Επιχειρησιακή Έρευνα: Στοχαστικά Μοντέλα 4/3/2021

Δεσμευμένη Μέση Τιμή – Πιθανογεννήτριες – Μετασχηματισμοί L–S

① Βασικά θέματα

- Ορισμοί
- Ιδιότητες
- Αντιστροφή Πιθανογ., Μετασχ. L–S

② Ασκ 1.4 | Φυλ. 1

$$P_X(z) = \frac{c}{6 - z - z^2}$$

1. $c = ;$
2. $E[X] = ;$
3. $f_X(k) = Pr[X=k]$, $k = 0, 1, 2, \dots$
4. $Pr[X \text{ άρτιος}] = ;$

$$P_X(z) = \frac{c}{6 - z - z^2}$$

$$P_X(1) = 1 \Rightarrow c = 4. \Rightarrow P_X(z) = \frac{4}{6 - z - z^2}$$

$$E[X] = P_X'(1) = \left.\frac{4(1+2z)}{(6 - z - z^2)^2}\right|_{z=1} = \frac{12}{16} = \frac{3}{4}.$$

Αντιστροφή πιθανογεννήτριας

Βήμα 1: Προσδιορ. άγνωστων παραμ. ✓

Βήμα 2: Παραγοντοπ. παρονομαστή : $6 - z - z^2 = -(z^2 + z - 6) = -(z+3)(z-2)$

$= (3+z)(2-z)$

Βήμα 3: Ανάλυση σε απλά κλάσματα:

$$\frac{4}{(3+z)(2-z)} = \frac{A}{3+z} + \frac{B}{2-z} \xRightarrow{\times(3+z)} \frac{4}{2-z} = A + B\cdot\frac{3+z}{2-z} \xRightarrow{z=-3} A = \frac{4}{5}$$

$$B = \left.\frac{4}{3+z}\right|_{z=2} \Rightarrow B = \frac{4}{5}.$$

Βήμα 4ο: Ανάπτυξη σε δυνάμεις του z. Χρειαζόμαστε

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$$

$$\sum_{k=0}^{\infty} \underbrace{\left[{n \atop k}\right]}_{\binom{n+k-1}{k}} z^k = \frac{1}{(1-z)^n}$$

$$P_X(z) = \frac{4/5}{3+z} + \frac{4/5}{2-z}$$

$$= \frac{4}{15} \cdot \frac{1}{1-\left(-\frac{z}{3}\right)} + \frac{4}{10} \cdot \frac{1}{1-\frac{z}{2}}$$

$$= \frac{4}{15} \sum_{k=0}^{\infty} \left(-\frac{1}{3}\right)^k z^k + \frac{4}{10} \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k z^k$$

$$\Rightarrow f_X(k) = \Pr[X=k] = \frac{4}{15}\left(-\frac{1}{3}\right)^k + \frac{2}{5} \cdot \left(\frac{1}{2}\right)^k, \quad k = 0, 1, 2, \ldots$$

$$\Pr[X \text{ άρτιος}] = \sum_{k=0}^{\infty} f_X(2k) = \ldots \ (\text{γεωμετρικά αθροίσματα})$$

$$\left.\begin{aligned} P_X(1) &= \sum_{k=0}^{\infty} f_X(k) = f_X(0)+f_X(1)+f_X(2)+\ldots \\ P_X(-1) &= \sum_{k=0}^{\infty} f_X(k)(-1)^k = f_X(0)-f_X(1)+f_X(2)-\ldots \end{aligned}\right\} \frac{P_X(1)+P_X(-1)}{2} = \overbrace{\sum_{k \text{ άρτιος}} f(k)}^{\Pr[X \text{ άρτιος}] \,=\, \frac{1+\frac{4}{6}}{2} \,=\, \frac{5}{6}}$$

③ Ασκ. 1.7 / φυλ. 2

$$P_X(z) = \frac{cz^2 - 31z + 36}{6(2-z)^2(3-z)}$$

1. $c = ;$ 2. $E[X] = ;$ 3. $f_X(k) = ;$ $k = 0, 1, 2, \ldots$

Βήμα 1: $P_X(1) = 1 \Rightarrow c + 5 = 6 \cdot 2 \Rightarrow c = 7$.

