) minors of $[sI-A, B]$, namely, $(s+R_2/L)^2$, $(1/L)(s+R_2/L)$ and $-(R_2/L)(s+R_2/L)$, from which we see that $\{-R_2/L\}$ is the input-decoupling zero. Similarly, we also see that $\{-R_2/L\}$ is the output-decoupling zero. Therefore, $\{-R_2/L\}$ is the input–output decoupling zero. Compare this with the results in Example 7.9 to verify (7.22).

(ii) When $R_1 = R_2 = R$, then $|P(s)| = (1/R)(s + R/L)^2$, which implies that the zeros of the system are at $\{-R/L, -R/L\}$. Proceeding as in (i), it can readily be shown that $\{-R/L\}$ is the input-decoupling zero and $\{-R/L\}$ is the output-decoupling zero. To determine which are the input–output decoupling zeros, one needs additional information to the zero location. This information can be provided by the left and right eigenvectors of the two zeros at $-R/L$ to determine that there is no input–output decoupling zero in this case (see Example 7.2).

In both cases (i) and (ii), $H(s)$ has been derived in Example 7.2. Verify relation (7.19).

Finally, note that there are characteristic vectors or zero directions, associated with each invariant and decoupling zero of the system $\{A, B, C, D\}$, just as there are characteristic vectors or eigenvectors, associated with each eigenvalue of A (pole of the system) (see [1, p. 306, Section 3.5]). For pole-zero cancellations to take place in the case of multi-input or output systems when the transfer function matrix is formed, not only the pole, zero locations must be the same but also their characteristic directions must be aligned.

7.5 Polynomial Matrix and Matrix Fractional Descriptions of Systems

In this section, representations of linear time-invariant systems based on polynomial matrices, called *Polynomial Matrix Description (PMD)* [or *Differential (Difference) Operator Representation (DOR)*] are introduced. Such representations arise naturally when differential (or difference) equations are used to describe the behavior of systems, and the differential (or difference) operator is introduced to represent the operation of differentiation (or of time-shift). Polynomial matrices in place of polynomials are involved since this approach is typically used to describe multi-input, multi-output systems. Note that state-space system descriptions only involve first-order differential (or difference) equations, and as such, PMDs include the state-space descriptions as special cases.

A rational function matrix can be written as a ratio or fraction of two polynomial matrices or of two rational matrices. If the transfer function matrix of a system is expressed as a fraction of two polynomial or rational matrices, this leads to a *Matrix Fraction(al) Description (MFD)* of the system. The MFDs that involve polynomial matrices, called polynomial MFDs, can be viewed as representations of internal realizations of the transfer function matrix; that is, they can be viewed as system PMDs of special form. *These polynomial fractional descriptions (PMFDs) help establish the relationship between internal and external system representations in a clear and transparent manner.* This can be used to advantage, for example, in the study of feedback control problems, leading to clearer understanding of the phenomena that occur when systems are interconnected in feedback configurations. The MFDs that involve ratios of rational matrices, in particular ratios of proper and stable rational matrices, offer convenient characterizations of transfer functions in feedback control problems.

MFDs that involve ratios of polynomial matrices and ratios of proper and stable rational matrices are essential in parameterizing all stabilizing feedback controllers. Appropriate selection of the parameters guarantees that a closed-loop system is not only stable, but it will also satisfy additional control criteria. This is precisely the approach taken in optimal control methods, such as H^∞-optimal control. Parameterizations of all stabilizing feedback controllers are studied in Chapter 10. We note that extensions of MFDs are also useful in linear, time-varying systems and in nonlinear systems. These extensions are not addressed here.

In addition to the importance of MFDs in characterizing all stabilizing controllers, and in H^∞-optimal control, PMFDs and PMDs have been used in other control design methodologies as well (e.g., self-tuning control). The use of PMFDs in feedback control leads in a natural way to the polynomial Diophantine matrix equation, which is central in control design when PMDs are used and which directly leads to the characterization of all stabilizing controllers. Finally, PMDs are generalizations of state-space descriptions, and the use of PMDs to characterize the behavior of systems offers additional insight and flexibility. Detailed treatment of all these issues may be found in [1, Chapter 7]. The development of the material in this section is concerned only with continuous-time systems; however, completely analogous results are valid for discrete-time systems and can easily be obtained by obvious modifications. In this section we emphasize PMFD and discuss controllability, observability, and stability.

An Important Comment on Notation. We will be dealing with matrices with entries polynomials in s or q, denoted by, e.g., $D(s)$ or $D(q)$, where s is the Laplace variable and $q \triangleq d/dt$, the differential operator. For simplicity of notation we frequently omit the argument s or q and we write D to denote the polynomial matrix on hand. When ambiguity may arise, or when it is important to stress the fact that the matrix in question is a polynomial matrix, the argument will be included.

7.5.1 A Brief Introduction to Polynomial and Fractional Descriptions

Below, the Polynomial Matrix Description (PMD) and the Matrix Fractional Description (MFD) of a linear, time-invariant system are introduced via a simple illustrating example.

Example 7.17. In the ordinary differential equation representation of a system given by

$$\begin{aligned} \ddot{y}_1(t) + y_1(t) + y_2(t) &= \dot{u}_2(t) + u_1(t), \\ \dot{y}_1(t) + \dot{y}_2(t) + 2y_2(t) &= \dot{u}_2(t), \end{aligned} \tag{7.23}$$

$y_1(t), y_2(t)$ and $u_1(t), u_2(t)$ denote, respectively, outputs and inputs of interest. We assume that appropriate initial conditions for the $u_i(t), y_i(t)$ and their derivatives at $t = 0$ are given.

By changing variables, one can express (7.23) by an equivalent set of first-order ordinary differential equations, in the sense that this set of equations will generate all solutions of (7.23), using appropriate initial conditions and the same inputs. To this end, let

$$x_1 = \dot{y}_1 - u_2, \quad x_2 = y_1, \quad x_3 = y_1 + y_2 - u_2. \tag{7.24}$$

Then (7.23) can be written as

$$\dot{x} = Ax + Bu, \quad y = Cx + Du, \tag{7.25}$$

where $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$, $u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$, $y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$, and

$$A = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 2 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 0 & -2 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

with initial conditions $x(0)$ calculated by using (7.24).

More directly, however, system (7.23) can be represented by

$$P(q)z(t) = Q(q)u(t), \quad y(t) = R(q)z(t) + W(q)u(t), \tag{7.26}$$

where $z(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix}$, $u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$, $y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$, and

$$P(q) = \begin{bmatrix} q^2+1 & 1 \\ q & q+2 \end{bmatrix}, \quad Q(q) = \begin{bmatrix} 1 & q \\ 0 & q \end{bmatrix}, \quad R(q) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad W(q) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

with $q \triangleq \frac{d}{dt}$, the differential operator. The variables $z_1(t), z_2(t)$ are called *partial state variables*, $z(t)$ denotes the *partial state* of the system description (7.26), and $u(t)$ and $y(t)$ denote the input and output vectors, respectively.

