B τ.κ.Β , $c \in \mathbb{R}\setminus\{0\}$

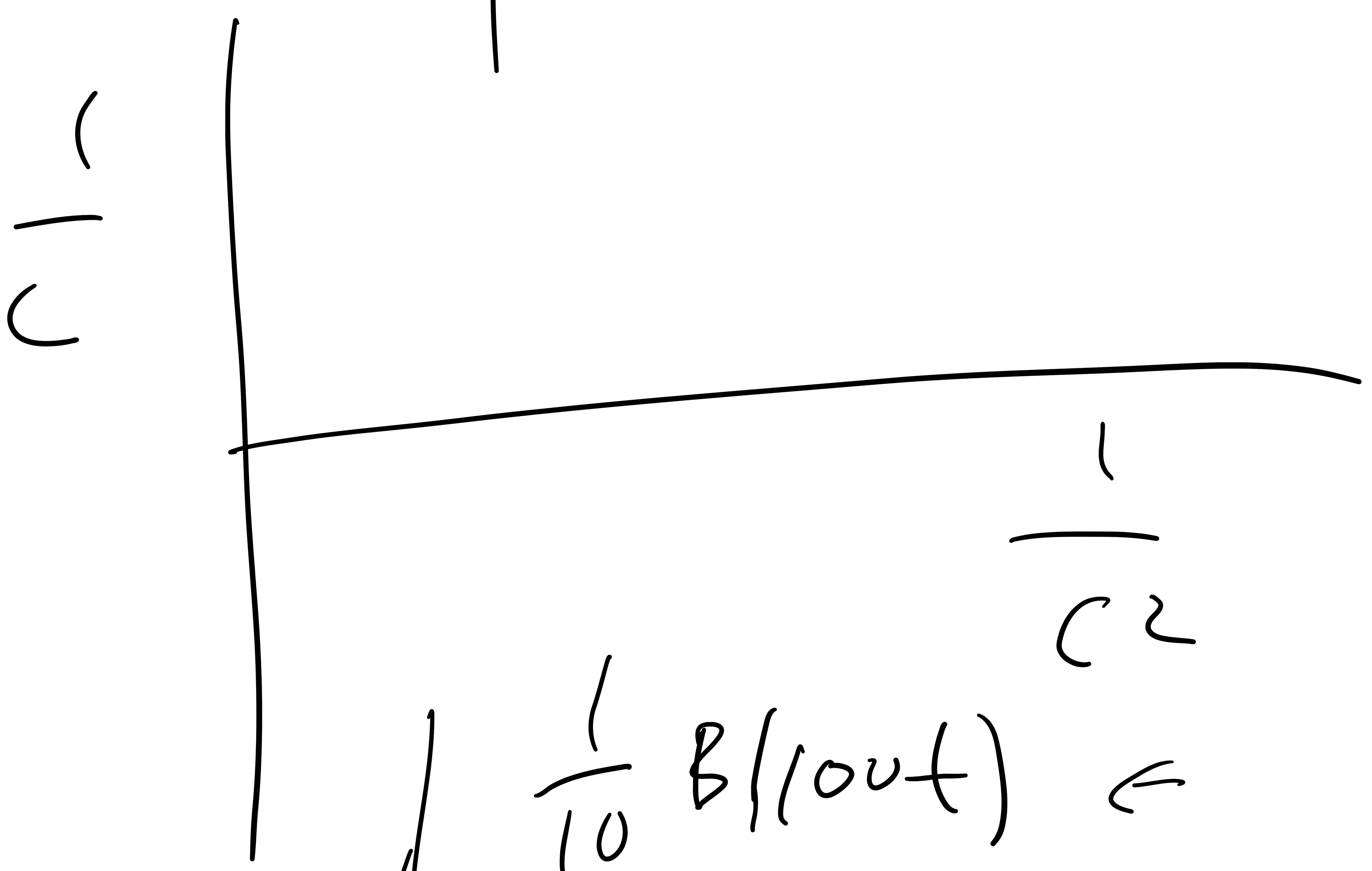
$X(t) = \frac{1}{c} B(c^2 t)$ είναι τ.κ.Β

