Chapter 8

Laplace Transform and Applications

This final chapter gives an introduction to the Laplace transform and some of its applications. §8.1 introduces two key properties that make the Laplace transform useful for differential equations: First, it behaves well with respect to differentiation, and second, a function can be recovered if its Laplace transform is known. The closely related Fourier transform also enjoys these properties. It was discussed in §4.3; see also the Internet Supplement for this chapter. §8.2 develops techniques for inverting Laplace transforms, while §8.3 considers some applications of Laplace transforms to ordinary differential equations.

8.1 Basic Properties of Laplace Transforms

The Laplace transform provides a powerful technique used in both pure and applied mathematics. For example, in control theory it has been an indispensable tool.[1] It is important, therefore, to have a good grasp of both its basic theory and its usefulness. Consider a (real- or complex-valued) function $f(t)$ defined on $[0, \infty[$. The ***Laplace transform*** of f is defined to be the function $\tilde{f}$ of a complex variable z given by

$$\tilde{f}(z) = \int_0^\infty e^{-zt} f(t)dt.$$

The Laplace transform $\tilde{f}$ is defined for those $z \in \mathbb{C}$ for which the integral converges. Other common notations for $\tilde{f}$ are $\mathcal{L}(f)$ or simply F.

For technical reasons, it will be convenient to impose a mild restriction on the functions we consider. We require that $f : [0, \infty[\to \mathbb{C}$ (or $\mathbb{R}$) be of ***exponential***

[1]See, for example, J. C. Willems and J. W. Polderman, *An Introduction to Mathematical Systems Theory and Control: A Behavioral Approach* (New York: Springer-Verlag, Texts in Applied Mathematics, 1997).

order. This means that there are constants $A > 0, B \in \mathbb{R}$, such that

$$|f(t)| \leq Ae^{tB}$$

for all $t \geq 0$. In other words, f should not grow too fast; for example, any polynomial satisfies this condition (Why?). *All functions considered in the remainder of this chapter will be assumed to be of exponential order.* It will also be assumed that on any finite interval $[0, a]$, f is bounded and integrable. (If, for example, we assume that f is piecewise continuous, this last condition will hold.)

Abscissa of Convergence The first important result in this chapter concerns the nature of the set on which $\tilde{f}(z)$ is defined and is analytic.

Theorem 8.1.1 (Convergence Theorem for Laplace Transforms) *Assume that $f : [0, \infty[\to \mathbb{C}$ (or $\mathbb{R}$) is of exponential order and let*

$$\tilde{f}(z) = \int_0^\infty e^{-zt} f(t) dt.$$

There exists a unique number $\sigma, -\infty \leq \sigma < \infty$, such that this integral converges if $\operatorname{Re} z > \sigma$ and diverges if $\operatorname{Re} z < \sigma$. Furthermore, $\tilde{f}$ is analytic on the set

$$A = \{z \mid \operatorname{Re} z > \sigma\}$$

and we have

$$\frac{d}{dz}\tilde{f}(z) = -\int_0^\infty te^{-zt} f(t) dt$$

*for $\operatorname{Re} z > \sigma$. The number σ is called the **abscissa of convergence**, and if we define the number ρ by*

$$\rho = \inf\{B \in \mathbb{R} \mid \text{there exists an } A > 0 \text{ such that } |f(t)| \leq Ae^{Bt}\},$$

then $\sigma \leq \rho$.

The set $\{z \mid \operatorname{Re} z > \sigma\}$ is called the ***half-plane of convergence***. (If $\sigma = -\infty$, this set is all of $\mathbb{C}$.) See Figure 8.1.1. In general, it is difficult to tell whether $\tilde{f}(z)$ will converge for z on the vertical line $\operatorname{Re} z = \sigma$. If there is any danger of confusion we can write $\sigma(f)$ for σ or $\rho(f)$ for ρ. A convenient way to compute $\sigma(f)$ is described in Worked Examples 8.1.12 and 8.1.13.

