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MORE ON THE SUPERPARTICULAR RATIOS IN MUSIC

G. D. HALSEY AND EDWIN HEWITT, University of Washington

There are ratios that are assigned without hesitation to the musical intervals that are the basis of traditional Western music. That is, these ratios denominate the relative acoustic frequency, or inversely, the length of violin string required to produce first one note and then the other of the interval. A recent article in this MONTHLY [6] by A. L. Leigh Silver presents an interesting discussion of this fact, and lists the following *superparticular* ratios along with their proper musical designations:

2/1 octave	9/8 major whole tone
3/2 perfect fifth	10/9 minor whole tone
4/3 perfect fourth	16/15 diatonic semitone
5/4 major third	25/24 chromatic semitone
6/5 minor third	81/80 common comma [or comma of Didymus].

In essence, these designations appear to have been known since the times of Zarlino and Descartes [**2**, p. 775]. With inversions, they account for all the common intervals except the tritone. The unstable character of the tritone sets it apart, as discussed, for example, by Hindemith [**3**, p. 81]. It can be expressed as a ratio by compounding suitable superparticular ratios. Whether it is assigned the ratio 64/45 or 45/32, depending on musical context, or indeed some other ratio, it is not superparticular, which is in keeping with its unique rôle in music.

Silver implies that the above ratios, limited to contain prime factors of 2, 3 and 5, are a finite sequence. It has been long known that the sequence actually terminates with 81/80: this was proved in 1897 by C. Størmer [**7**]. Størmer also proved a more general theorem [**8**], as follows. (We are indebted to Professor Ivan Niven for this reference.) *Let* A, B, $M_1,\cdots,M_m$, N_1, $N_2,\cdots,N_n$ *be given positive integers. Then the equations*

$$AM_1^{x_1}M_2^{x_2}\cdots M_m^{x_m}-BN_1^{y_1}N_2^{y_2}\cdots N_n^{y_n}=\pm 1,\ \pm\ 2$$

admit only a finite number of solutions, all of which can be computed from the smallest positive solutions u_k *of Pell's equation*

$$t_1^2-D_1u_1^2=1,\cdots,t_r^2-D_ru_r^2=1$$

for certain D_k*'s that can be written down in terms of* A, B, M_i, *and* N_j.

D. H. Lehmer [**4**] has recently given a new proof of Størmer's theorem for prime M_j's and $A=B=1$ (excluding ± 2 on the right side) and has published complete tables for the primes 2, 3, 5, $\cdots$, 41. (Professor Donald R. Snow has kindly given us this reference.)

It may be of some interest to give a short derivation of Størmer's theorem for our case. The pairs of integers $(x, x+1)$ for which x and $x+1$ are divisible only by 2, 3, or 5 are (1,2), (2,3), (3,4), (4,5), (5,6), (8,9), (15,16), (24,25), (80,81). That is to say, all possible superparticular ratios derived from the first three primes were long ago identified by musical theory.

We establish the result by checking all possible cases. We first note that if

$$(*)\qquad 2^a3^b5^c-2^{a'}3^{b'}5^{c'}=\pm 1,$$

(all exponents nonnegative integers), then $aa'=bb'=cc'=0$, since the left side has absolute value at least $2^{|a-a'|}$ if $a\neq a'$, for example. A moment's thought shows that the only possible solutions of the equation (*) are the following, where a, b, c denote positive integers:

(0) $1=2^1-1$;

(1) $2^a=3^b\pm 1$; (2) $2^a=5^b\pm 1$;

(3) $3^a=5^b\pm 1$; (4) $2^a3^b=5^c\pm 1$;

(5) $2^a5^b=3^c\pm 1$; (6) $3^a5^b=2^c\pm 1$.