$\frac{1}{\cancel{c^2}} \cancel{c^2} t$

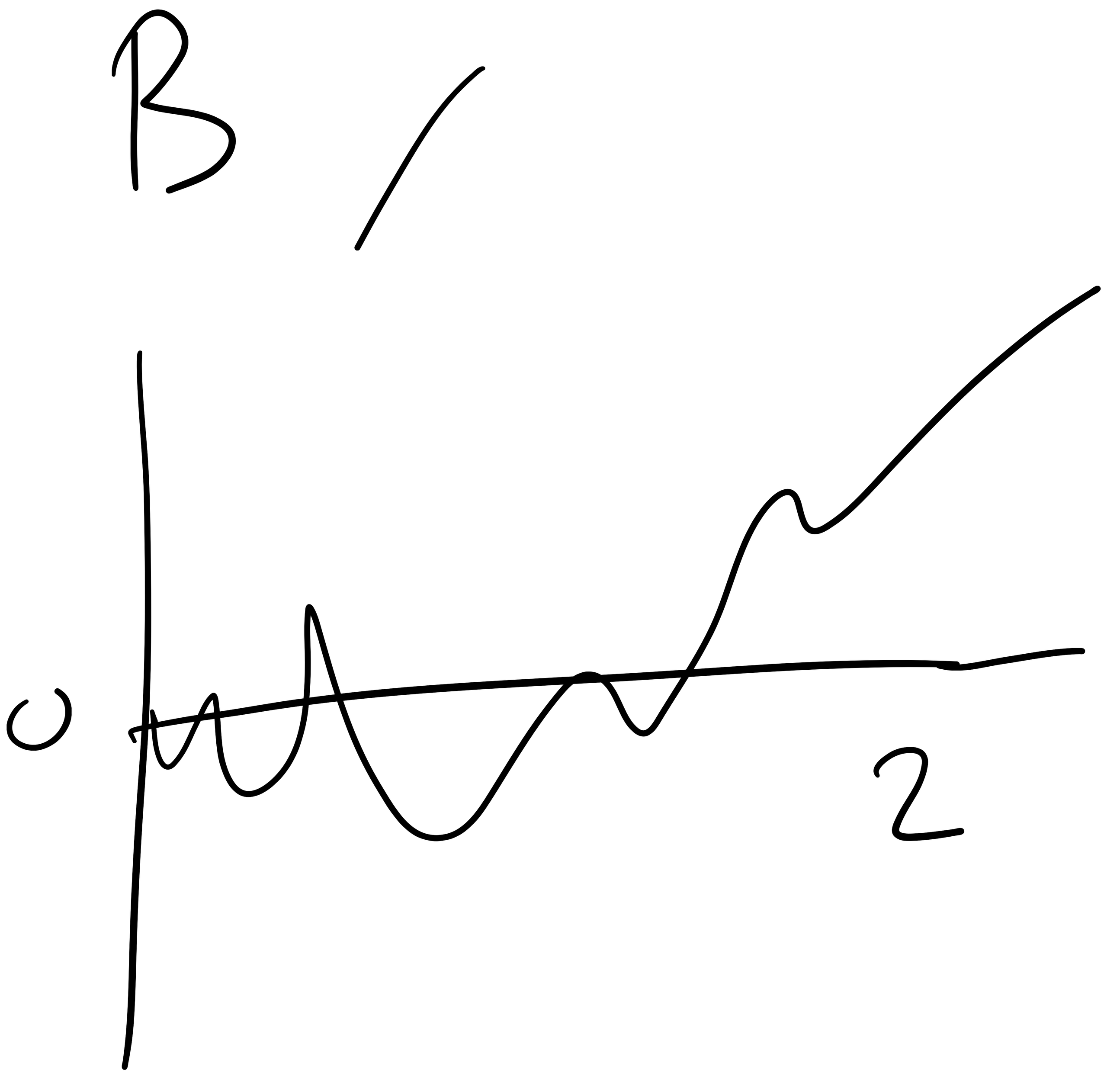
$c = -1$

$-B(t)$

$c = 10$

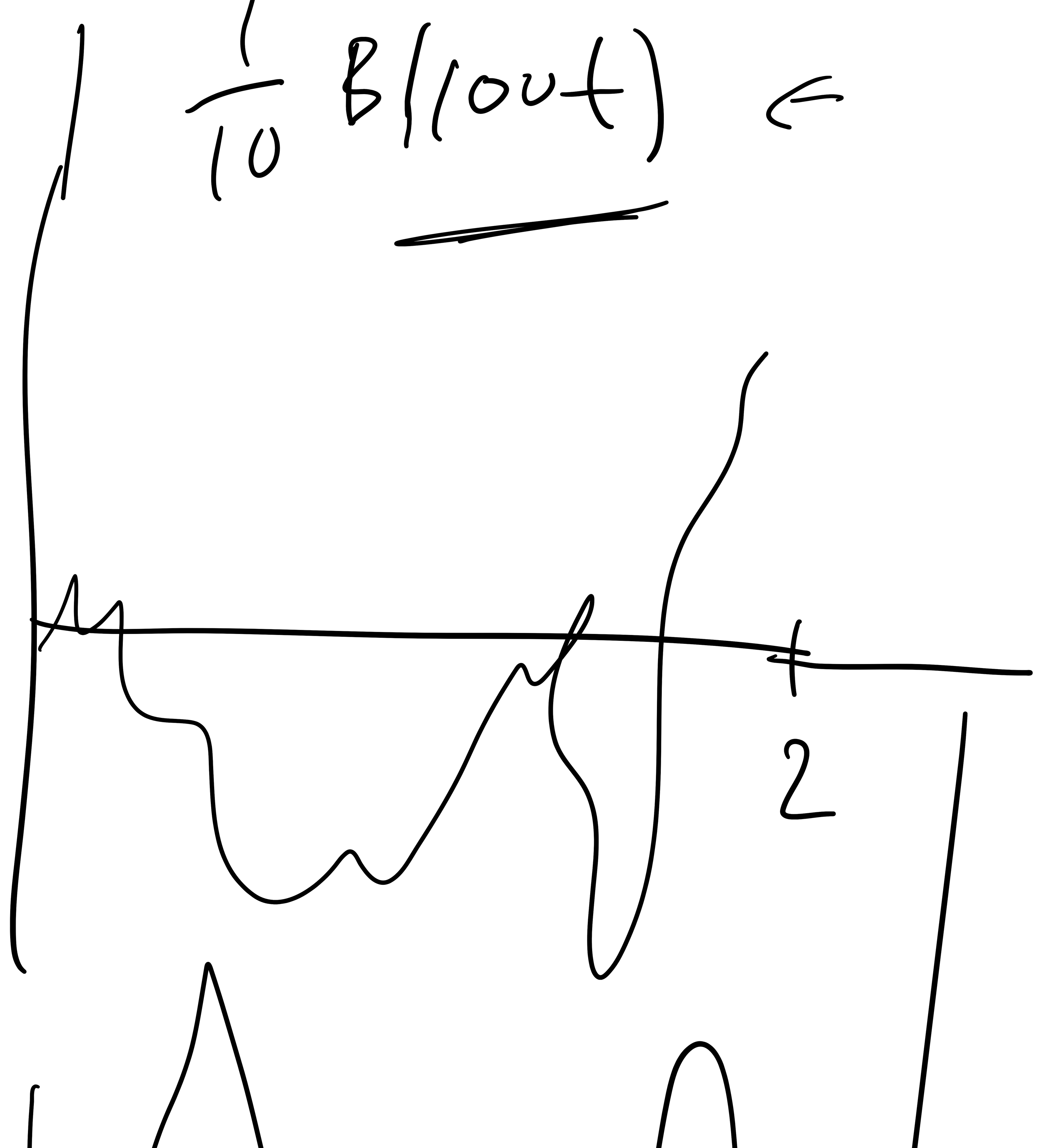


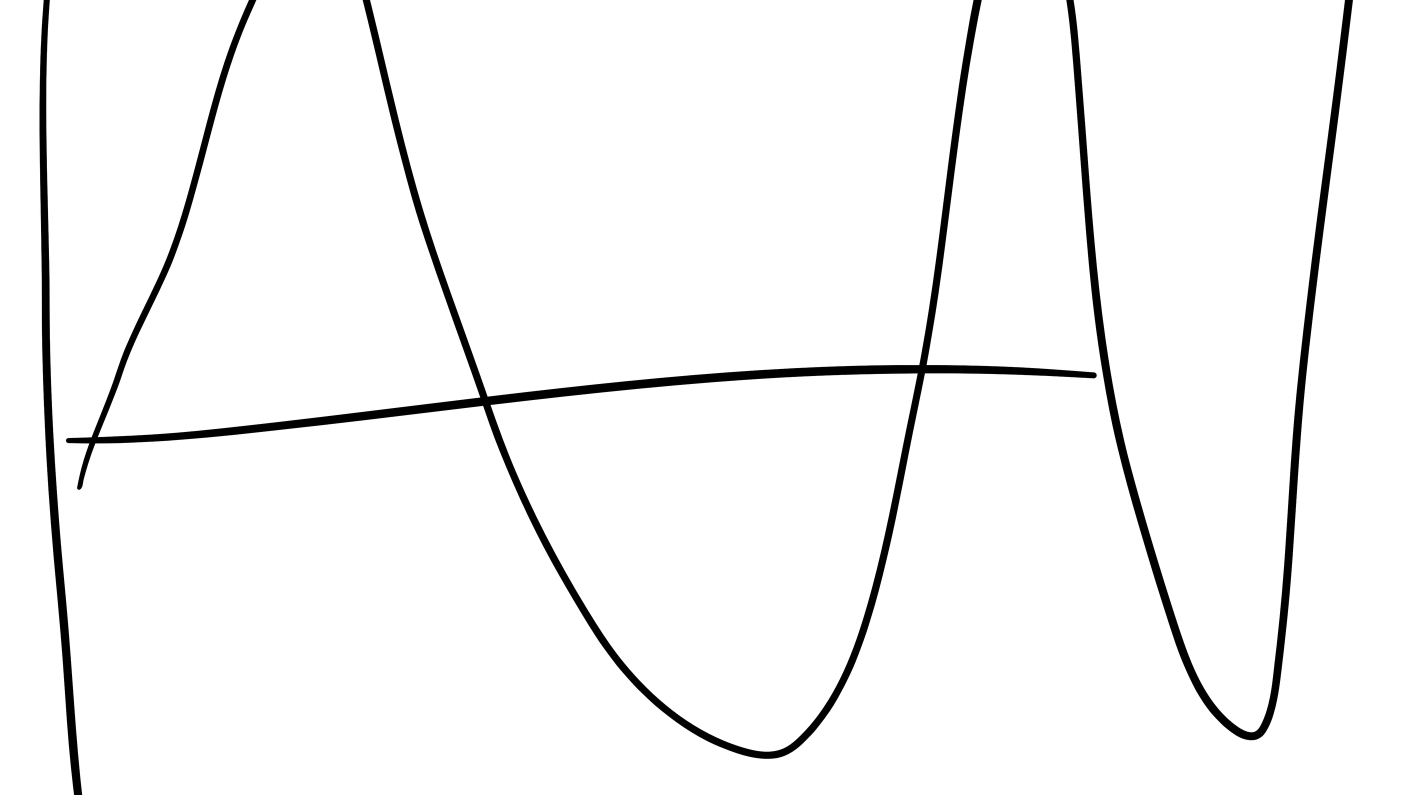
B



$\frac{1}{10} B(10t)$