Polynomial Matrix Descriptions (PMDs)

Representation (7.26), also denoted as $\{P(q), Q(q), R(q), W(q)\}$, is an example of a *Polynomial Matrix Description (PMD)* of a system. Note that the state-space description (7.25) is a special case of (7.26). To see this, write (7.25) as

$$(qI - A)x(t) = Bu(t), \quad y(t) = Cx(t) + Du(t). \tag{7.27}$$

Clearly, description $\{qI - A, B, C, D\}$ is a special case of the general Polynomial Matrix Description $\{P(q), Q(q), R(q), W(q)\}$ with

$$P(q) = qI - A, Q(q) = B, R(q) = C, W(q) = D. \tag{7.28}$$

The above example points to the fact that a PMD of a system can be derived in a natural way from differential (or difference) equations that involve variables that are directly connected to physical quantities. By this approach, it is frequently possible to study the behavior of physical variables directly without having to transform the system to a state-space description. The latter may involve (state) variables that are quite removed from the physical phenomena they represent, thus losing physical insight when studying a given problem. The price to pay for this additional insight is that one has to deal with differential (or difference) equations of order greater than one. This typically adds computational burdens. We note that certain special forms of PMDs, namely the polynomial Matrix Fractional Descriptions, are easier to deal with than general forms. However, a change of variables may again be necessary to obtain such forms.

Consider a general PMD of a system given by

$$P(q)z(t) = Q(q)u(t), \quad y(t) = R(q)z(t) + W(q)u(t), \tag{7.29}$$

with $P(q) \in R[q]^{l\times l}, Q(q) \in R[q]^{l\times m}$, and $R(q) \in R[q]^{p\times l}, W(q) \in R[q]^{p\times m}$, where $R[q]^{l\times l}$ denotes the set of $l \times l$ matrices with entries that are real polynomials in q. The transfer function matrix $H(s)$ of (7.29) can be determined by taking the Laplace transform of both sides of the equation assuming zero initial conditions ($z(0) = \dot{z}(0) = \cdots = 0, u(0) = \dot{u}(0) = \cdots = 0$). Then

$$H(s) = R(s)P^{-1}(s)Q(s) + W(s). \tag{7.30}$$

For the special case of state-space representations, $H(s)$ in (7.30) assumes the well-known expression $H(s) = C(sI - A)^{-1}B + D$. For the study of the relationship between external and internal descriptions, (7.30) is not particularly convenient. There are, however, special cases of (7.30) that are very convenient to use in this regard. In particular, it can be shown [1, Section 7.3] that if the system is controllable, then there exists a representation equivalent to (7.29), which is of the form

$$D_c(q)z_c(t) = u(t), \quad y(t) = N_c(q)z_c(t), \tag{7.31}$$

where $D_c(q) \in R[q]^{m\times m}$ and $N_c(q) \in R[q]^{p\times m}$. Representation (7.31) is obtained by letting $Q(q) = I_m$ and $W(q) = 0$ in (7.29) and using D and N instead of P and R. Equation (7.30) now becomes

$$H(s) = N_c(s)D_c(s)^{-1}, \tag{7.32}$$

where $N_c(s)$ and $D_c(s)$ represent the matrix numerator and matrix demonimator of the transfer function, respectively. Similarly, if the system is observable, there exists a representation equivalent to (7.29), which is of the form

$$D_o(q)z_o(t) = N_o(q)u(t), \quad y(t) = z_o(t), \tag{7.33}$$

where $D_o(q) \in R[q]^{p\times p}$ and $N_o(q) \in R[q]^{p\times m}$. Representation (7.33) is obtained by letting in (7.29) $R(q) = I_p$ and $W(q) = 0$ with $P(q) = D_o(q)$ and $Q(q) = N_o(q)$. Here,

$$H(s) = D_o^{-1}(s)N_o(s). \tag{7.34}$$

Note that (7.32) and (7.34) are generalizations to the MIMO case of the SISO system expression $H(s) = n(s)/d(s)$. As $H(s) = n(s)/d(s)$ can be derived directly from the differential equation $d(q)y(t) = n(q)u(t)$, by taking the Laplace transform and assuming that the initial conditions are zero, (7.34) can be derived directly from (7.33).

Returning now to (7.25) in Example 7.17, notice that the system is observable (state observable from the output y). Therefore, the system in this case can be represented by a description of the form $\{D_o, N_o, I_2, 0\}$. In fact, (7.26) is such a description, where D_o and N_o are equal to P and Q, respectively, i.e., $D_o(q) = \begin{bmatrix} q^2+1 & 1 \\ q & q+2 \end{bmatrix}$, and $N_o(q) = \begin{bmatrix} 1 & q \\ 0 & q \end{bmatrix}$. The transfer function matrix is given by

$$\begin{aligned} H(s) = C(sI-A)^{-1}B + D &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} s & 0 & 1 \\ -1 & s & 0 \\ 0 & -2 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 0 & -2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ &= D_o^{-1}(s)N_o(s) = \begin{bmatrix} s^2+1, & 1 \\ s, & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & s \\ 0 & s \end{bmatrix} \\ &= \tfrac{1}{s^3+2s^2+2} \begin{bmatrix} s+2 & -1 \\ -s & s^2+1 \end{bmatrix} \begin{bmatrix} 1 & s \\ 0 & s \end{bmatrix} = \tfrac{1}{s^3+2s^2+2} \begin{bmatrix} s+2 & s(s+1) \\ -s & s(s^2-s+1) \end{bmatrix}. \end{aligned}$$

Matrix Fractional Descriptions (MFDs) of System Transfer Matrices

A given $p \times m$ proper, rational transfer function matrix $H(s)$ of a system can be represented as

$$H(s) = N_R(s)D_R^{-1}(s) = D_L^{-1}(s)N_L(s), \tag{7.35}$$

where $N_R(s) \in R[s]^{p\times m}$, $D_R(s) \in R[s]^{m\times m}$ and $N_L(s) \in R[s]^{p\times m}$, $D_L(s) \in R[s]^{p\times p}$. The pairs $\{N_R(s), D_R(s)\}$ and $\{D_L(s), N_L(s)\}$ are called *Polynomial Matrix Fractional Descriptions (PMFDs)* of the system transfer matrix with $\{N_R(s), D_R(s)\}$ termed a *right Fractional Description* and $\{D_L(s), N_L(s)\}$ a *left Fractional Description.* Notice that in view of (7.32), the right Polynomial Matrix Fractional Description (rPMFD) corresponds to the controllable Polynomial Matrix Description (PMD) given in (7.31). That is, $\{D_R, I_m, N_R, 0\}$, or

$$D_R(q)z_R(t) = u(t), \quad y(t) = N_R(q)z_R(t), \tag{7.36}$$

is a controllable PMD of the system with transfer function $H(s)$. The subscript c was used in (7.31) and (7.32) to emphasize the fact that N_c, D_c originated from an internal description that was controllable. In (7.35) and (7.36), the subscript R is used to emphasize that $\{N_R, D_R\}$ is a right fraction representation of the external description $H(s)$.