Βήμα 2: Παραγοντοπ. του παρον. ✓

Βήμα 3: $$P_X(z) = \frac{7z^2 - 31z + 36}{6(2-z)^2(3-z)} = \frac{A}{(2-z)^2} + \frac{B}{2-z} + \frac{C}{3-z}$$

$$= \frac{\cancel{6}(3-z)A + \cancel{6}(2-z)(3-z)B + \cancel{6}(2-z)^2 C}{\cancel{6}(2-z)^2(3-z)}$$

$$= \frac{(3A + 6B + 4C) + (-A - 5B - 4C)z + (B + C)z^2}{(2-z)^2(3-z)}$$

Άρα: $$\left.\begin{aligned} 3A + 6B + 4C &= 36/6 = 6 \\ -A - 5B - 4C &= -31/6 \\ B + C &= 7/6 \end{aligned}\right\} \Rightarrow A = \frac{1}{3},\ B = \frac{1}{6},\ C = 1.$$

Γενική παρατήρηση:

Ρητή συνάρτηση

$$\frac{N(z)}{P(z)} = \frac{N(z)}{\alpha (z-p_1)^{a_1}(z-p_2)^{a_2}\cdots(z-p_k)^{a_k}}$$

$$= \text{πολυων} + \frac{\text{σταθ}}{z-p_1} + \frac{\text{σταθ.}}{(z-p_1)^2} + \frac{\text{σταθ}}{(z-p_1)^3} + \cdots + \frac{\text{σταθ}}{(z-p_1)^{a_1}}$$

$$+ \frac{\text{σταθ}}{z-p_2} + \frac{\text{σταθ}}{(z-p_2)^2} + \cdots + \frac{\text{σταθ}}{(z-p_2)^{a_2}} + \cdots$$

$$P_x(z) = \frac{1/3}{(2-z)^2} + \frac{1/6}{2-z} + \frac{1}{3-z} = \frac{1}{12}\cdot\frac{1}{(1-\frac{z}{2})^2} + \frac{1}{12}\cdot\frac{1}{1-\frac{z}{2}} + \frac{1}{3}\cdot\frac{1}{1-\frac{z}{3}}$$

$$= \frac{1}{12}\sum_{k=0}^{\infty}\binom{2+k-1}{k}\left(\frac{1}{2}\right)^k z^k + \frac{1}{12}\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k z^k + \frac{1}{3}\sum_{k=0}^{\infty}\left(\frac{1}{3}\right)^k z^k$$

$$\frac{1}{(1-z)^n} = \sum_{k=0}^{\infty}\left[\begin{matrix} n \\ k \end{matrix}\right] z^k, \qquad \left[\begin{matrix} n \\ k \end{matrix}\right] = \binom{n+k-1}{k}$$

$$\Rightarrow f_x(k) = \frac{1}{12}\underbrace{\binom{2+k-1}{k}}_{\binom{k+1}{k} = k+1}\left(\frac{1}{2}\right)^k + \frac{1}{12}\left(\frac{1}{2}\right)^k + \frac{1}{3}\left(\frac{1}{3}\right)^k, \quad k=0,1,\ldots$$

④ Ασκ. 15 / Φυλ. 1

$X \sim Exp(\lambda)$, $Y \sim Exp(\lambda)$, $Z \sim Exp(\mu)$, $\lambda \neq \mu$, ανεξ. $W = X+Y+Z$. $\tilde{F}_W(s), f_W(x)$

$$\tilde{F}_W(s) = \tilde{F}_X(s)\tilde{F}_Y(s)\tilde{F}_Z(s) = \left(\frac{\lambda}{\lambda+s}\right)^2 \cdot \frac{\mu}{\mu+s} = \frac{\lambda^2\mu}{(\lambda+s)^2(\mu+s)}$$

$$= \frac{A}{(s+\lambda)^2} + \frac{B}{s+\lambda} + \frac{C}{s+\mu}$$

$$\xrightarrow{\times(s+\lambda)^2} \frac{\lambda^2\mu}{\mu+s} = A + B(s+\lambda) + \frac{C}{s+\mu}(s+\lambda)^2 \xRightarrow{s=-\lambda} A = \frac{\lambda^2\mu}{\mu-\lambda}$$

$$\times(s+\lambda) \quad \underbrace{\frac{\lambda^2\mu}{(\lambda+s)(\mu+s)} - \frac{\lambda^2\mu}{(\mu-\lambda)(s+\lambda)}}_{=} = B + \frac{C}{s+\mu}(s+\lambda)$$

$$\frac{\lambda^2\mu(\mu-\lambda) - \lambda^2\mu(\mu+s)}{(\mu-\lambda)(s+\lambda)(s+\mu)} = B + \frac{C}{s+\mu}(s+\lambda) \xRightarrow{s=-\lambda} B = -\frac{\lambda^2\mu}{(\mu-\lambda)^2}$$

$$\frac{\lambda^2\mu(\cancel{\mu-\lambda-\mu-s})^{-1}}{(\mu-\lambda)\cancel{(s+\lambda)}(s+\mu)}$$