$\frac{1}{10} B(100t)$ ←





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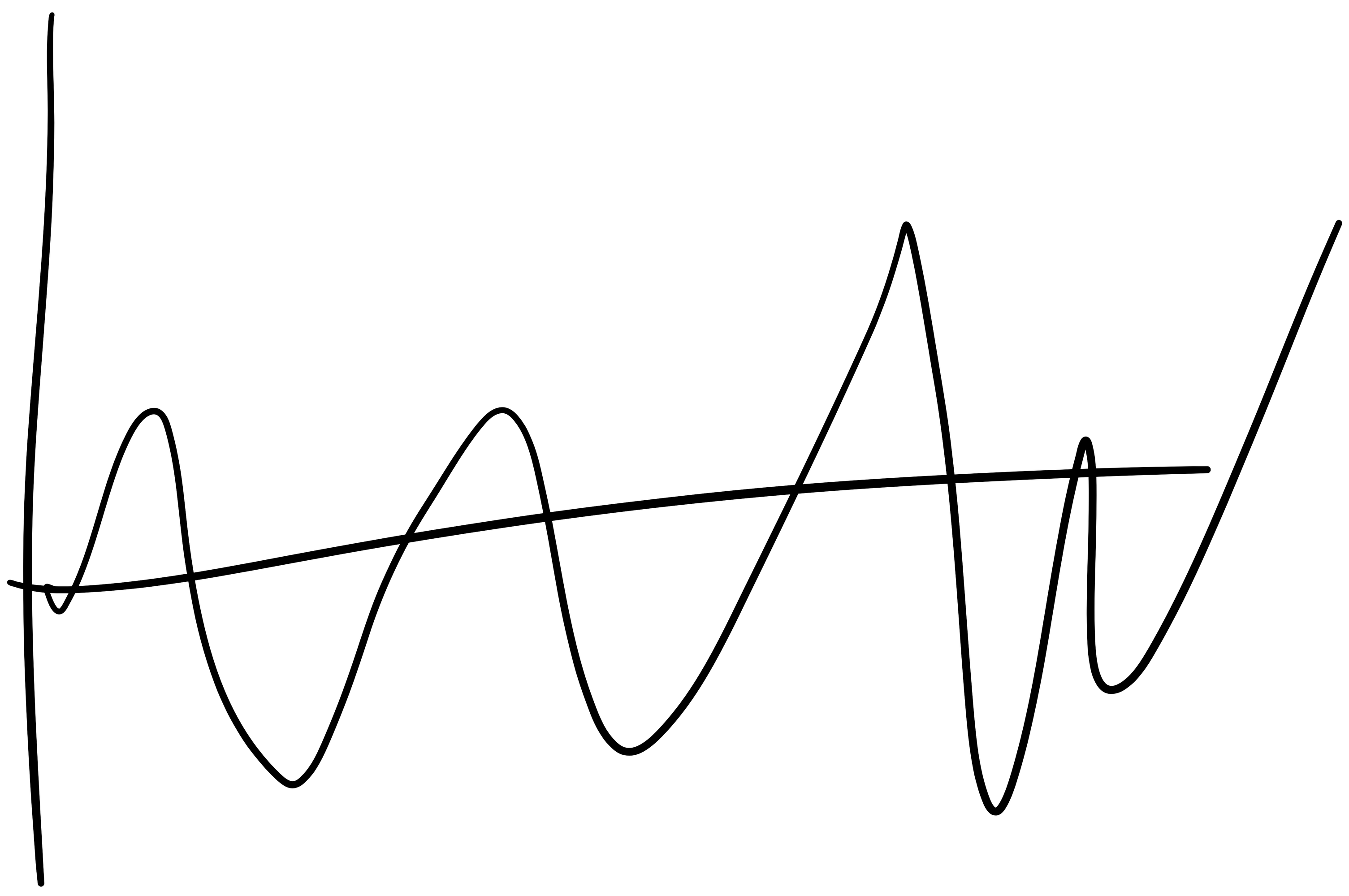
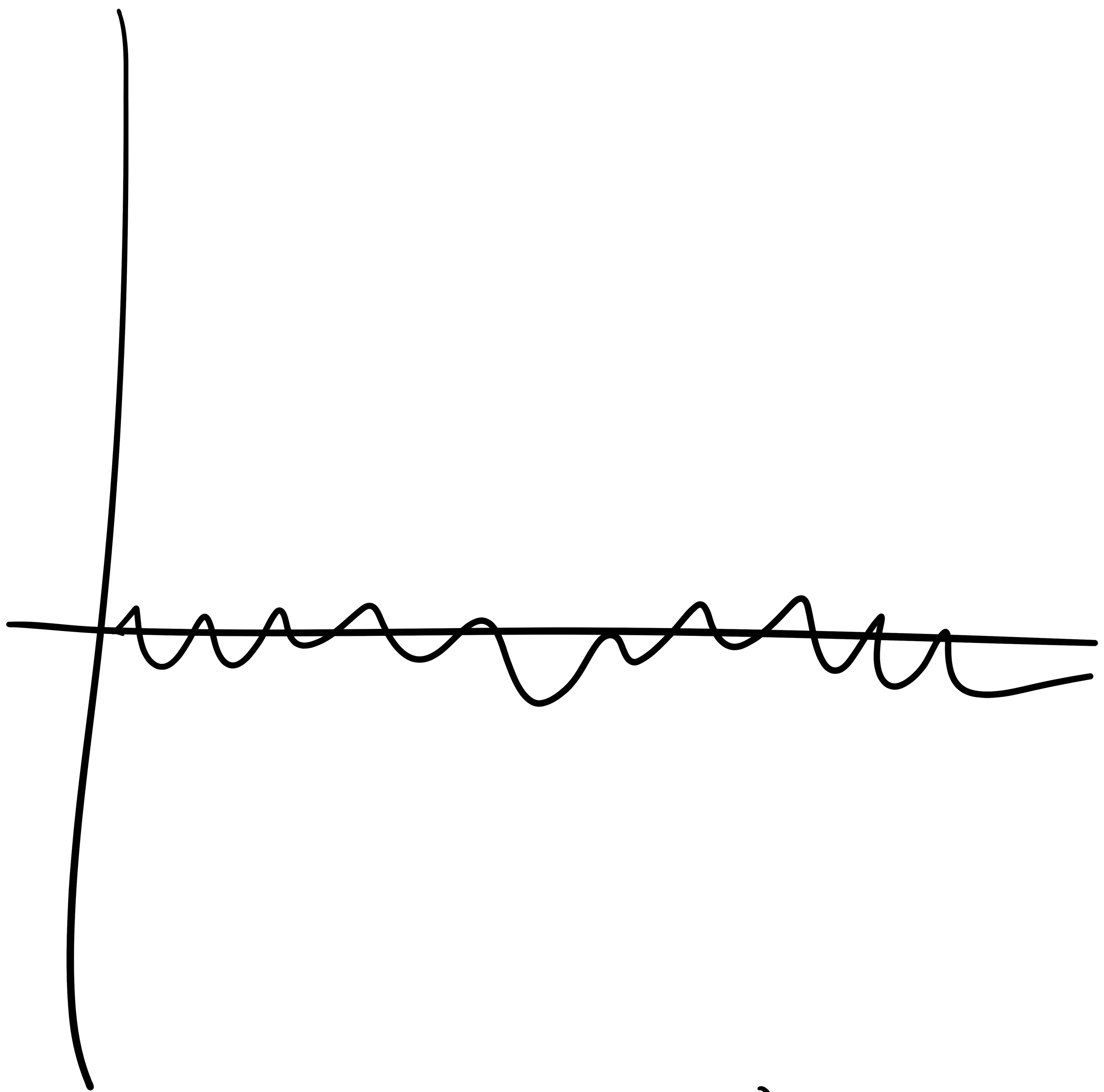
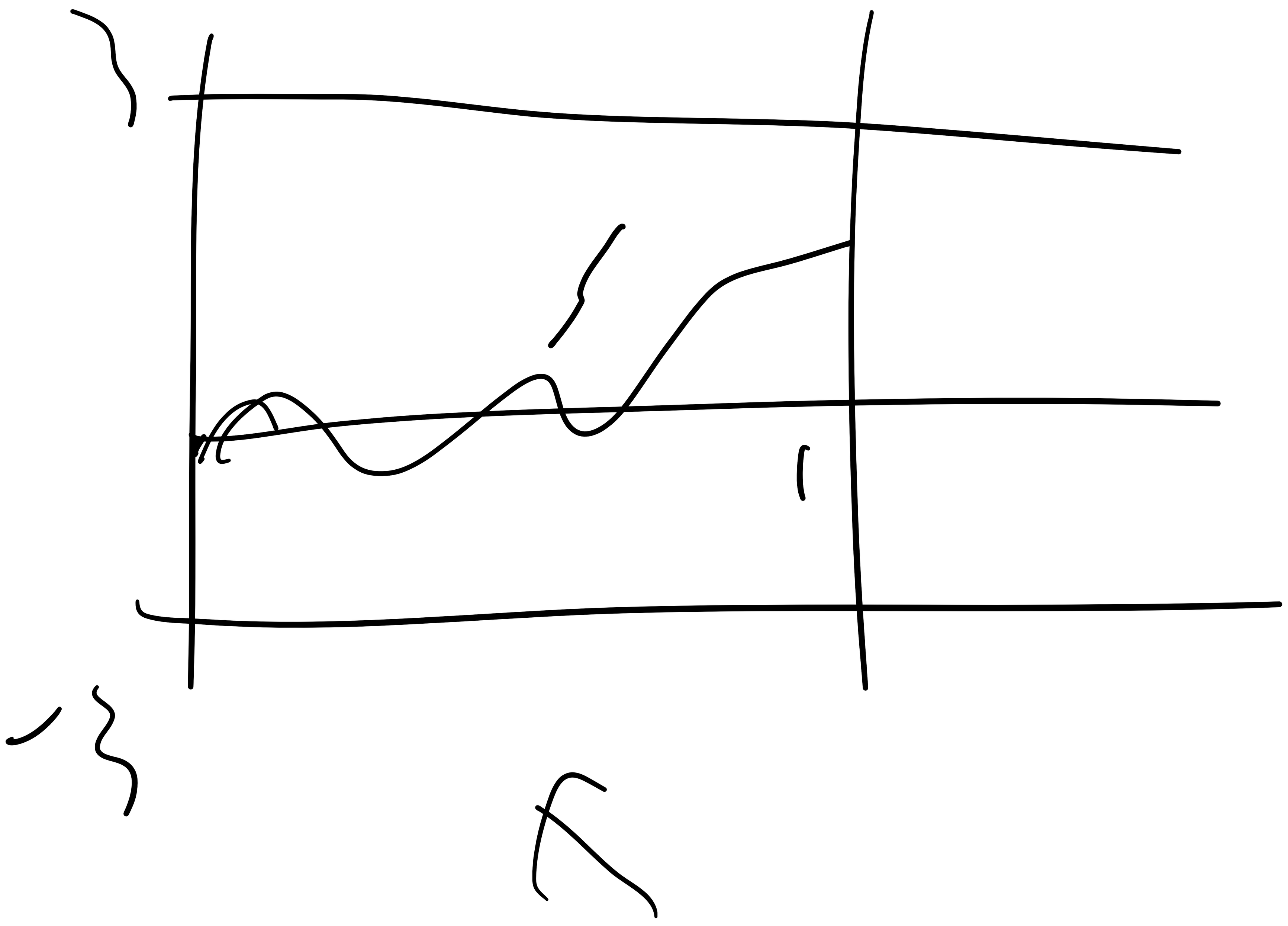