Similarly, in view of (7.34), the left Polynomial Matrix Fractional Description (lPMFD) corresponds to the observable Polynomial Matrix Description (PMD) given in (7.33). That is, $\{D_L, N_L, I_p, 0\}$, or

$$D_L(q)z_L(t) = N_L(q)u(t), \quad y(t) = z_L(t), \tag{7.37}$$

is an observable PMD of the system with transfer function $H(s)$. Comments analogous to the ones made above concerning controllable and right fractional descriptions (subscripts c and R) can also be made here concerning the subscripts o and L.

An *MFD* of a transfer function may not consist necessarily of ratios of polynomial matrices. In particular, given a $p \times m$ proper transfer function matrix $H(s)$, one can write

$$H(s) = \widehat{N}_R(s)\widehat{D}_R^{-1}(s) = \widehat{D}_L^{-1}(s)\widehat{N}_L(s), \tag{7.38}$$

where $\widehat{N}_R, \widehat{D}_R, \widehat{D}_L, \widehat{N}_L$ are proper and stable rational matrices. To illustrate, in the example considered above, $H(s)$ can be written as

$$\begin{aligned}
H(s) &= \frac{1}{s^3 + 2s^2 + 2}\begin{bmatrix} s+2 & s(s+1) \\ -s & s(s^2 - s + 1 \end{bmatrix} \\
&= \left[\begin{bmatrix} (s+1)^2 & 0 \\ 0 & s+2 \end{bmatrix}^{-1}\begin{bmatrix} s^2+1 & 1 \\ s & s+2 \end{bmatrix}\right]^{-1}\left[\begin{bmatrix} (s+1)^2 & 0 \\ 0 & s+2 \end{bmatrix}^{-1}\begin{bmatrix} 1 & s \\ 0 & s \end{bmatrix}\right] \\
&= \begin{bmatrix} \frac{s^2+1}{(s+1)^2} & \frac{1}{(s+1)^2} \\ \frac{s}{s+2} & 1 \end{bmatrix}^{-1}\begin{bmatrix} \frac{1}{(s+1)^2} & \frac{s}{(s+1)^2} \\ 0 & \frac{s}{s+2} \end{bmatrix} = \widehat{D}_L^{-1}(s)\widehat{N}_L(s).
\end{aligned}$$

Note that $\widehat{D}_L(s)$ and $\widehat{N}_L(s)$ are proper and stable rational matrices.

Such representations of proper transfer functions offer certain advantages when designing feedback control systems. They are discussed further in [1, Section 7.4D].

7.5.2 Coprimeness and Common Divisors

Coprimeness of polynomial matrices is one of the most important concepts in the polynomial matrix representation of systems since it is directly related to controllability and observability.

A polynomial $g(s)$ is a *common divisor* (cd) of polynomials $p_1(s), p_2(s)$ if and only if there exist polynomials $\tilde{p}_1(s), \tilde{p}_2(s)$ such that

$$p_1(s) = \tilde{p}_1(s)g(s), \quad p_2(s) = \tilde{p}_2(s)g(s). \tag{7.39}$$

The highest degree cd of $p_1(s), p_2(s), g^*(s)$, is a *greatest common divisor* (gcd) of $p_1(s), p_2(s)$. It is unique within multiplication by a nonzero real number. Alternatively, $g^*(s)$ is a gcd of $p_1(s), p_2(s)$ if and only if *any* cd $g(s)$ of $p_1(s), p_2(s)$ is a divisor of $g^*(s)$ as well; that is,

$$g^*(s) = m(s)g(s) \tag{7.40}$$

with $m(s)$ a polynomial. The polynomials $p_1(s), p_2(s)$ are *coprime* (cp) if and only if a gcd $g^*(s)$ is a nonzero real.

The above can be extended to matrices. In this case, both right divisors and left divisors must be defined, since in general, two polynomial matrices do not commute. Note that one may talk about right or left divisors of polynomial matrices only when the matrices have the same number of columns or rows, respectively.

An $m \times m$ matrix $G_R(s)$ is a *common right divisor* (crd) of the $p_1 \times m$ polynomial matrix $P_1(s)$ and the $p_2 \times m$ matrix $P_2(s)$, if there exist polynomial matrices $P_{1R}(s), P_{2R}(s)$ so that

$$P_1(s) = P_{1R}(s)G_R(s), \quad P_2(s) = P_{2R}(s)G_R(s). \tag{7.41}$$

Similarly, a $p \times p$ polynomial matrix $G_L(s)$ is a *common left divisor* (cld) of the $p \times m_1$ polynomial matrix $\widehat{P}_1(s)$ and the $p \times m_2$ matrix $\widehat{P}_2(s)$, if there exist polynomial matrices $\widehat{P}_{1L}(s), \widehat{P}_{2L}(s)$ so that

$$\widehat{P}_1(s) = G_L(s)\widehat{P}_{1L}(s), \quad \widehat{P}_2(s) = G_L(s)\widehat{P}_{2L}(s). \tag{7.42}$$

Also $G_R^*(s)$ is a *greatest common right divisor* (gcrd) of $P_1(s)$ and $P_2(s)$ if and only if any crd $G_R(s)$ is an rd of $G_R^*(s)$. Similarly, $G_L^*(s)$ is a *greatest common left divisor* (gcld) of $\widehat{P}_1(s)$ and $\widehat{P}_2(s)$ if and only if any cld $G_L(s)$ is a ld of $G_L^*(s)$. That is,

$$G_R^*(s) = M(s)G_R(s), \quad G_L^*(s) = G_L(s)N(s), \tag{7.43}$$

with $M(s)$ and $N(s)$ polynomial matrices and $G_R(s)$ and $G_L(s)$ any crd and cld of $P_1(s), P_2(s)$, respectively.

Alternatively, it can be shown that any crd $G_R^*(s)$ of $P_1(s)$ and $P_2(s)$ [or a cld $G_L^*(s)$ of $\widehat{P}_1(s)$ and $\widehat{P}_2(s)$] with determinant of the highest degree possible is a gcrd (gcld) of the matrices. It is unique within a pre-multiplication (post-multiplication) by a unimodular matrix. Here it is assumed that $G_R(s)$ is nonsingular. Note that if rank $\begin{bmatrix} P_1(s) \\ P_2(s) \end{bmatrix} = m$ (a $(p_1+p_2)\times m$ matrix), which is a typical case in polynomial matrix system descriptions, then rank $G_R(s) = m$; that is, $G_R(s)$ is nonsingular.

The polynomial matrices $P_1(s)$ and $P_2(s)$ are *right coprime* (rc) if and only if a gcrd $G_R^*(s)$ is a unimodular matrix. Similarly, $\widehat{P}_1(s)$ and $\widehat{P}_2(s)$ are *left coprime* (lc) if and only if a gcld $G_2^*(s)$ is a unimodular matrix.

