Εκθετική κατανομή - Ανανεωτική Θεωρία: Βασικοί υπολογισμοί.

① Βασικά θέματα

- Ιδιότητες Εκθετικής
- Βασικοί υπολογισμοί ανανεωτικής θεωρίας: $F_{S_k}(t)$, $p_n(t)$, $m_X(t)$

χρόνος ζωής

$$\lambda_X(t) = \lim_{h\to 0^+} \frac{Pr[t<X\le t+h \mid X>t]}{h} = \lim_{h\to 0^+} \frac{Pr[t<X\le t+h]}{h\, Pr[X>t]} = \frac{f_X(t)}{1-F_X(t)}$$

Ρυθμός βλάβης για μη-αρν. τ.μ.

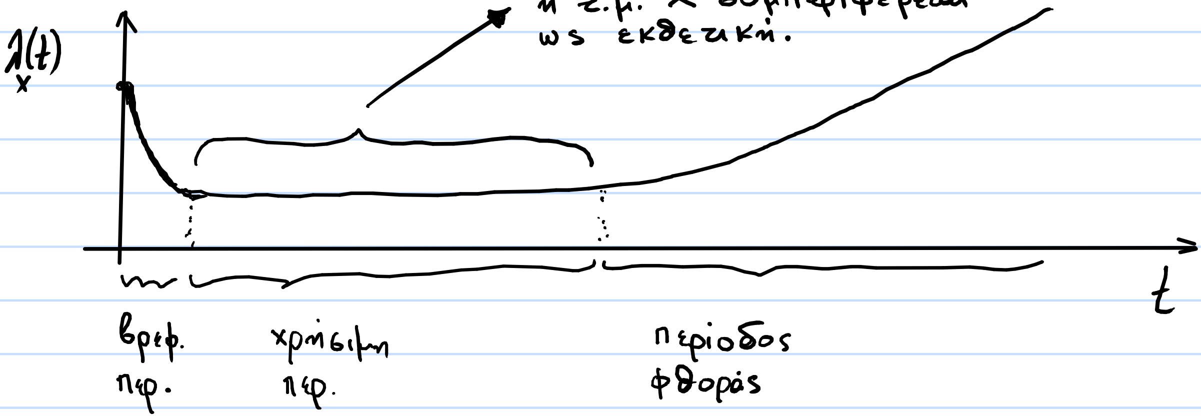
$$= \frac{\lambda e^{-\lambda t}}{\cancel{1}-(\cancel{1}-e^{-\lambda t})} = \underbrace{\lambda, \text{ ανεξ. του } t.}_{\text{Μόνο για την } Exp(\lambda)}$$

για $X \sim Exp(\lambda)$

Μόνο για την $Exp(\lambda)$ (ισοδύναμος τρόπος διατύπωσης αμνήμονης)

Συνήθης Ρυθμός βλάβης:

Σε αυτό το διάστημα η τ.μ. X συμπεριφέρεται ως εκθετική.



Υπολογισμοί για $m_X(t)$:

$$m_X(t) = \sum_{k=1}^{\infty} F_X^{*k}(t)$$

$$\tilde{m}_X(s) = \frac{\tilde{F}_X(s)}{1 - \tilde{F}_X(s)}$$

$$m_X(t) = F_X(t) + \int_0^t F(t-x)\, dm_X(x)$$

② **Ασκ. 2.2 / Φυλ. 1**

$\{N(t)\}$ ανανεωτική διαδικ. με κατανομή ενδ. χρόνων Erlang(r,λ).

$$X_i \sim Erlang(r,\lambda) \qquad f_{X_i}(t) = \frac{\lambda^r}{(r-1)!}\, t^{r-1} e^{-\lambda t}, \quad t \geq 0$$

$$F_{X_i}(t) = \int_0^t \frac{\lambda^r}{(r-1)!}\, u^{r-1} e^{-\lambda u}\, du = 1 - \sum_{j=0}^{r-1} e^{-\lambda t} \frac{(\lambda t)^j}{j!}$$