$\frac{1}{10} B(10t)$



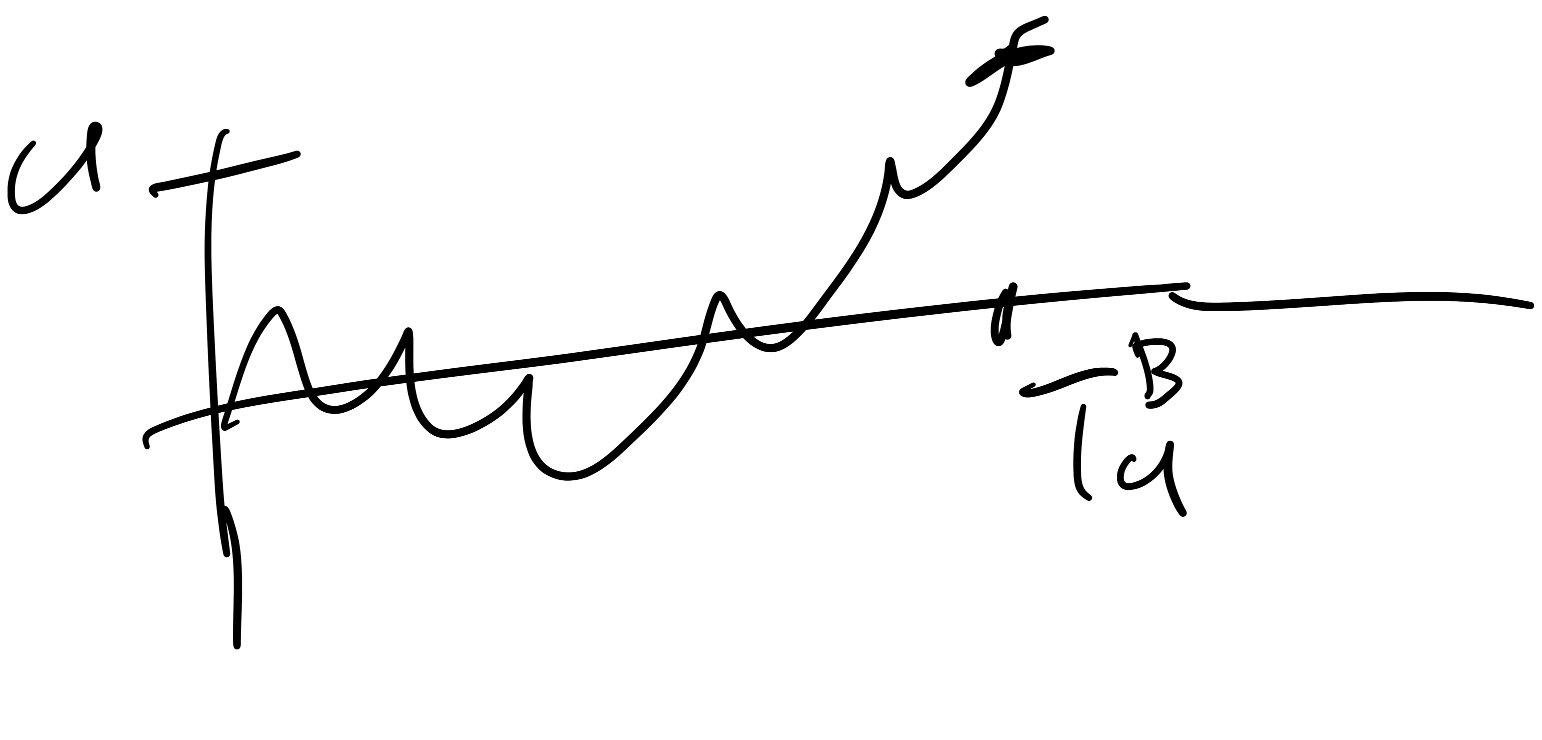


$B(1) > 10$

$X(1) > 10$

Εφαρμογή $\quad a \neq 0, \; B$ τ.κ.B

$$T_a^B = \inf\{t \geq 0 : B(t) = a\}$$



Ισχύει $T_a \overset{d}{=} a^2 T_1$

$\uparrow$

Απόδ

Έστω $X(t) = \dfrac{B(a^2 t)}{a}$ $\quad \forall t \geq 0$

είναι τ.κ.Β

$$X(t) = 1 \iff B(a^2 t) = a$$

$s = a^2 t$

$$\{t : X(t) = 1\} = \frac{1}{a^2}\{s : B(s) = a\}$$

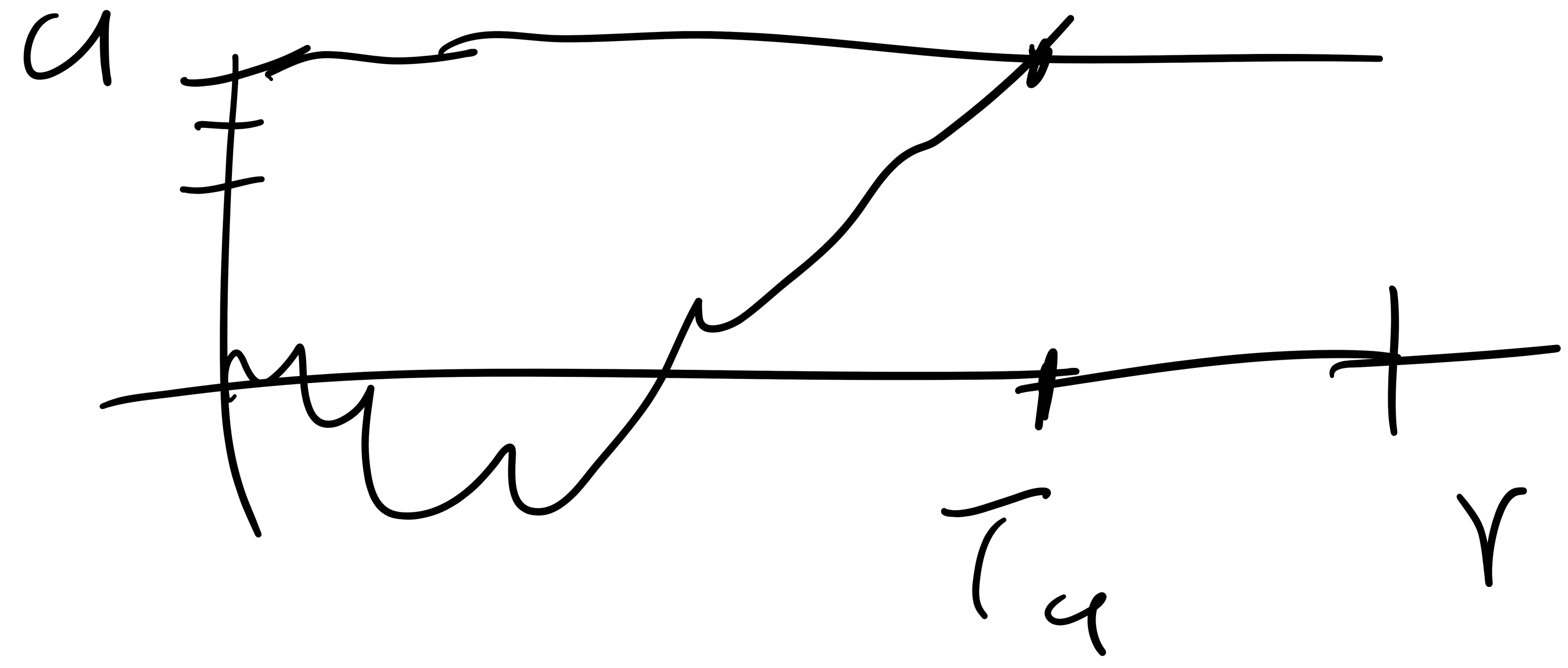
$$T_1^X = \frac{1}{a^2} T_a^B$$

$$T_a^B = a^2 T_1^X \overset{d}{=} a^2 T_1^B$$

Το T_a είναι τ.μ.