For the two equations (1), we know the solutions $(a,b)=(1,1),(1,0),(2,1),(3,2)$. We shall show that there are no others. Assuming that there are other solutions, we may suppose that $a>3$ and that a is the least value that yields a solution of the equations in (1). Plainly we have $b>2$, and so $2^a\equiv\pm 1 \pmod 9$. Since $2^a\equiv 1$ (mod 9) if and only if $a\equiv 0 \pmod 6$ and $2^a\equiv -1 \pmod 9$ if and only if $a\equiv 3$ (mod 6), it follows that $a\equiv 0 \pmod 3$: $a=3a'$. Thus we have

$$2^{3a'}\pm 1=(2^{a'}\pm 1)x=3^b$$

for some positive integer x, and so unique factorization shows that $2^{a'}\pm 1=3^{b'}$.

The minimum condition on a and the restriction $a > 3$ show that $a' = 2$ or 3, i.e., $a = 6$ or 9. Since $2^6 \pm 1 = 63, 65$ and $2^9 \pm 1 = 511, 513$, we see that (1) admits no solutions besides those listed above.

For the two equations (2), we know one solution, namely $(a, b) = (2,1)$. If there are others, suppose that we have the least exponent $b > 1$. Thus we have $5^b \equiv \pm 1 \pmod 8$, and since $5^{2k+1} \equiv 5 \pmod 8$, $5^{2k} \equiv 1 \pmod 8$, we see that $2^a = 5^b + 1$ has no solution. For $2^a = 5^b - 1$, we get $2^a = 5^{2b'} - 1$, $(5^{b'} + 1)(5^{b'} - 1) = 2^a$. Now argue as in the discussion of equations (1).

The equations (3) trivially have no solutions since one side is even and the other is odd.

In the case of equations (4) consider the equation $2^a 3^b = 5^c - 1$. We know the solution $(a, b, c) = (3, 1, 2)$. Plainly we must have $c > 1$, and so

$$2^a 3^b = 2^2 \sum_{j=0}^{c-1} 5^j,$$

which implies that $a \geqq 2$. Since $\sum_{j=0}^{c-1} 5^j \equiv 0 \pmod 3$, c has to be even, $c = 2c'$, and we have

$$2^a 3^b = (5^{c'} - 1)(5^{c'} + 1).$$

The number $5^{c'} + 1$ is congruent to 2 modulo 4. Unique factorization and the last equality yield

$$5^{c'} + 1 = 2 \cdot 3^{b'},\ 5^{c'} - 1 = 2^{a-1} \cdot 3^{b-b'}$$

for some integer b' such that $1 \leqq b' \leqq b$. Subtracting, we find

$$1 = 3^b - 2^{a-2} \cdot 3^{b-b'}.$$

Plainly we must have $a > 2$, and also either $b' = 0$ or $b = b'$. Since $b' \geqq 1$, we have

$$5^{c'} - 1 = 2^{a-1},$$

which by the above solution of (2) implies that $a - 1 = 2$, $c' = 1$. Thus $2^3 3^1 = 5^2 - 1$ is the only solution of (4−).

Next consider the equation $2^a 3^b = 5^c + 1$, for which we know the solution $2^1 3^1 = 5^1 + 1$. Assuming that there is a solution with $c > 1$, we may suppose that we have the solution with the least value of $c > 1$. Since the right side is congruent to 2 modulo 4, we must have $a = 1$. Since $2 \cdot 3^b \equiv 1 \pmod 5$ if and only if $b \equiv 1 \pmod 4$, we have $b = 4b' + 1$ with $b' \geqq 0$. Since $5^c \equiv 1 \pmod 3$ if and only if c is odd, we have $c = 2c' + 1$, with $c' \geqq 0$, and so our equation is

$$\begin{aligned} 2 \cdot 3^{4b'+1} &= 5^{2c'+1} + 1 \\ &= 6 \sum_{j=0}^{2c'} (-1)^j 5^j, \end{aligned}$$