Example 7.18. Let $P_1 = \begin{bmatrix} s(s+2) & 0 \\ 0 & (s+1)^2 \end{bmatrix}$, $P_2 = \begin{bmatrix} (s+1)(s+2) & s+1 \\ 0 & s(s+1) \end{bmatrix}$. Two distinct common right divisors are $G_{R_1} = \begin{bmatrix} 1 & 0 \\ 0 & s+1 \end{bmatrix}$ and $G_{R_2} = \begin{bmatrix} s+2 & 0 \\ 0 & 1 \end{bmatrix}$ since $\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \left[\begin{array}{cc} s(s+2) & 0 \\ 0 & s+1 \\ \hline (s+1)(s+2) & 1 \\ 0 & s \end{array}\right] G_{R_1} = \left[\begin{array}{cc} s & 0 \\ 0 & (s+1)^2 \\ \hline s+2 & s+1 \\ 0 & s(s+1) \end{array}\right] G_{R_2}$. A greatest common right divisor (gcrd) is $G_R^* = \begin{bmatrix} s+2 & 0 \\ 0 & s+1 \end{bmatrix} = \begin{bmatrix} s+2 & 0 \\ 0 & 1 \end{bmatrix} G_{R_1} = \begin{bmatrix} 1 & 0 \\ 0 & s+1 \end{bmatrix} G_{R_2}$. Now, $\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} G_R^{*-1} = \begin{bmatrix} P_{1R}^* \\ P_{2R}^* \end{bmatrix} = \left[\begin{array}{cc} s & 0 \\ 0 & s+1 \\ \hline s+1 & 1 \\ 0 & s \end{array}\right]$ where P_{1R}^* and P_{2R}^* are right coprime (rc). Note that a greatest common left divisor (gcld) of P_1 and P_2 is $G_L^* = \begin{bmatrix} 1 & 0 \\ 0 & s+1 \end{bmatrix}$. Both G_R^* and G_L^* can be determined using an algorithm to derive the Hermite form of $\begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$; see [1, p. 532].

Remarks

It can be shown that two square $p \times p$ nonsingular polynomial matrices with determinants that are prime polynomials are both right and left coprime. The converse of this is not true; that is, two right coprime polynomial matrices do not necessarily have prime determinant polynomials. A case in point is Example 7.18, where P_{1R}^* and P_{2R}^* are right coprime; however, $\det P_{1R}^* = \det P_{2R}^* = s(s+1)$.

Left and right coprimeness of two polynomial matrices (provided that the matrices are compatible) are quite distinct properties. For example, two matrices can be left coprime but not right coprime, and vice versa (refer to Example 7.19).

Example 7.19. $P_1 = \begin{bmatrix} s(s+2) & 0 \\ 0 & s+1 \end{bmatrix}$ and $P_2 = \begin{bmatrix} (s+1)(s+2) & 1 \\ 0 & s \end{bmatrix}$ are left coprime but not right coprime since a gcrd is $G_R^* = \begin{bmatrix} s+2 & 0 \\ 0 & 1 \end{bmatrix}$ with $\det G_R^* = (s+2)$.

Finally, we note that all of the above definitions apply also to more than two polynomial matrices. To see this, replace in all definitions P_1, P_2 by $P_1, P_2, \ldots, P_k$. This is not surprising in view of the fact that the $p_1 \times m$ matrix $P_1(s)$ and the $p_2 \times m$ matrix $P_2(s)$ consist of p_1 and p_2 rows, respectively, each of which can be viewed as a $1 \times m$ polynomial matrix; that is, instead of, e.g., the coprimeness of P_1 and P_2, one could speak of the coprimeness of the $(p_1 + p_2)$ rows of P_1 and P_2.

How to Determine a Greatest Common Right Divisor

Lemma 7.20. *Let $P_1(s) \in R[s]^{p_1 \times m}$ and $P_2(s) \in R[s]^{p_2 \times m}$ with $p_1 + p_2 \geq m$. Let the unimodular matrix $U(s)$ be such that*

$$U(s) \begin{bmatrix} P_1(s) \\ P_2(s) \end{bmatrix} = \begin{bmatrix} G_R^*(s) \\ 0 \end{bmatrix}. \tag{7.44}$$

Then $G_R^*(s)$ is a greatest common right divisor (gcrd) of $P_1(s), P_2(s)$.

Proof. Let

$$U = \begin{bmatrix} \bar{X} & \bar{Y} \\ -\widetilde{P}_2 & \widetilde{P}_1 \end{bmatrix}, \tag{7.45}$$

with $\bar{X} \in R[s]^{m \times p_1}, \bar{Y} \in R[s]^{m \times p_2}, \widetilde{P}_2 \in R[s]^{q \times p_1}$, and $\widetilde{P}_1 \in R[s]^{q \times p_2}$, where $q \triangleq (p_1 + p_2) - m$. Note that $\bar{X}, \bar{Y}$ and $\widetilde{P}_2, \widetilde{P}_1$ are left coprime (lc) pairs. If they were not, then $\det U \neq \alpha$, a nonzero real number. Similarly, $\bar{X}, \widetilde{P}_2$ and $\bar{Y}, \widetilde{P}_1$ are right coprime (rc) pairs. Let

$$U^{-1} = \begin{bmatrix} \bar{P}_1 & -\widetilde{Y} \\ \bar{P}_2 & \widetilde{X} \end{bmatrix}, \tag{7.46}$$

where $\bar{P}_1 \in R[s]^{p_1 \times m}, \bar{P}_2 \in R[s]^{p_2 \times m}$ are rc and $\widetilde{X} \in R[s]^{p_2 \times q}, \widetilde{Y} \in R[s]^{p_1 \times q}$ are rc. Equation (7.44) implies that

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = U^{-1} \begin{bmatrix} G_R^* \\ 0 \end{bmatrix} = \begin{bmatrix} \bar{P}_1 \\ \bar{P}_2 \end{bmatrix} G_R^*; \tag{7.47}$$

i.e., G_R^* is a common right divisor of P_1, P_2. Equation (7.44) implies also that

$$\bar{X}P_1 + \bar{Y}P_2 = G_R^*. \tag{7.48}$$

This relationship shows that any crd G_R of P_1, P_2 will also be a right divisor of G_R^*. This can be seen directly by expressing (7.48) as $MG_R = G_R^*$, where M is a polynomial matrix. Thus, G_R^* is a crd of P_1, P_2 with the property that any crd G_R of P_1, P_2 is a rd of G_R^*. This implies that G_R^* is a gcrd of P_1, P_2. ■

Example 7.21. Let $P_1 = \begin{bmatrix} s(s+2) & 0 \\ 0 & (s+1)^2 \end{bmatrix}$, $P_2 = \begin{bmatrix} (s+1)(s+2) & s+1 \\ 0 & s(s+1) \end{bmatrix}$. Then

$$U\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} \bar{X} & \bar{Y} \\ -\widetilde{P}_2 & \widetilde{P}_1 \end{bmatrix}\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \left[\begin{array}{cc:cc} -(s+2) & -1 & s+1 & 0 \\ s+1 & 1 & -s & 0 \\ \hdashline -(s+1)^2 & -s & s(s+1) & 0 \\ -(s+1) & 0 & s & -1 \end{array}\right]\begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$$

$$= \left[\begin{array}{cc} s+2 & 0 \\ 0 & s+1 \\ \hdashline 0 & 0 \\ 0 & 0 \end{array}\right] = \left[\begin{array}{c} G_R^* \\ \hdashline 0 \end{array}\right].$$

In view of Lemma 7.20, $G_R* = \begin{bmatrix} s+2 & 0 \\ 0 & s+1 \end{bmatrix}$ is a gcrd (see also Example 7.18).