$T_a : \Omega \to [0, \infty]$ Για $r \geq 0$

$$\{T_a \leq r\} = \bigcup_{s \in [0,r]} \{B(s) = a\}$$ ← δεν μας κάνει



έστω ότι $a > 0$.

$$\{T_a \le r\} = \bigcap_{n \ge 1} \bigcup_{\substack{q \le r \\ q \in \mathbb{Q}}} \underbrace{\left\{ B(q) \ge a - \frac{1}{n} \right\}}_{\in \mathcal{F}}$$

$\hookrightarrow \in \mathcal{F}$

<u>Πρόταση</u> (Αντιστροφή χρόνου)

B τ.κ.Β., θέτουμε

$$X(t) = \begin{cases} t\,B\left(\frac{1}{t}\right) & t > 0 \\ 0 & t = 0 \end{cases}$$

Τότε η X <u>είναι</u> τ.κ.Β

X , $\underline{B}$

$(B(t_1), \quad B(t_2)) \leftarrow \qquad C(t_i, t_j) = t_i \wedge t_j$

$(X(t_1), \quad X(t_2)) \qquad b, \; C$

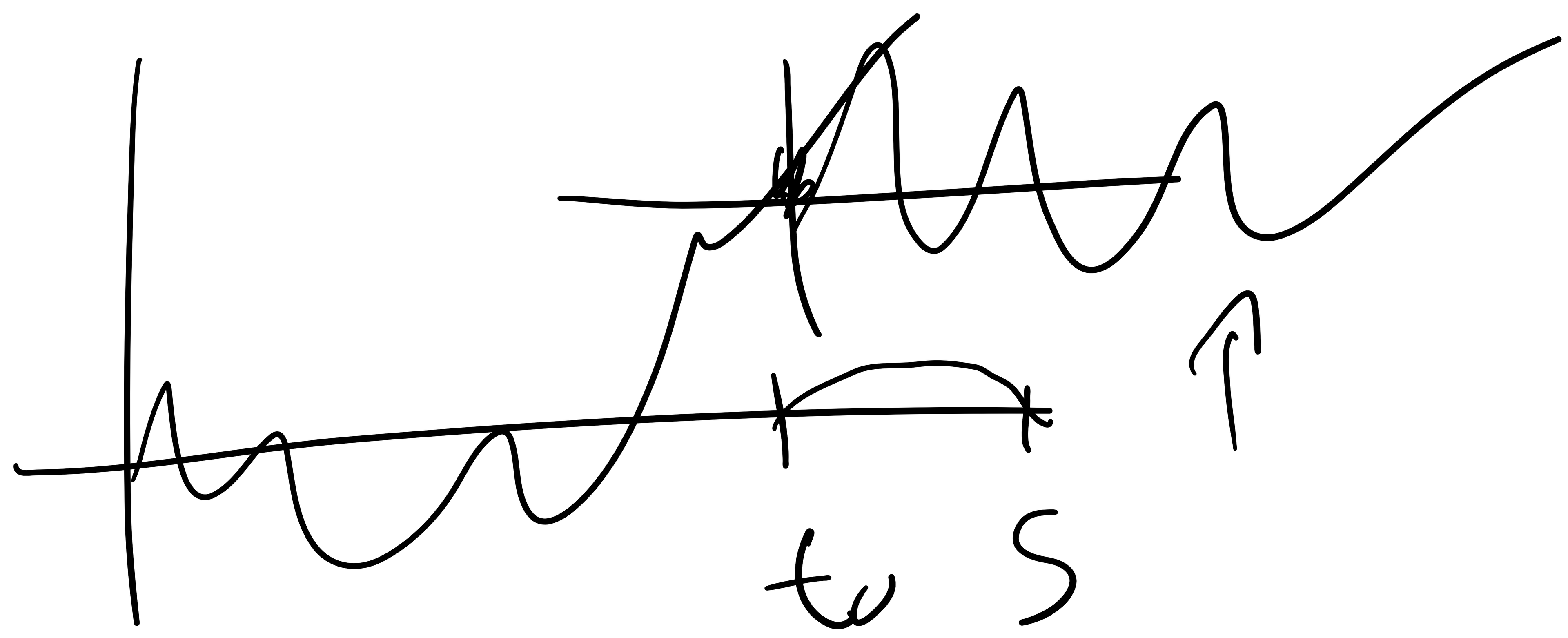
Για $0 < s < t$

$$\mathrm{cov}(X(s), X(t)) = \mathrm{cov}\left(sB\left(\tfrac{1}{s}\right), tB\left(\tfrac{1}{t}\right)\right)$$

$$= st\left(\frac{1}{s} \wedge \frac{1}{t}\right) = st \cdot \frac{1}{t} = s = s \wedge t$$

$$\lim_{t \to 0^+} tB\left(\frac{1}{t}\right) = 0$$

$$\lim_{t \to \infty} \frac{B(t)}{t} = 0$$



Για $0 < t_0$. Η $\{B(s) - B(t_0) : s \geq t_0\}$

είναι ανεξάρτητη από την $\{B(s) : s \in [0, t_0]\}$

(άρα από την $\mathcal{F}^0_{t_0}$)