$F_{S_k}(t) = ;$

$P_k(t) = ;$

$m(t) = ;$

$\tilde{F}_{S_k}(s) = ;$

$\tilde{P}_k(s) = ;$

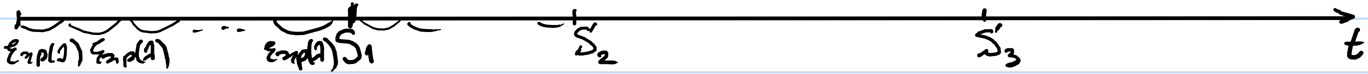
$\tilde{m}(s) = ;$

$$S_k \sim Erlang(kr,\lambda) \qquad \underbrace{\{S_{k+1} \leq t\}}_{} \subseteq \underbrace{\{S_k \leq t\}}_{}$$

$$F_{S_k}(t) = 1 - \sum_{j=0}^{kr-1} e^{-\lambda t} \frac{(\lambda t)^j}{j!}$$

$$P_k(t) = Pr[\underbrace{N(t) = k}_{}] = Pr[\overbrace{S_k \leq t < S_{k+1}}^{\{S_k \leq t\} \setminus \{S_{k+1} \leq t\}}]$$

$$= Pr[S_k \leq t] - Pr[S_{k+1} \leq t] = F_{S_k}(t) - F_{S_{k+1}}(t)$$



Άρα

$$P_k(t) = F_X^{*k}(t) - F_X^{*(k+1)}(t)$$

$$= \left(\cancel{1} - \sum_{j=0}^{kr-1} e^{-\lambda t}\frac{(\lambda t)^j}{j!}\right) - \left(\cancel{1} - \sum_{j=0}^{(k+1)r-1} e^{-\lambda t}\frac{(\lambda t)^j}{j!}\right)$$

$$= \sum_{j=kr}^{(k+1)r-1} e^{-\lambda t}\frac{(\lambda t)^j}{j!}$$

$$m(t) = \sum_{k=1}^{\infty} F_X^{*k}(t) = \sum_{k=1}^{\infty}\left(1 - \sum_{j=0}^{kr-1} e^{-\lambda t}\frac{(\lambda t)^j}{j!}\right)$$

$$\tilde{F}_{S_k}(s) = \left(\tilde{F}_X(s)\right)^k = \left(\frac{\lambda}{\lambda+s}\right)^{kr} \qquad \tilde{P}_k(s) = \left(1-\tilde{F}_X(s)\right)\tilde{F}_X(s)^k = \left(1-\left(\frac{\lambda}{\lambda+s}\right)^r\right)\left(\frac{\lambda}{\lambda+s}\right)^{kr}$$

$$\tilde{m}_X(s) = \frac{\tilde{F}_X(s)}{1-\tilde{F}_X(s)} = \frac{\left(\frac{\lambda}{\lambda+s}\right)^2}{1-\left(\frac{\lambda}{\lambda+s}\right)^2} = \frac{\lambda^2}{(\lambda+s)^2-\lambda^2}.$$

③ Ασκ. 2.4 / ΦυΛ. 1

$\{N(t)\}$ ανανε. διαδικασία με $X_1, X_2, \dots$ ενδιαμ. χρόνους.

$$X_i = \begin{cases} 0 & \text{με πιθ. } p \\ Exp(\lambda) & \text{με πιθ. } 1-p \end{cases}$$

$$F_{X_i}(t) = p + (1-p)(1-e^{-\lambda t}),\ t \geq 0.$$

$$\tilde{F}_{X_i}(s) = p + (1-p)\cdot \frac{\lambda}{\lambda+s} = \frac{p(\lambda+s)+(1-p)\lambda}{\lambda+s} = \frac{\lambda+ps}{\lambda+s}$$

$\tilde{F}_{S_k}(s)$, $\tilde{P}_k(s)$, $\tilde{m}_x(s) = ;$

$$\tilde{F}_{S_k}(s) = (\tilde{F}_x(s))^k, \quad \tilde{P}_k(s) = (1-\tilde{F}_x(s))(\tilde{F}_x(s))^k, \quad \tilde{m}_x(s) = \frac{\tilde{F}_x(s)}{1-\tilde{F}_x(s)}$$