Note that in order to derive (7.44) and thus determine a gcrd G_R^* of P_1 and P_2, one could use the algorithm to obtain the Hermite form [1, p. 532]. Finally, note also that if the Smith form of $\begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$ is known, i.e., $U_L \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} U_R = S_P = \begin{bmatrix} \mathrm{diag}[\epsilon_i] & 0 \\ 0 & 0 \end{bmatrix}$, then $(\mathrm{diag}[\epsilon_i], 0)U_R^{-1}$ is a gcrd of P_1 and P_2 in view of Lemma 7.20. When rank $\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = m$, which is the case of interest in systems, then a gcrd of P_1 and P_2 is $\mathrm{diag}[\epsilon_i]U_R^{-1}$.

Criteria for Coprimeness

There are several ways of testing the coprimeness of two polynomial matrices as shown in the following Theorem.

Theorem 7.22. *Let $P_1 \in R[s]^{p_1 \times m}$ and $P_2 \in R[s]^{p_2 \times m}$ with $p_1 + p_2 \geq m$. The following statements are equivalent:*

(a) P_1 and P_2 are right coprime.
(b) A gcrd of P_1 and P_2 is unimodular.

(c) There exist polynomial matrices $X \in R[s]^{m\times p_1}$ and $Y \in R[s]^{m\times p_2}$ such that

$$XP_1 + YP_2 = I_m. \tag{7.49}$$

(d) The Smith form of $\begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$ is $\begin{bmatrix} I \\ 0 \end{bmatrix}$.

(e) rank $\begin{bmatrix} P_1(s_i) \\ P_2(s_i) \end{bmatrix} = m$ *for any complex number s_i.*

(f) $\begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$ constitutes m columns of a unimodular matrix.

Proof. See [1, p. 538, Section 7.2D, Theorem 2.4]. ■

Example 7.23. (a) The polynomial matrices $P_1 = \begin{bmatrix} s & 0 \\ 0 & s+1 \end{bmatrix}$, $P_2 = \begin{bmatrix} s+1 & 1 \\ 0 & s \end{bmatrix}$ are right coprime in view of the following relations. To use condition (b) of the above theorem, let $U \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \left[\begin{array}{cc|cc} -(s+2) & -1 & s+1 & 0 \\ s+1 & 1 & -s & 0 \\ \hline -(s+1)^2 & -s & s(s+1) & 0 \\ -(s+1) & 0 & s & -1 \end{array}\right] \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} =$ $\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline 0 & 0 \\ 0 & 0 \end{array}\right] = \left[\begin{array}{c} G_R^* \\ \hline 0 \end{array}\right]$. Then $G_R^* = I_2$, which is unimodular. Applying condition (c)
$XP_1 + YP_2 = \begin{bmatrix} -(s+2) & -1 \\ s+1 & 1 \end{bmatrix} P_1 + \begin{bmatrix} s+1 & 0 \\ -s & 0 \end{bmatrix} P_2 = I_2$.

To use (d), note that the invariant polynomials of $\begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$ are $\epsilon_1 = \epsilon_2 = 1$; and the Smith form is then $\begin{bmatrix} I_2 \\ 0 \end{bmatrix}$. To use condition (e), note that the only complex values s_i that may reduce the rank of $\begin{bmatrix} P_1(s_i) \\ P_2(s_i) \end{bmatrix}$ are those for which $\det P_1(s_i)$ or $\det P_2(s_i) = 0$; i.e., $s_1 = 0$ and $s_2 = -1$. For these values we have rank $\begin{bmatrix} P_1(s_1) \\ P_2(s_1) \end{bmatrix} =$ rank $\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} = 2$ and rank $\begin{bmatrix} P_1(s_2) \\ P_2(s_2) \end{bmatrix} =$ rank $\begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} = 2$; i.e., both are of full rank.

The following Theorem 7.24 is the corresponding to Theorem 7.22 result for (left coprime) proper and stable matrices. Note that $\widehat{U}$ proper and stable is a unimodular matrix if $\widehat{U}^{-1}$ is also a proper and stable matrix.

Theorem 7.24. *Let $\widehat{P}_1 \in R[s]^{p\times m_1}$ and $\widehat{P}_2 \in R[s]^{p\times m_2}$ with $m_1 + m_2 \geq p$. The following statements are equivalent:*

(a) $\widehat{P}_1$ *and* $\widehat{P}_2$ *are left coprime.*
(b) A gcld of $\widehat{P}_1$ *and* $\widehat{P}_2$ *is unimodular.*
(c) There exist polynomial matrices $\widehat{X} \in R[s]^{m_1 \times p}$ *and* $\widehat{Y} \in R[s]^{m_2 \times p}$ *such that*

$$\widehat{P}_1 \widehat{X} + \widehat{P}_2 \widehat{Y} = I_p. \tag{7.50}$$

(d) The Smith form of $[\widehat{P}_1, \widehat{P}_2]$ *is* $[I, 0]$.
(e) $\operatorname{rank}[\widehat{P}_1(s_i), \widehat{P}_2(s_i)] = p$ *for any complex number* s_i.
(f) $[\widehat{P}_1, \widehat{P}_2]$ *are* p *rows of a unimodular matrix.*

Proof. The proof is completely analogous to the proof of Theorem 7.22 and is omitted. ■

7.5.3 Controllability, Observability, and Stability

Consider now the Polynomial Matrix Description

$$P(q)z(t) = Q(q)u(t), \quad y(t) = R(q)z(t) + W(q)u(t), \tag{7.51}$$

where $P(q) \in R[q]^{l \times l}$, $Q(q) \in R[q]^{l \times m}$, $R(q) \in R[q]^{p \times l}$, and $W(q) \in R[q]^{p \times m}$.

Assume that the PMD given in (7.51) is equivalent to some state-space representation

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t), \tag{7.52}$$

where $A \in R^{n \times n}$, $B \in R^{n \times m}$, $C \in R^{p \times n}$, and $D \in R^{p \times m}$ [1, p. 553, Section 7.3A].

Controllability

Definition 7.25. *The representation* $\{P, Q, R, W\}$ *given in (7.51) is said to be* controllable *if its equivalent state-space representation* $\{A, B, C, D\}$ *given in (7.52) is state controllable.*

Theorem 7.26. *The following statements are equivalent:*

(a) $\{P, Q, R, W\}$ *is controllable.*
(b) The Smith form of $[P, Q]$ *is* $[I, 0]$.
(c) $\operatorname{rank}[P(s_i), Q(s_i)] = l$ *for any complex number* s_i.
(d) P, Q *are left coprime.*

Proof. See [1, p. 561, Theorem 3.4]. ■

The right Polynomial Matrix Fractional Description, $\{D_R, I_m, N_R\}$, is controllable since D_R and I are left coprime.