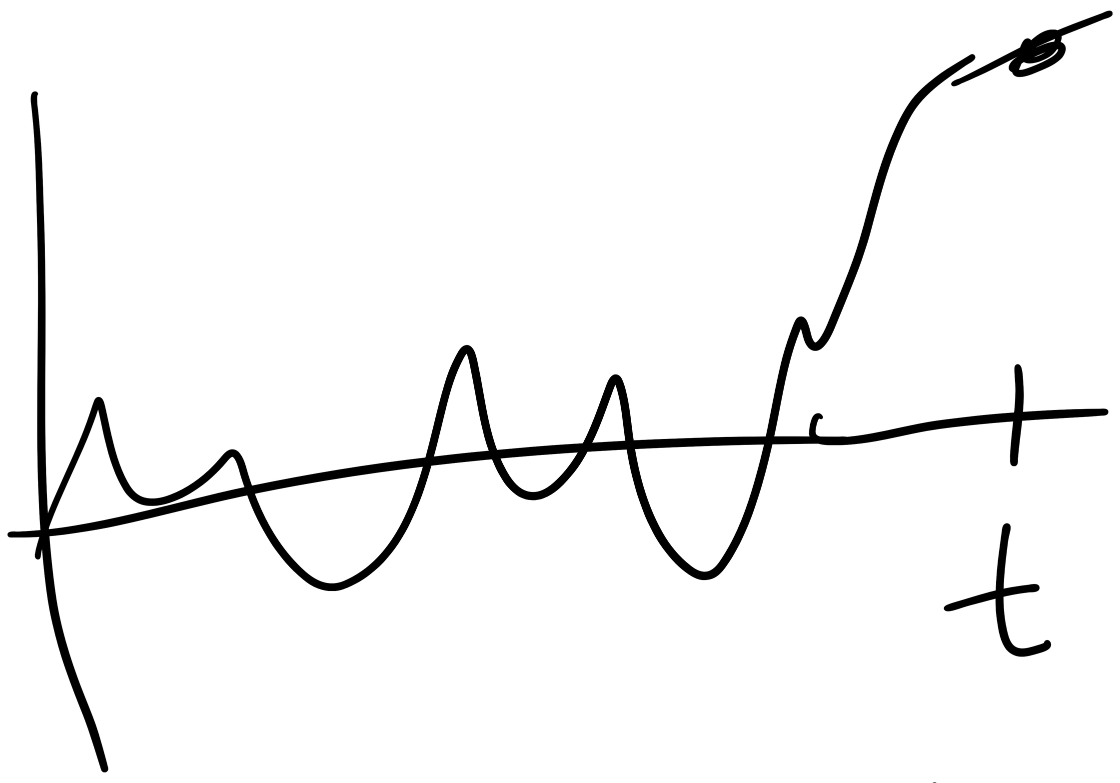
Αυτό πάει $B(s) - B(t_0) = X(s - t_0)$

[illegible] $X(r) = B(t_0 + r) - B(t_0)$ $\quad r \geq 0$

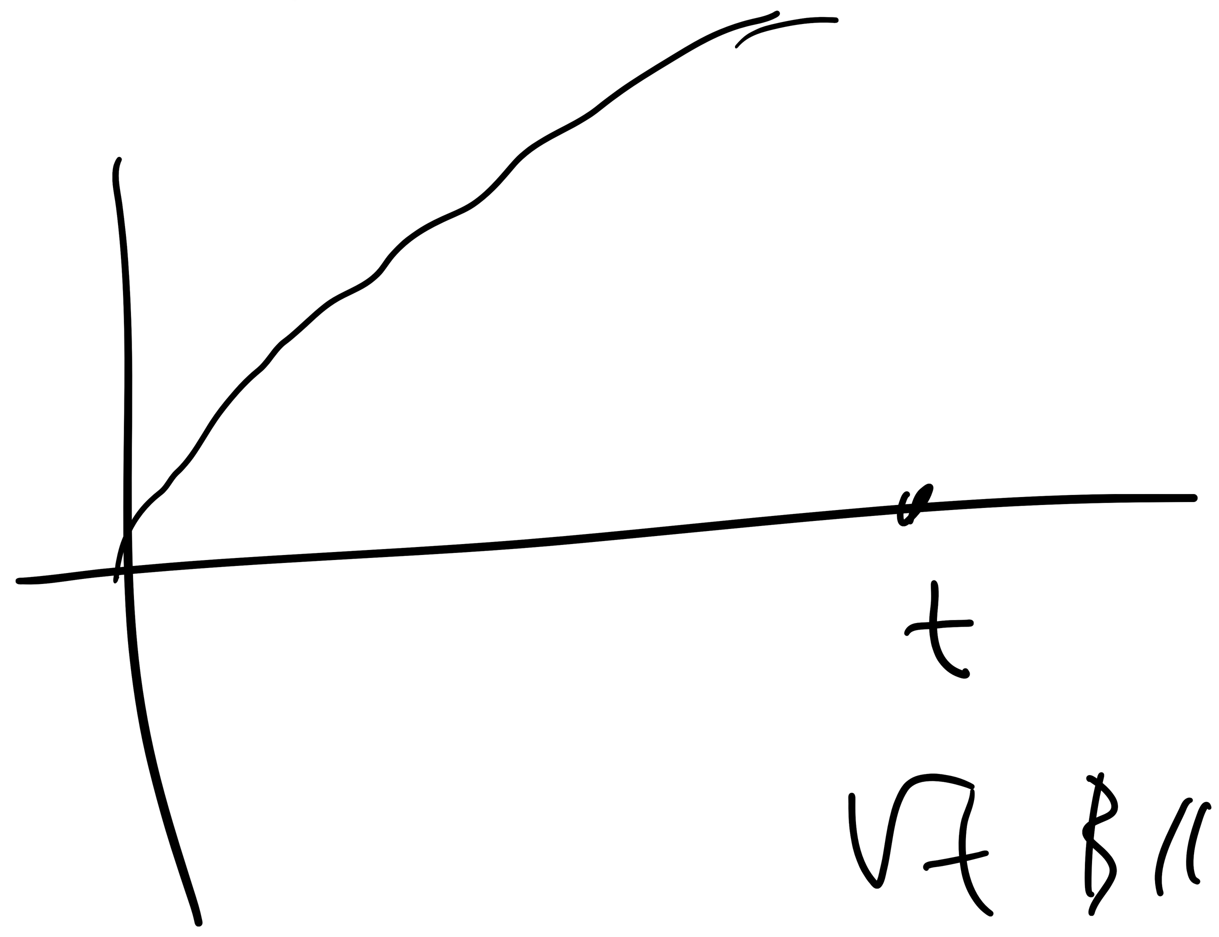
$X \perp\!\!\!\perp \mathcal{F}_{t_0}$

$$B(t) \overset{d}{=} \sqrt{t}\,\underbrace{B(1)}_{\sim N(0,1)} = \sqrt{t}\, Z$$

[illegible] t

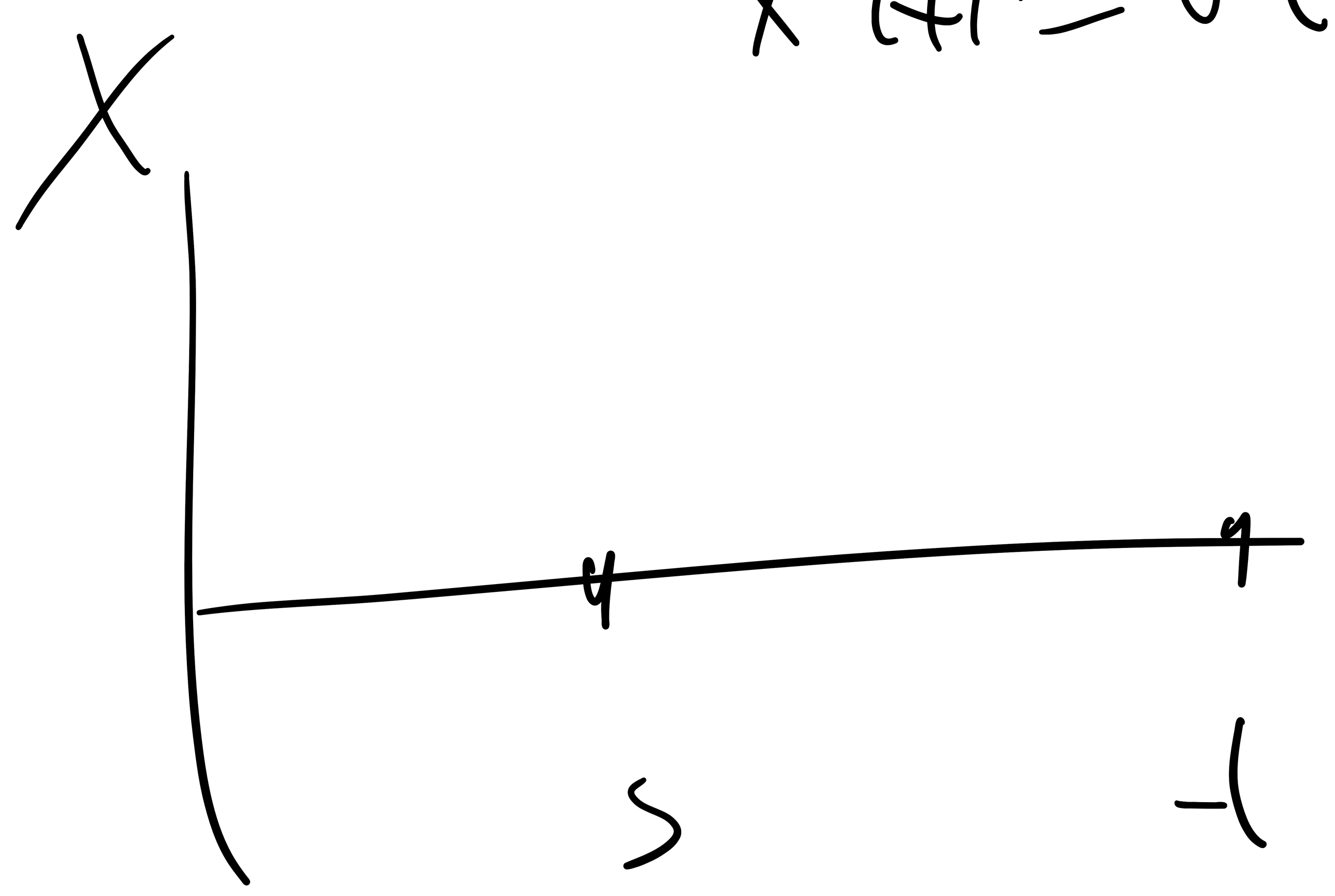


$B(t)$



$\sqrt{t}\, B(1)$

$X(t) = \sqrt{t}\, B(1)$



$B(s)$ $\quad B(t)$ $\qquad X(s)$ $\quad X(t)$

$$E\left(\int_0^1 e^{B(s)}\,ds\right) = \int_0^1 E\left(e^{B(s)}\right) ds$$

$$= \int_0^1 E\left(e^{\sqrt{s}\,B(1)}\right) ds = \int_0^1 e^{\frac{1}{2}(\sqrt{s})^2}\, ds$$

$$= \int_0^1 e^{\frac{s}{2}}\, ds$$

$$\int_0^1 e^{B(s)}\, ds \overset{d}{=} \int_0^1 e^{\sqrt{s}\,B(1)}\, ds$$

Άσκηση 5.5 $\qquad B$ Τ.Κ.Β., $t>0$

$$X := \int_0^t B(s)\, ds$$

Ν.δ. ότι $\quad X \sim N\left(0, \frac{t^3}{3}\right)$

$$X = \lim_{n\to\infty} \overbrace{\frac{t}{n}\sum_{k=1}^{n} B\left(k\frac{t}{n}\right)}^{\text{ισχύει}} = X_n$$

$$EX \underset{d}{=} \int_0^t E(\underbrace{B(s)}_{\sim N(0,s)})\, ds = \int_0^t 0\, ds = 0$$

$$\int_0^t E|\underbrace{B(s)}_{\sqrt{s}B(1)}|\, ds = \int_0^t \sqrt{s}\, \overbrace{E|B(1)|}^{\in(0,\infty)}\, ds$$