i.e.,

$$3^{4b'} = \sum_{j=0}^{2c'} (-1)^j 5^j.$$

If $b' = 0$, we have $c' = 0$ and we are at our known solution $a = b = c = 1$. If $b' > 0$, we argue as follows. Since $-5 \equiv 1 \pmod 3$, we have

$$\sum_{j=0}^{2c'} (-1)^j 5^j \equiv 2c' + 1 \pmod 3,$$

and so $2c' + 1 \equiv 0 \pmod 3$. That is, c has the form $3(2d+1)$, and our original equation has the form

$$\begin{aligned} 2 \cdot 3^{4b'+1} &= 5^{3(2d+1)} + 1 \\ &= (5^{2d+1} + 1)(5^{2(2d+1)} - 5^{2d+1} + 1). \end{aligned}$$

Applying unique factorization, we see that there is a b'' such that

$$2 \cdot 3^{4b''+1} = 5^{2d+1} + 1.$$

Since $c = 3(2d+1)$ is the least value of $c > 1$ yielding a solution of (4+), we see that $2d + 1 = 1$, $c = 3$. Since $5^3 + 1 = 2 \cdot 3^2 \cdot 7$, we have proved that (4 +) has only one solution, $2^1 \cdot 3^1 = 5^1 + 1$.

For the equation (5), we have only the solutions $(a, b, c) = (4, 1, 4)$ and $(1, 1, 2)$.

The equation (6−): $3^a 5^b = 2^c - 1$ has only the solution $(1, 1, 4)$ and the equation (6 +): $3^a 5^b = 2^c + 1$ has no solutions at all. The proofs are like those gone through above and are omitted.

Although ratios that involve the number 7 are foreign to the true musical intervals, in at least one instance, Hindemith [loc.cit. p. 82] uses two such ratios in a tentative analysis of the dominant seventh chord. There he ascribes the ratios 7/5 or 10/7 to the tritone. Although these ratios are not superparticular, the interval that characterizes their difference (50/49) is superparticular. Therefore, it is of some mild interest for musical theory to list the solutions of Størmer's equation for the primes {2,3,5,7} and ± 1. A computation yields: (6,7), (7,8), (14,15), (20,21), (27,28), (35,36) (48,49), (49,50), (63,64), (125,126), (224,225), (2400,2401), (4374,4375). Størmer [7] has shown that these are the only adjacent pairs for the primes 2,3,5,7. Lehmer [4] has a complete table for the primes 2,3,5, ⋯, 41.

There is a generalization of part of Størmer's theorem, which follows readily from a theorem of A. Baker [1]. (We are indebted to Professor James Jordan for the reference to Baker's article.) Given any finite set P of primes and any fixed positive integer a, there are only a finite number of pairs (x, y) of positive integers such that $|x - y| \leqq a$ and x and y admit as prime factors only numbers from P.

Finally we note the interesting paper of Pólya [5], where analogues of part of Størmer's theorem are taken up.

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CORRECTION TO "RECONSTRUCTING AN EVOLUTIONARY TREE"

(This MONTHLY, 79(1972), 596–603)

DAVID SANKOFF

The figure on p. 597 should be labelled FIG. 2 and should appear on p. 600; the figure which appears on p. 600 should be labelled FIG. 1a and FIG. 1b and should appear on p. 597.

MATHEMATICAL NOTES

EDITED BY ROBERT GILMER

The present backlog for this Department is substantial. Until further notice, new manuscripts cannot be accepted. This moratorium will probably continue until June 1, 1973; authors are requested to hold their manuscripts pending a further announcement.

COMPLEMENTS AND COMMENTS

ROBERT GILMER

We are grateful to readers who are willing to share with us their comments on articles appearing in the Notes Section. Such comments enhance the value of the Monthly. The information we have received during the past year includes the following.

Calculus. J. D. Riley notes that the necessary hypothesis $f(0) = 0$ has been omitted in the article by F. Cunningham and N. Grossman (September, 1971, pp. 781–3) concerning Young's inequality.