Observability

Observability can be introduced in a completely analogous manner to controllability. This leads to the following concept and result.

Definition 7.27. *The representation $\{P, Q, R, W\}$ given in (7.51) is said to be* observable *if its equivalent state-space representation $\{A, B, C, D\}$ given in (7.52) is state observable.* ■

Theorem 7.28. *The following statements are equivalent:*

(a) $\{P, Q, R, W\}$ *is observable.*
(b) The Smith form of $\begin{bmatrix} P \\ R \end{bmatrix}$ *is* $\begin{bmatrix} I \\ 0 \end{bmatrix}$.
(c) $\operatorname{rank} \begin{bmatrix} P(s_i) \\ R(s_i) \end{bmatrix} = l$ *for any complex number* s_i.
(d) P, R *are right coprime.*

Proof. It is analogous to the proof of Theorem 7.26. ■

The left Polynomial Matrix Fractional Description (PMFD), $\{D_L, N_L, I_p\}$, is observable since D_L and I_p are right coprime.

Stability

Definition 7.29. *The representation $\{P, Q, R, W\}$ given in (7.51) is said to be* asymptotically stable *if for its equivalent state-space representation $\{A, B, C, D\}$ given in (7.52) the equilibrium $x = 0$ of the free system $\dot{x} = Ax$ is asymptotically stable.*

Theorem 7.30. *The representation $\{P, Q, R, W\}$ is asymptotically stable if and only if $Re\lambda_i < 0$, $i = 1, \ldots, n$, where λ_i, $i = 1, \ldots, n$ are the roots of* $\det P(s)$*; the λ_i are the eigenvalues or poles of the system.*

Proof. See [1, p. 563, Theorem 3.6]. ■

7.5.4 Poles and Zeros

Poles and zeros can be defined in a completely analogous way for system (7.51) as was done in Section 7.4 for state-space representations.

It is straightforward to show that

$$\{\text{poles of } H(s)\ \} \subset \{\text{roots of } \det P(s)\ \}. \tag{7.53}$$

The roots of $\det P$ are the *eigenvalues or the poles of the system* $\{P, Q, R, W\}$ and are equal to the eigenvalues of A in any equivalent state-space representation $\{A, B, C, D\}$. Relation (7.53) becomes an equality when the system is controllable and observable, since in this case the poles of the transfer function matrix H are exactly those eigenvalues of the system that are both controllable and observable.

Consider the *system matrix* or *Rosenbrock Matrix* of the representation $\{P, Q, R, W\}$,

$$S(s) = \begin{bmatrix} P(s) & Q(s) \\ -R(s) & W(s) \end{bmatrix}. \tag{7.54}$$

The *invariant zeros of the system* are the roots of the invariant zero polynomial, which is the product of all the invariant factors of $S(s)$. The *input-decoupling*, *output-decoupling*, and the *input–output decoupling* zeros of $\{P, Q, R, W\}$ can be defined in a manner completely analogous to the state-space case. For example, the roots of the product of all invariant factors of $[P(s), Q(s)]$ are the input-decoupling zeros of the system; they are also the uncontrollable eigenvalues of the system. Note that the input-decoupling zeros are the roots of $\det G_L(s)$, where $G_L(s)$ is a gcld of all the columns of $[P(s), Q(s)] = G_L(s)[\bar{P}(s), \bar{Q}(s)]$. Similar results hold for the output-decoupling zeros.

The *zeros of* $H(s)$, also called the *transmission zeros of the system*, are defined as the roots of *the zero polynomial of* $H(s)$,

$$z_H(s) = \epsilon_1(s) \ldots \epsilon_r(s), \tag{7.55}$$

where the ϵ_i are the numerator polynomials in the Smith–McMillan form of $H(s)$. When $\{P, Q, R, W\}$ is controllable and observable, the zeros of the system, the invariant zeros, and the transmission zeros coincide.

Consider the representation $D_R z_R = u, y = N_R z_R$ with $D_R \in R[s]^{m\times m}$ and $N_R \in R[s]^{p\times m}$ and notice that in this case the Rosenbrock matrix (7.54) can be reduced via elementary column operations to the form

$$\begin{bmatrix} D_R & I \\ -N_R & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ -D_R & I \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} = \begin{bmatrix} 0 & I \\ -N_R & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & -N_R \end{bmatrix}.$$

In view of the fact that the invariant factors of S do not change under elementary matrix operations, the nonunity invariant factors of S are the nonunity invariant factors of N_R. Therefore, the *invariant zero polynomial of the system* equals the product of all invariant factors of N_R and its roots are the *invariant zeros of the system.* Note that when $\operatorname{rank} N_R = p \le m$, the invariant zeros of the system are the roots of $\det G_L$, where G_L is the gcld of all the columns of N_R; i.e., $N_R = G_L \bar{N}_R$. When N_R, D_R are right coprime, the system is controllable and observable. In this case it can be shown that the *zeros of* H $(= N_R D_R^{-1})$, also called the *transmission zeros of the system*, are equal to the *invariant zeros* (and to the *system zeros* of $\{D_R, I, N_R\}$) and can be determined from N_R. In fact, the zero polynomial of the system, $z_s(s)$, equals $z_H(s)$, the zero polynomial of H, which equals $\epsilon_1(s) \ldots \epsilon_r(s)$, the product of the invariant factor of N_R; i.e.,

$$z_s(s) = z_H(s) = \epsilon_1(s) \ldots \epsilon_r(s). \tag{7.56}$$

The pole polynomial of $H(s)$ is

$$p_H(s) = k \quad \det D_R(s), \tag{7.57}$$

where $k \in R$.

When $H(s)$ is square and nonsingular and $H(s) = N_R(s)D_R^{-1}(s) = D_L^{-1}(s)N_L(s)$ rc and lc, respectively, the poles of $H(s)$ *are the roots of* $\det D_R(s)$ *or of* $\det D_L(s)$ *and the* zeros of $H(s)$ *are the roots of $N_R(s)$ or of $N_L(s)$. An important well-known special case is the case of a scalar $H(s) = n(s)/d(s)$, where the poles of $H(s)$ are the roots of $d(s)$ and the zeros of $H(s)$ are the roots of $n(s)$.*