$$= E|\ \ |\int_0^t \sqrt{s}\, ds < \infty$$

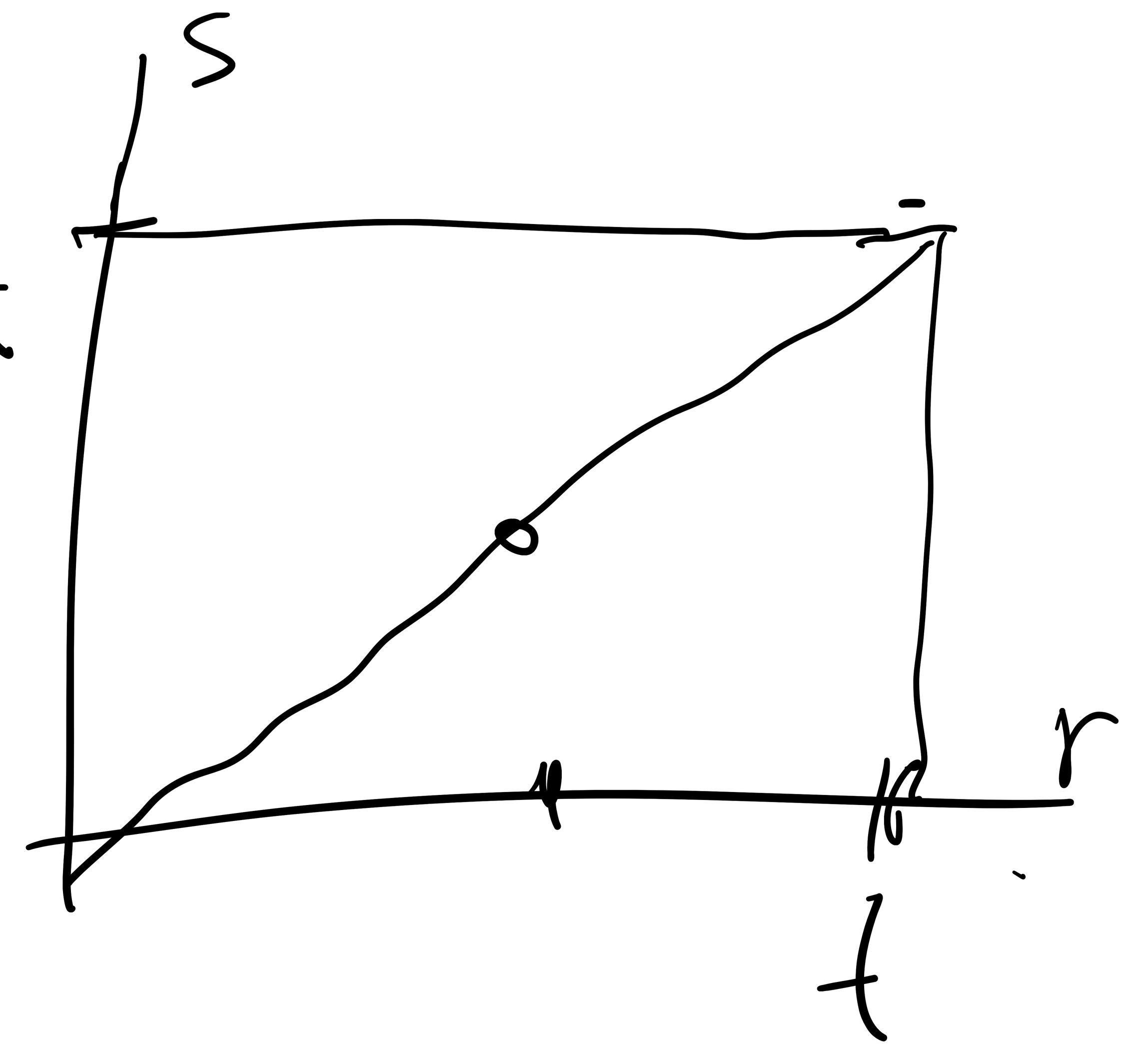
$$Var(X) = E(X^2) = E\left[\cancel{\left(\int_0^t B(s)\, ds\right)^2}\right]$$

$$= E\left(\int_0^t \underbrace{\int_0^t B(s)\, \underline{B(r)}\, ds\, dr} \right)$$

$$\left(\left(\int_{\mathbb{R}} e^{-x^2} dx \right)^2 = \int_{\mathbb{R}^2} e^{-x^2-y^2}\, dx\, dy \right)$$

$$= \int_0^t \int_0^t E\left(B(s) B(r) \right) ds\, dr$$

$$= \int_0^t \int_0^t (s \wedge r)\, ds\, dr$$



$$= \int_0^t \left(\int_0^r ds + \int_r^t ds \right) dr$$

$$= \int_0^t \left(\int_0^r s\, ds + \int_r^t r\, ds \right) dr$$

$$= \int_0^t \left(\underbrace{\frac{r^2}{2}} + \underline{r}\,(t - \underline{r}) \right) dr$$

$$= -\frac{t^3}{6} + t\,\frac{t^2}{2} = t^3\left(\frac{1}{2} - \frac{1}{6} \right) = \frac{t^3}{3}$$

$X = \lim_{n \to +\infty} X_n$ με π.θ. 1. Αρα

$X_n \Rightarrow X$ $\quad X_n \sim N(0, \sigma_n^2)$

$$\varphi_{X_n}(t) = e^{-\frac{t^2}{2}\sigma_n^2} \to \underline{\underline{e^{-\frac{t^2}{2}A}}}$$

$\sigma_n^2 \to A$

$N(\mu_n, \sigma_n^2)$ $\quad N(\mu, \sigma^2)$

$\mu_n \downarrow \mu$, $\sigma_n^2 \downarrow \sigma^2$

<u>Δύο τρόποι</u>

1) $\sigma_n^2 \to \frac{t^3}{3}$ $\quad N(0, \frac{t^3}{3})$

2) Δείχνουμε ότι X κανονική

$X_n \Rightarrow X$ είναι

$$\varphi_X(t) = \lim_{n \to \infty} \varphi_{X_n}(t) = \lim_{n \to \infty} e^{-\frac{t^2}{2}\sigma_n^2}$$

$\Rightarrow$ $A = \lim_{n \to \infty} \sigma_n^2$ υπάρχει $\in [0, \infty]$

Αν $A = \infty$, τότε $\varphi_X(t) = 1_{t=0}$

Αν $A = 0$, τότε $\varphi_X(t) \to 1 = \varphi_0(t)$

Άρα $X \stackrel{d}{=} 0$

Αποδείξω $E(X^2) = \frac{t^3}{3}$

Άρα $A \in (0, \infty)$

$\varphi_X(t) = e^{-\frac{t^2}{2} A}$

Δηλ. $X \sim N(0, A)$

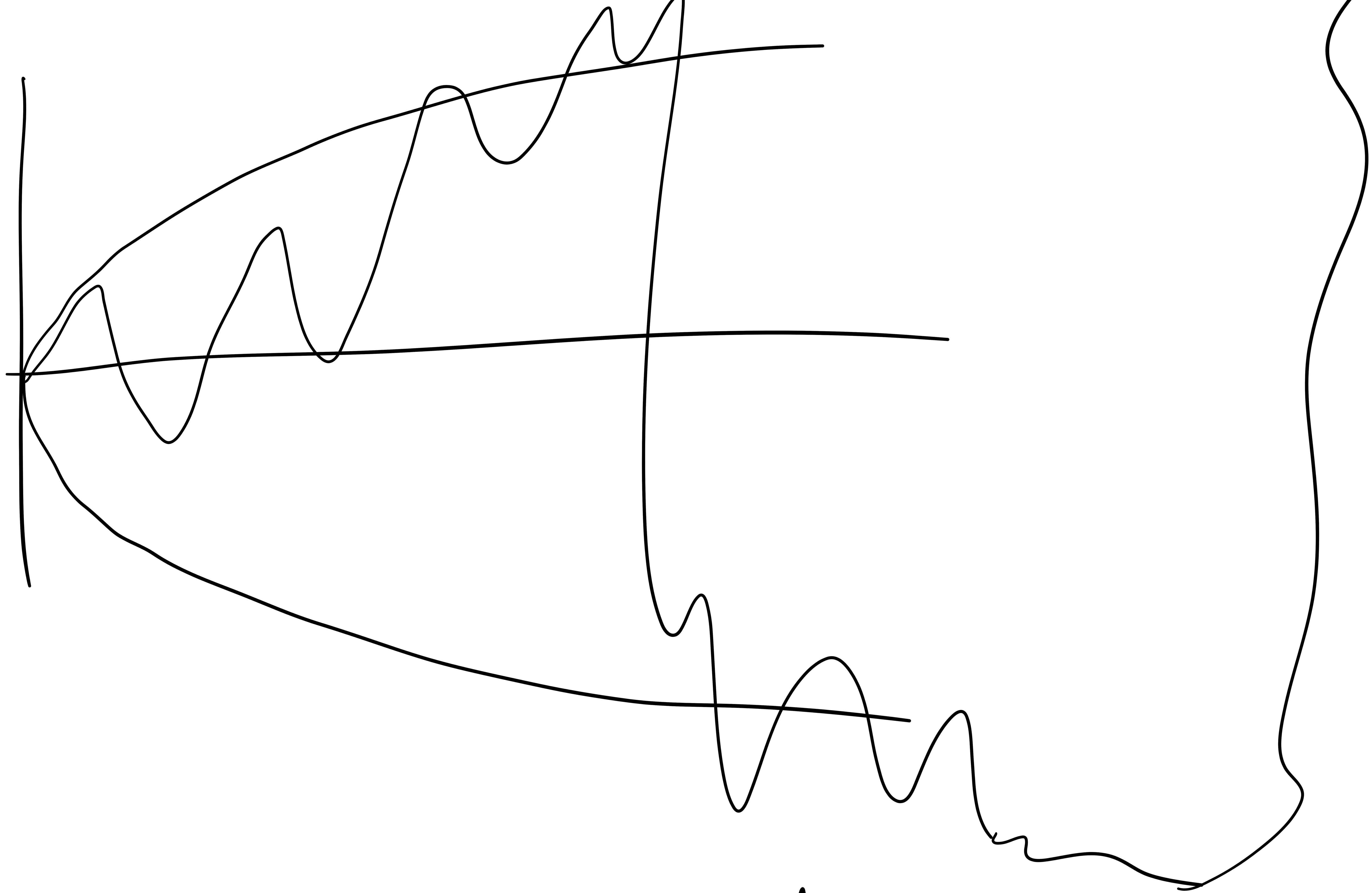
$EX^2 = \frac{t^3}{3} \Rightarrow A = \frac{t^3}{3}$