$m(t) = ;$

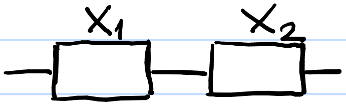
$$\tilde{F}_{S_k}(s) = \left(\frac{\lambda+ps}{\lambda+s}\right)^k \qquad \tilde{P}_k(s) = \left(1 - \frac{\lambda+ps}{\lambda+s}\right)\left(\frac{\lambda+ps}{\lambda+s}\right)^k$$

$$\tilde{m}_x(s) = \frac{\frac{\lambda+ps}{\lambda+s}}{1-\frac{\lambda+ps}{\lambda+s}} = \frac{\lambda+ps}{\cancel{\lambda}+s-\cancel{\lambda}-ps} = \frac{\lambda+ps}{(1-p)s} = \frac{\lambda}{1-p}\cdot\frac{1}{s} + \frac{p}{1-p}$$

$$m_x(t) = \frac{\lambda}{1-p}\cdot t + \frac{p}{1-p}.$$

$$\int_0^\infty e^{-st}\,dt = \frac{1}{s}$$

④ Ασκ. 1.6 / Φυλ. 1



$\sim$ $\mathcal{E}xp(\lambda)$ $\quad$ $\sim$ $\mathcal{E}rlang(n,\mu)$

ανεξ.

Μέσος χρόνος ζωής =;

Χρόνος ζωής συστήματος: $\overbrace{\min(X_1, X_2)}^{X}$

$$E[\underbrace{\min(X_1, X_2)}_{X}] = \int_0^\infty (1 - F_X(t))dt$$

$$= \int_0^\infty \Pr[X > t]dt = \int_0^\infty \Pr[X_1 > t, X_2 > t]dt$$

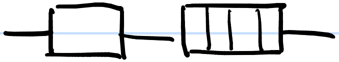
$$= \int_0^\infty \Pr[X_1 > t]\Pr[X_2 > t]dt$$

$$= \int_0^\infty e^{-\lambda t} \int_t^\infty \frac{\mu^n}{(n-1)!} u^{n-1} e^{-\mu u} du\, dt \underset{0 \le t \le u}{=} \int_0^\infty \frac{\mu^n}{(n-1)!} u^{n-1} e^{-\mu u} \int_0^u e^{-\lambda t} dt \quad du$$

$$= \int_0^\infty \frac{\mu^n}{(n-1)!} u^{n-1} e^{-\mu u} \cdot \frac{1 - e^{-\lambda u}}{\lambda} du = \frac{1}{\lambda} - \frac{\mu^n}{(n-1)!\,\lambda} \int_0^\infty u^{n-1} e^{-(\lambda+\mu)u} du$$

$$= \frac{1}{\lambda} - \frac{\mu^n}{\lambda(\lambda+\mu)^n} = \frac{1}{\lambda}\left(1 - \left(\frac{\mu}{\lambda+\mu}\right)^n\right).$$

2ος τρόπος



$$\underset{X_1}{\sim \mathrm{Exp}(\lambda)} \qquad \underset{X_2}{\sim \mathrm{Erlang}(n,\mu)}$$

$$X_2 = Y_1 + Y_2 + \dots + Y_n, \qquad Y_1 \sim \mathrm{Exp}(\mu),\ Y_2 \sim,\ \dots,\ Y_n \sim$$

$$\text{Μέσος χρόνος ζωής}\,(\mathrm{Exp}(\lambda), \mathrm{Erlang}(n,\mu)) = \frac{1}{\lambda+\mu} + \cancel{\frac{\lambda}{\lambda+\mu}\cdot 0} + \frac{\mu}{\lambda+\mu}\cdot \text{Μέσος χρόνος ζωής}\,(\mathrm{Exp}(\lambda), \mathrm{Erlang}(n-1,\mu))$$

$$= \frac{1}{\lambda+\mu} + \frac{\mu}{\lambda+\mu}\cdot\frac{1}{\lambda+\mu} + \frac{\mu}{\lambda+\mu}\cdot\frac{\mu}{\lambda+\mu}\cdot\left(\text{Μέσος χρ. ζωής}\ \mathrm{Exp}(\lambda), \mathrm{Erlang}(n-2,\mu)\right)$$

$$= \dots = \frac{1}{\lambda}\left(1 - \left(\frac{\mu}{\lambda+\mu}\right)^n\right).$$