Παραδ: Ρίψη νομισμ. 2 φορές

$X = \# K$

$Y = \# \Gamma$ στην $1^{η}$

$A =$ Άρτιο $\# K$.

$$\rightarrow P_X(x) = \begin{cases} \frac{1}{4} & x=0 \\ \frac{1}{2} & x=1 \\ \frac{1}{4} & x=2 \end{cases}$$

$P_{X|A}(x) \neq P_X(x)$

$\Downarrow$

X, A εξαρτημένα

$$\rightarrow P_{X|A}(x) = \begin{cases} \frac{1}{2} & , x=0 \\ \frac{1}{2} & , x=2 \end{cases}$$

$$P_{X|A} = \frac{P(X=x, A)}{P(A)}, \quad P(A) = \frac{1}{2}$$

$Y = \#\ \Gamma$ στην 1η

A : Αρτιο $\#K$.

$$P_Y(y) = \begin{cases} 1/2 & , y=0 \\ 1/2 & , y=1 \end{cases}$$

$$P_{Y|A}(y) = \frac{P(Y=y, A)}{P(A)} = \begin{cases} \dfrac{P(\overbrace{Y=0, A}^{KK}) = 1/4}{1/2} , & y=0 \\ \dfrac{P(\overbrace{Y=1, A}^{\Gamma\Gamma}) = 1/4}{1/2} , & y=1 \end{cases} = \begin{cases} \frac{1}{2} & , y=0 \\ \frac{1}{2} & , y=1 \end{cases}$$

$P(A) = 1/2$

$P_{Y|A}(y) = P_Y(y) \Rightarrow Y, A$ ανεξάρτητα

$E[X+Y] = E[X] + E[Y]$ <u>Πάντα</u>

$E[XY] = E[X]E[Y] \Leftarrow X, Y$ ανεξ.

<u>Αποδ:</u>

$$E[\underbrace{X+Y}_{g(X,Y)}] = \sum_x \sum_y \underbrace{(x+y)}_{g(x,y)} p_{X,Y}(x,y)$$

$$= \sum_x \sum_y \underline{x}\, p_{X,Y}(x,y) + \sum_x \sum_y y\, p_{X,Y}(x,y)$$

$$= \sum_x x \sum_y p_{X,Y}(x,y) + \sum_y y \sum_x p_{X,Y}(x,y)$$

$$= \sum_x x\, p_X(x) + \sum_y y\, p_Y(y)$$

$$= E[X] + E[Y]$$

Νοε 19-3:49 μμ

$$E[XY] = \sum_x \sum_y xy \, p_{X,Y}(x,y)$$

$$= \sum_x x \sum_y y \, \underline{\underline{p_{X,Y}(x,y)}}$$

ανεξ

$$= \sum_x x \sum_y y \, \underline{\underline{p_X(x)\, p_Y(y)}}$$

$$= \underbrace{\sum_x x \, p_X(x)}_{E[X]} \; \underbrace{\sum_y y \, p_Y(y)}_{E[Y]}$$

$E[XY] = E[X]E[Y] \not\Rightarrow X, Y$ ανεξ.

$p_X(x) = \frac{1}{3}, \quad x = -1, 0, 1$

$Y = X^2.$

$\boxed{X, Y \text{ εξαρτημ.}}$

$E[XY] = E[X^3] = 0$

$E[X] = 0$

$\Rightarrow \boxed{E[XY] = E[X]E[Y]}$

$Var[X+Y] \neq Var[X] + Var[Y] \leftarrow$ Γενικά

$Var[X+Y] = Var[X] + Var[Y] \Leftarrow X, Y$ ανεξ.

Αποδ:

$$Var[X+Y] = E[(X+Y)^2] - (E[X+Y])^2$$

$$= E[X^2 + 2XY + Y^2] - (E[X] + E[Y])^2$$

$Var[X] = E[X^2] - (E[X])^2$

$$= E[X^2] + 2E[XY] + E[Y^2] - (E[X])^2 - 2E[X]E[Y] - (E[Y])^2$$

$$= Var[X] + Var[Y] + 2(E[XY] - E[X]E[Y])$$

Ανεξ. $\rightarrow$ $= Var[X] + Var[Y]$.

Γενικ για πολλές τ.μ. της Ανεξ:

$X_1, X_2, \dots, X_n$ ανεξ

$$\Updownarrow$$

$$P(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n)$$

$$= P(X_1 = x_1)\, P(X_2 = x_2) \cdots P(X_n = x_n)$$

$$\forall\, x_1, x_2, \dots, x_n.$$

Διωνυμική $Bin(n,p)$

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad 0 \le k \le n.$$

$$\left.\begin{aligned} E[X] &= \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k} = \ldots \\ Var[X] &= E[X^2] - (E[X])^2 \\ & \qquad \uparrow \\ & \sum_{k=0}^{n} k^2 \binom{n}{k} p^k (1-p)^{n-k} = \ldots \end{aligned}\right\} \underline{\underline{\text{Δύσκολα}}}$$

Όμως

$$X = \sum_{i=1}^{n} X_i \qquad X_i = \begin{cases} 1 & \text{με πιθ. } p \\ 0 & \text{με πιθ. } 1-p \end{cases} = \#\ \text{επιτ. στην δοκ. } i$$

X_i ανεξάρτητες

$$X = \sum_{i=1}^{n} X_i \implies$$

$$E[X] = \sum_{i=1}^{n} E[X_i] = np.$$

$$X = \sum_{i=1}^{n} X_i, \quad X_i \text{ ανεξ.} \implies$$

$$Var[X] = \sum_{i=1}^{n} Var[X_i] = np(1-p).$$

$$E[X_i] = 0\cdot(1-p) + 1\cdot p = p$$

$$E[X_i^2] = 0^2(1-p) + 1^2\cdot p = p$$

$$Var[X_i] = E[X_i^2] - (E[X_i])^2 = p - p^2 = p(1-p)$$

Δειγματικός Μέσος

$X_1, X_2, \dots, X_n$ ανεξ. και $\boxed{\text{ισόνομες}}$

Παρατηρήσεις || ?

$\boxed{P_{X_i}(x) \text{ είναι όλες ίδιες}}$

$\bar{X}_n = \dfrac{X_1 + \dots + X_n}{n}$: Δειγμ. μέσος ή μέσος όρος.

$E[X_i] = \mu$ Πληθυσμ. μέσος

$Var[X_i] = \sigma^2$ Πληθυσμ. διασπορά

$$E[X_i] = \mu \qquad Var[X_i] = \sigma^2$$

$$E[\bar{X}_n] = E\left[\frac{X_1 + X_2 + \cdots + X_n}{n}\right]$$

$$= \frac{1}{n} \cdot \big(\underbrace{E[X_1] + E[X_2] + \cdots + E[X_n]}_{n\mu}\big) = \mu.$$

$$Var[\bar{X}_n] = Var\left[\frac{X_1 + X_2 + \cdots + X_n}{n}\right]$$

$$= \frac{1}{n^2}\big(Var[X_1] + Var[X_2] + \cdots + Var[X_n]\big)$$

$$= \frac{1}{n^2} \cdot n\, Var[X_i] = \frac{1}{n^2} \cdot n \cdot \sigma^2 = \frac{\sigma^2}{n} \downarrow 0 \ \underset{n \to \infty}{\text{καθώς}}$$