<u>Πρόταση</u> B τ.κ.Β. με π.θ. 1

Ισχύει

$\varliminf_{t \to \infty} \frac{B(t)}{\sqrt{t}} = -\infty$, $\varlimsup_{t \to \infty} \frac{B(t)}{\sqrt{t}} = \infty$

$\frac{B(t)}{t} \to 0$ $\qquad$ $\varliminf B(t) = -\infty$, $\limsup B(t) = \infty$



Αποδ

Δείχνουμε ότι

$$\varliminf_{\substack{n\to\infty \\ n\in\mathbb{N}}} \frac{B(n)}{\sqrt{n}} = -\infty \qquad \varlimsup_{\substack{n\to\infty \\ n\in\mathbb{N}}} \frac{B(n)}{\sqrt{n}} = \infty$$

Θέτουμε $X_k = B(k) - B(k-1)$, $k = 1, \ldots, n$

Ανεξάρτητες, ισόνομες, $X_1 \sim N(0,1)$

$$B(n) = X_1 + X_2 + \quad + X_n$$

Δείχνουμε το $\varlimsup \frac{S_n}{\sqrt{n}} = \infty$.

Για $c > 0$, θέτουμε

$$A_n(c) = \{ B(n)/\sqrt{n} \geq c \}$$

$$P(A_n(c)) = P(\underset{\sim N(0,1)}{Z} \geq c) = 1-\Phi(c) > 0$$

$$P(\underbrace{\{\overline{\lim_{n\to\infty}} \frac{B(n)}{\sqrt{n}} \geq c\}}_{A(c)}) \geq P(\limsup_{n\geq 1} A_n(c))$$

$$\geq \overline{\lim}\, P(A_n(c)) = 1-\Phi(c) > 0$$

$$P(A(c)) > 0$$

$$\mathcal{C}_n = \sigma(\{X_n, X_{n+1}, \dots\})$$

$$\mathcal{C}_\infty = \bigcap \mathcal{C}_n$$

$$A(c) \in \mathcal{C}_\infty$$

S_{n_0}

$$Y = \overline{\lim_{n\to\infty}} \frac{S_n}{\sqrt{n}} = \overline{\lim_{n\to\infty}} \underbrace{\frac{X_{n_0+1} + \dots + X_n}{\sqrt{n}}}_{\mathcal{C}_{n_0+1}\text{ mensurável}}$$

$$\{Y \geq c\} \in \mathcal{C}_{n_0+1} \quad \forall n_0$$

$$\dots \in \mathcal{C}_\infty$$

$\therefore \; P(A(r)) = 1$

$\Rightarrow \; P\Big(\bigcap_{r\in\mathbb{N}} A(r)\Big) = 1$

$$\overline{\lim_{n\to\infty}} \frac{B(n)}{\sqrt{n}} \geq r \quad \forall r\in\mathbb{N}$$

$$\hookrightarrow \; \overline{\lim_{\substack{n\to\infty\\ n\in\mathbb{N}}}} \frac{B(n)}{\sqrt{n}} = \infty$$

Για το $\underline{\lim}$.

Η $X(t) = -B(t)$ είναι Τ.Κ.Β

$$1 = P\Big(\overline{\lim_{t\to\infty}} \frac{X(t)}{\sqrt{t}} = \infty\Big)$$

$$= P\Big(-\underline{\lim_{t\to\infty}} \frac{B(t)}{\sqrt{t}} = \infty\Big)$$

$$= P\Big(\underline{\lim} \frac{B(t)}{\sqrt{t}} = -\infty\Big)$$

Υπολογίστε την ΣΔ

$$\int_0^t B^2(s)\,ds$$

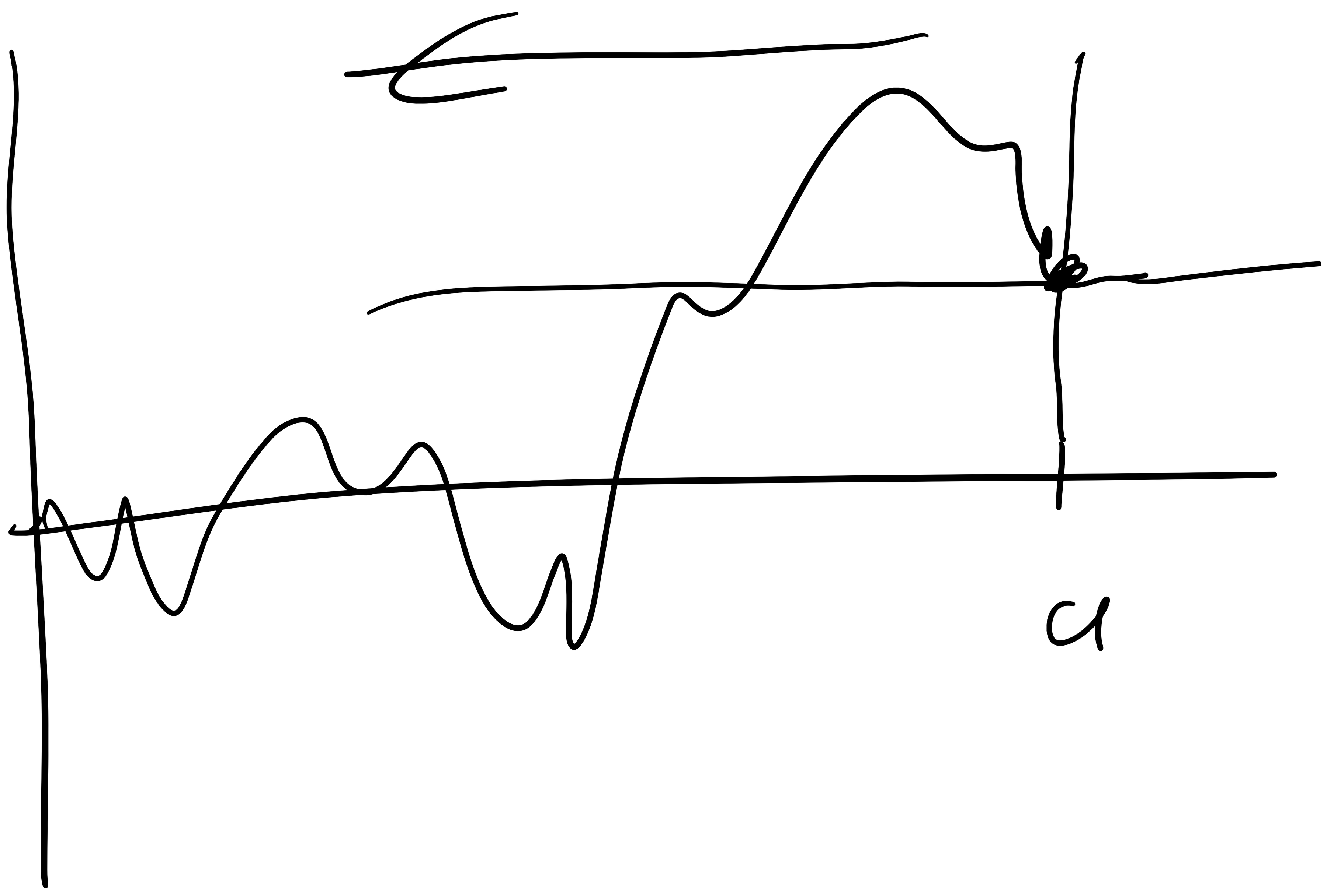
Άσκηση 5.7

$a>0$, B τ.κ.Β.

Τότε η $X(t) = B(a-t) - B(a)$ $\forall t\in[0,a]$

είναι τ.κ.Β στο $[0,a]$.

Λύση



$B(a-t)-B(a)$

i) Για $0 \le t_1 < \dots < t_n$

$X(t_1),\ X(t_2)-X(t_1),\ \dots,\ X(t_n)-X(t_{n-1})$

είναι

$B(a-t_1)-B(a),\ B(a-t_2)-B(a-t_1),\ \dots,$

$B(a-t_n)-B(a-t_{n-1})$ ← ανεξ.

$a-t_n,\ a-t_{n-1},\ \dots,\ a-t_2,\ a-t_1,\ a$

ii) Пи $0 \leq s < t$

$$X(t) - X(s) = B(a-t) - B(a-s)$$

$$= -\underbrace{(B(a-s) - B(a-t))}_{\sim N(0,\, t-s)}$$

$$\sim N(0,\, t-s)$$

$$N(0,\, (a-t)-(a-s)) = N(0,\, \overset{co}{s-t})$$

iii) $X(t) = B(a-t) - B(a)$ см. xij