Για το C:

$$\xrightarrow[s=-\mu]{\times(s+\mu)} \quad C = \frac{\lambda^2\mu}{(\mu-\lambda)^2}$$

Υπενθυμίσεις: $X \sim Exp(\theta) \Rightarrow \tilde{F}_X(s) = \frac{\theta}{s+\theta}$

$X \sim Erlang(n,\theta) \Rightarrow \tilde{F}_X(s) = \left(\frac{\theta}{s+\theta}\right)^n$

Άρα: $$\tilde{F}_W(s) = \frac{\mu}{\mu-\lambda}\cdot\frac{\lambda^2}{(s+\lambda)^2} - \frac{\lambda\mu}{(\mu-\lambda)^2}\cdot\frac{\lambda}{s+\lambda} + \frac{\lambda^2}{(\mu-\lambda)^2}\cdot\frac{\mu}{s+\mu}$$

$$\Rightarrow F_W(w) = \frac{\mu}{\mu-\lambda}\left(\text{σ.κ. της } Erlang(2,\lambda)\right) - \frac{\lambda\mu}{(\mu-\lambda)^2}\cdot\left(\text{σ.κ. της } Exp(\lambda)\right) + \frac{\lambda^2}{(\mu-\lambda)^2}\cdot\left(\text{σ.κ. της } Exp(\mu)\right)$$

$$= \frac{\mu}{\mu-\lambda}\left(1-e^{-\lambda w}-\lambda w e^{-\lambda w}\right) - \frac{\lambda\mu}{(\mu-\lambda)^2}\left(1-e^{-\lambda w}\right) + \frac{\lambda^2}{(\mu-\lambda)^2}\left(1-e^{-\mu w}\right) \quad w \geq 0$$

$$f_W(w) = \frac{\mu}{\mu-\lambda}\cdot\frac{\lambda^2}{1!} w e^{-\lambda w} - \frac{\lambda\mu}{(\mu-\lambda)^2}\lambda e^{-\lambda w} + \frac{\lambda^2}{(\mu-\lambda)^2}\mu e^{-\mu w}, \quad w \geq 0.$$

⑤ Ασκ. 1.3 / Φυλ. 2

$X|U=p \sim Bin(n,p)$ $\qquad f_X(x) = ;\quad x=0,1,2,\dots,n$

$U \sim Uniform((0,1))$ $\qquad E[X]$

$$f_X(x) = Pr[X=x] = \int_0^1 Pr[X=x \mid U=p]\cdot \underset{\underset{1}{\|}}{f_U(p)}\, dp = \int_0^1 \binom{n}{x} p^x (1-p)^{n-x}\, dp$$

$$= \binom{n}{x} \int_0^1 p^x (1-p)^{n-x} dp = \binom{n}{x}\cdot B(x+1,\, n+1-x)$$

$$= \binom{n}{x}\cdot \frac{x!\,(n-x)!}{(n+1)!} = \frac{1}{n+1} \rightsquigarrow X \sim \text{Διακρ. Ομοιομ. } \{0,1,\dots,n\}$$

$$E[X] = E[E[X|U]] = E[nU] = n\,E[U] = \frac{n}{2}. \quad \left(\begin{array}{l} E[X|U=p] = np \\ E[X|U] = nU \end{array}\right)$$

$$B(a,b) = \int_0^1 p^{a-1}(1-p)^{b-1} dp = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)} \overset{\substack{\text{ακερ} \\ a,b}}{=} \frac{(a-1)!\,(b-1)!}{(a+b-1)!}$$

Εναλλακτικά: $E[X] = \sum_{x=0}^{n} x f_X(x) = \sum_{x=0}^{n} x \cdot \frac{1}{n+1} = \frac{n\cancel{(n+1)}}{2} \cdot \frac{1}{\cancel{n+1}} = \frac{n}{2}$