7.6 Summary and Highlights

- The transfer function
$$H(s) = C_1(sI - A_{11})^{-1}B_1 + D \tag{7.4}$$
and the impulse response
$$H(t, 0) = C_1 e^{A_{11}t} B_1 + D\delta(t) \tag{7.5}$$
depend only on the controllable and observable parts of the system, (A_{11}, B_1, C_1). Similar results hold for the discrete-time case in (7.6) and (7.7).
- Since
$$\{\text{eigenvalues of } A\} \supset \{\text{poles of } H(s)\}, \tag{7.9}$$
internal stability always implies BIBO stability but not necessarily vice versa. Recall that the system is stable in the sense of Lyapunov (or internally stable) if and only if all eigenvalues of A have negative real parts; the system is BIBO stable if and only if all poles of $H(s)$ have negative real parts. BIBO stability implies internal stability only when the eigenvalues of A are exactly the poles of $H(s)$, which is the case when the system is both controllable and observable.
- When $H(s) = C(sI - A)^{-1}B + D$ and (A, B) is controllable and (A, C) is observable, then
$$\{\text{eigenvalues of } A \text{ (poles of the system)}\} = \{\text{poles of } H(s)\}, \tag{7.58}$$
$$\{\text{zeros of the system}\} = \{\text{zeros of } H(s)\}. \tag{7.22}$$
When the system is not controllable and observable
$$\begin{aligned}&\{\text{eigenvalues of } A \text{ (poles of the system)}\} = \\ &\{\text{poles of } H(s)\} \cup \{\text{uncontrollable and/or unobservable eigenvalues}\}.\end{aligned} \tag{7.20}$$
- If the system $\{A, B, C, D\}$ is not both controllable and observable, then the uncontrollable and/or unobservable eigenvalues cancel out when the transfer functions $H(s) = C(sI - A)^{-1}B + D$ is determined.

Poles and Zeros

- The Smith–McMillan form of a transfer function matrix $H(s)$ is

$$SM_H(s) = \begin{bmatrix} \tilde{\Lambda}(s) & 0 \\ 0 & 0 \end{bmatrix}, \tag{7.12}$$

  with $\tilde{\Lambda}(s) \triangleq \operatorname{diag}\left[\frac{\epsilon_1(s)}{\psi_1(s)}, \dots, \frac{\epsilon_r(s)}{\psi_r(s)}\right]$ are the invariant factors of $N(s)$ in $H(s) = \frac{1}{d(s)}N(s)$ and $r = \operatorname{rank} H(s)$.
- The characteristic or pole polynomial of $H(s)$ is

$$p_H(s) = \psi_1(s) \cdots \psi_r(s). \tag{7.13}$$

  p_H is also the monic least common denominator of all nonzero minors of $H(s)$. The roots of $p_H(s)$ are the poles of $H(s)$.
- The zero polynomial of $H(s)$ is

$$z_H(s) = \epsilon_1(s)\epsilon_2(s) \cdots \epsilon_r(s). \tag{7.18}$$

  The roots of $z_H(s)$ are the zeros of $H(s)$ (or the transmission zeros of the system). When $H(s) = N(s)D(s)^{-1}$ a right coprime polynomial factorization, the zeros of $H(s)$ are the invariant zeros of $N(s)$. When $N(s)$ is square, the zeros are the roots of $|N(s)|$.

Polynomial Matrix Descriptions

- PMDs are given by

$$P(q)z(t) = Q(q)u(t), \quad y(t) = R(q)z(t) + W(q)u(t), \tag{7.29}$$

  where $q \triangleq d/dt$ the differential operator ($qz = \dot{z}$). PMDs are, in general, equivalent to state-space representations of the form

$$\dot{x} = Ax + Bu, \quad y = Cx + D(q)u,$$

  and so they are more general than the $\{A, B, C, D\}$ descriptions.
- The transfer function matrix is

$$H(s) = R(s)P^{-1}(s)Q(s) + W(s). \tag{7.30}$$

- The system is *controllable* if and only if (P, Q) are *left coprime* (lc). It is *observable* if and only if (P, R) are *right coprime* (rc). (See Theorems 7.26 and 7.28.) The system is *asymptotically stable* if all the eigenvalues of the system, the roots of $|P(q)|$, have negative real parts. (See Theorem 7.30.)

- Polynomial Matrix Fractional Descriptions (PMFDs) are given by

$$H(s) = N_R(s)D_R^{-1}(s) = D_L^{-1}(s)N_L(s), \tag{7.35}$$

  where (N_R, D_R) are rc and (D_L, N_L) are lc. They correspond to the PMD

$$D_R(q)z_R(t) = u(t), \quad y(t) = N_R(q)z_R(t) \tag{7.36}$$

  and

$$D_L(q)z_L(t) = N_L(q)u(t), \quad y(t) = z_L(t), \tag{7.37}$$

  which are both controllable and observable representations.
- Proper and stable Matrix Fractional Descriptions (MFDs) are given by

$$H(s) = \widehat{N}_R(s)\widehat{D}_R(s)^{-1} = \widehat{D}_L^{-1}(s)\widehat{N}_L(s), \tag{7.38}$$

  where $\widehat{N}_R, \widehat{D}_R, \widehat{D}_L, \widehat{N}_L$ are proper and stable matrices with $(\widehat{N}_R, \widehat{D}_R)$ rc and $(\widehat{D}_L, \widehat{N}_L)$ lc.

7.7 Notes

The role of controllability and observability in the relation between propertiers of internal and external descriptions are found in Gilbert [2], Kalman [4], and Popov [5]. For further information regarding these historical issues, consult Kailath [3] and the original sources.

Multivariable zeros have an interesting history. For a review, see Schrader and Sain [7] and the references therein. Refer also to Vardulakis [8]. Polynomial matrix descriptions were used by Rosenbrock [6] and Wolovich [9]. See [1, Sections 7.6 and 7.7] for extensive notes and references.

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Exercises

7.1. Consider the system $\dot{x} = Ax + Bu, y = Cx + Du$.

(a) Show that only controllable modes appear in $e^{At}B$ and therefore in the zero-state response of the state.
(b) Show that only observable modes appear in Ce^{At} and therefore in the zero-input response of the system.
(c) Show that only modes that are both controllable and observable appear in $Ce^{At}B$ and therefore in the impulse response and the transfer function matrix of the system. Consider next the system $x(k+1) = Ax(k)+Bu(k)$, $y(k) = Cx(k) + Du(k)$.
(d) Show that only controllable modes appear in A^kB, only observable modes in CA^k, and only modes that are both controllable and observable appear in CA^kB [that is, in $H(z)$].
(e) Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $C = [1, 1, 0]$, and $D = 0$. Verify the results obtained in (d).

7.2. In the circuit of Example 7.2, let $R_1R_2C = L$ and $R_1 = R_2 = R$. Determine $x(t) = [x_1(t), x_2(t)]^T$ and $i(t)$ for unit step input voltage, $v(t)$, and initial conditions $x(0) = [a, b]^T$. Comment on your results.

7.3. (a) Consider the state equation $\dot{x} = Ax + Bu, x(0) = x_0$, where $A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$. Determine $x(t)$ as a function of $u(t)$ and x_0, and verify that the uncontrollable modes do not appear in the zero-state response but do appear in the zero-input response.
(b) Consider the state equation $x(k + 1) = Ax(k) + Bu(k)$ and $x(0) = x_0$, where A and B are as in (a). Demonstrate for this case results corresponding to (a).
In (a) and (b), determine $x(t)$ and $x(k)$ for unit step inputs and $x(0) = [1,\ 1,\ 1]^T$.

7.4. (a) Consider the system $\dot{x} = Ax + Bu, y = Cx$ with $x(0) = x_0$, where $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $C = [1, 1]$. Determine $y(t)$ as a function of $u(t)$ and x_0, and verify that the unobservable modes do not appear in the output.
(b) Consider the system $x(k+1) = Ax(k)+Bu(k), y(k) = Cx(k)$ with $x(0) = x_0$, where A, B, and C are as in (a). Demonstrate for this case results that correspond to (a).
In (a) and (b), determine and plot $y(t)$ and $y(k)$ for unit step inputs and $x(0) = 0$.

7.5. Consider the system $x(k+1) = Ax(k) + Bu(k), y(k) = Cx(k)$, where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & -1/2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \qquad C = [1,\ 1,\ 0].$$

Determine the eigenvalues that are uncontrollable and/or unobservable. Determine $x(k), y(k)$ for $k \geq 0$, given $x(0)$ and $u(k)$, $k \geq 0$, and show that only controllable eigenvalues (resp., modes) appear in $A^k B$, only observable ones appear in CA^k, and only eigenvalues (resp., modes) that are both controllable and observable appear in $CA^k B$ [in $H(z)$].

7.6. Given is the system $\dot{x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} u,\ y = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} x.$

(a) Determine the uncontrollable and the unobservable eigenvalues (if any).
(b) What is the impulse response of this system? What is its transfer function matrix?
(c) Is the system asymptotically stable?

7.7. Given is the transfer function matrix $H(s) = \begin{bmatrix} \frac{s-1}{s} & 0 & \frac{s-2}{s+2} \\ 0 & \frac{s+1}{s} & 0 \end{bmatrix}$.

(a) Determine the Smith–McMillan form of $H(s)$ and its characteristic (pole) polynomial and minimal polynomial. What are the poles of $H(s)$?
(b) Determine the zero polynomial of $H(s)$. What are the zeros of $H(s)$?

7.8. Let $H(s) = \begin{bmatrix} \frac{s^2+1}{s^2} \\ \frac{s+1}{s^3} \end{bmatrix}$.

(a) Determine the Smith–McMillan form of $H(s)$ and its characteristic (pole) polynomial and minimal polynomial. What are the poles of $H(s)$?
(b) Determine the zero polynomial of $H(s)$. What are the zeros of $H(s)$?

7.9. A rational function matrix $R(s)$ may have, in addition to finite poles and zeros, *poles and zeros at infinity* ($s = \infty$). To study the poles and zeros at infinity, the bilinear transformation

$$s = \frac{b_1 w + b_0}{a_1 w + a_0}$$

with $a_1 \neq 0, b_1 a_0 - b_0 a_1 \neq 0$ may be used, where b_1/a_1 is not a finite pole or zero of $R(s)$. This transformation maps the point $s = b_1/a_1$ to $w = \infty$ and the point of interest, $s = \infty$, to $w = -a_0/a_1$. The rational matrix $\widehat{R}(w)$ is now obtained as

$$\widehat{R}(w) = R\left(\frac{b_1 w + b_0}{a_1 w + a_0}\right),$$

and the finite poles and zeros of $\widehat{R}(w)$ are determined. The poles and zeros at $w = -a_0/a_1$ are the poles and zeros of $R(s)$ at $s = \infty$. Note that frequently a good choice for the bilinear transformation is $s = 1/w$; that is, $b_1 = 0, b_0 = 1$ and $a_1 = 1, a_0 = 0$.

(a) Determine the poles and zeros at infinity of

$$R_1(s) = \frac{1}{s+1}, \qquad R_2(s) = s, \qquad R_3(s) = \begin{bmatrix} 1 & 0 \\ s+1 & 1 \end{bmatrix}.$$

Note that a rational matrix may have both poles and zeros at infinity.

(b) Show that if $R(s)$ has a pole at $s = \infty$, then it is not a proper rational function ($\lim_{s\to\infty} R(s) \to \infty$).

7.10. Consider the polynomial matrices $P(s) = \begin{bmatrix} s^2+s & -s \\ -s^2-1 & s^2 \end{bmatrix}$, $R(s) = \begin{bmatrix} s & 0 \\ -s-1 & 1 \end{bmatrix}$.

(a) Are they right coprime (rc)? If they are not, find a greatest common right divisor (gcrd).
(b) Are they left coprime (lc)? If they are not, find a greatest common left divisor (gcld).

7.11. (a) Show that two square and nonsingular polynomial matrices, the determinants of which are coprime polynomials, are both right and left coprime. *Hint:* Assume they are not, say, right coprime and then use the determinants of their gcrd to arrive at a contradiction.
(b) Show that the opposite is not true; i.e., two right (left) coprime polynomial matrices do not necessarily have determinants which are coprime polynomials.

7.12. Let $P(s)$ be a polynomial matrix of full column rank, and let $y(s)$ be a given polynomial vector. Show that the equation $P(s)x(s) = y(s)$ will have a polynomial solution $x(s)$ for any $y(s)$ if and only if the columns of $P(s)$ are lc, or equivalently, if and only if $P(\lambda)$ has full column rank for any complex number λ.

7.13. Consider $P(q)z(t) = Q(q)u(t)$ and $y(t) = R(q)z(t) + W(q)u(t)$, where

$$P(q) = \begin{bmatrix} q^3-q & q^2-1 \\ -q-2 & 0 \end{bmatrix}, \quad Q(q) = \begin{bmatrix} q-1 & -2q+2 \\ 1 & 3q \end{bmatrix},$$
$$R(q) = \begin{bmatrix} 2q^2+q+2 & 2q \\ -q-2 & 0 \end{bmatrix}, \quad W(q) = \begin{bmatrix} -1 & 3q+4 \\ -1 & -3q \end{bmatrix},$$

with $q \triangleq \frac{d}{dt}$.

(a) Is this system representation controllable? Is it observable?

(b) Find the transfer function matrix $H(s)(\hat{y}(s) = H(s)\hat{u}(s))$.
(c) Determine an equivalent state-space representation $\dot{x} = Ax + Bu$, $y = Cx + Du$, and repeat (a) and (b) for this representation.

7.14. Use system theoretic arguments to show that two polynomials $d(s) = s^n + d_{n-1}s^{n-1} + \cdots + d_1 s + d_0$ and $n(s) = n_{n-1}s^{n-1} + n_{n-2}s^{n-2} + \cdots + n_1 s + n_0$ are coprime if and only if

$$\operatorname{rank}\begin{bmatrix} C_c \\ C_c A_c \\ \vdots \\ C_c A_c^{n-1} \end{bmatrix} = n,$$

where $A_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -d_0 & -d_1 & -d_2 & & -d_{n-1} \end{bmatrix}$ and $C_c = [n_0, n_1, \ldots, n_{n-1}]$.

7.15. Consider the system $Dz = u, y = Nz$, where $D = \begin{bmatrix} s^2 & 0 \\ 0 & s^3 \end{bmatrix}$ and $N = [s^2 - 1, s + 1]$.

(a) Is the system controllable? Is it observable? Determine all uncontrollable and/or unobservable eigenvalues, if any.
(b) Determine the invariant and the transmission zeros of the system.