The proof of this theorem and more detailed convergence results are given at the end of this section. The basic idea is that if $\operatorname{Re} z > \rho$, then A and B may be selected with $\rho < B < \operatorname{Re} z$ and $|f(t)| < Ae^{Bt}$. The improper integral for $\tilde{f}(z)$ converges by comparison with $\int_0^\infty Ae^{(B - \operatorname{Re} z)t}\, dt$.

The map $f \mapsto \tilde{f}$ is linear in the sense that $(af + bg)\tilde{} = a\tilde{f} + b\tilde{g}$, valid for $\operatorname{Re} z > \max[\sigma(f), \sigma(g)]$. It is also true that the map is one-to-one; that is, $\tilde{f} = \tilde{g}$ implies that $f = g$; in other words, a function $\phi(z)$ is the Laplace transform of at most one function.

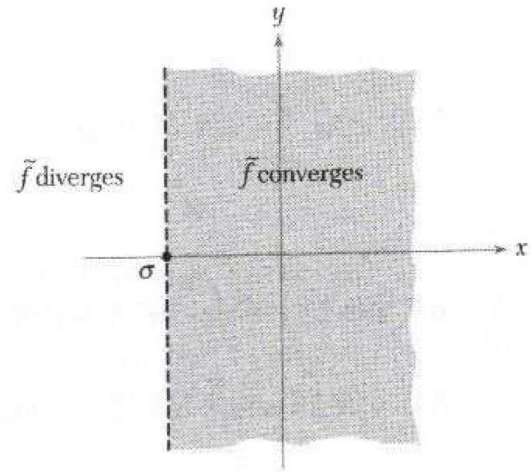

Figure 8.1.1: Half plane of convergence of the Laplace transform.

Theorem 8.1.2 (Laplace Transforms) *Suppose that the functions f and h are continuous and that $\tilde{f}(z) = \tilde{h}(z)$ for $\operatorname{Re} z > \gamma_0$ for some γ_0. Then $f(t) = h(t)$ for all $t \in [0, \infty[$* .

This theorem is not as simple as it seems. We do not have enough mathematical tools to give a complete proof, but the main ideas are given at the end of the section. Using ideas from integration theory, we could extend the result of the uniqueness theorem to discontinuous functions as well, but we would have to modify what we mean by "equality of functions." For example, if $f(t)$ is changed at a single value of t, then $\tilde{f}$ is unchanged.

The uniqueness theorem enables us to give a meaningful answer to the problem "Given $g(z)$, find $f(t)$ such that $\tilde{f} = g$," because it makes clear that there can be at most one such (continuous) f. We call f the ***inverse Laplace transform*** of g; methods for finding f when g is given are considered in §8.2.

Laplace Transforms of Derivatives The main utility of Laplace transforms is that they enable us to transform differential problems into algebraic problems. When the latter are solved, the answers to the original problems are obtained by using the inverse Laplace transform. The procedure is based on the following theorem.

Proposition 8.1.3 *Let $f(t)$ be continuous on $[0, \infty[$ and piecewise C^1, that is, piecewise continuously differentiable. Then for $\operatorname{Re} z > \rho$ (as defined in the convergence theorem (8.1.1)),*

$$\widetilde{\left(\frac{df}{dt}\right)}(z) = z\tilde{f}(z) - f(0).$$

Proof By definition,

$$\left(\frac{df}{dt}\right)\tilde{}(z) = \int_0^\infty e^{-zt}\frac{df}{dt}(t)dt.$$

Integrating by parts, we get

$$\lim_{t_0\to\infty}\left(e^{-zt}f(t)\big|_0^{t_0}\right) + \int_0^\infty ze^{-zt}f(t)dt.$$

By definition of $\rho, |e^{-Bt_0}\cdot f(t_0)| \le A$ for some $B < \operatorname{Re} z$. Thus, we get

$$|e^{-zt_0}\cdot f(t_0)| = |e^{-(z-B)t_0}||e^{-Bt_0}\cdot f(t_0)| \le e^{-(\operatorname{Re} z - B)t_0}A,$$

which approaches 0 as $t_0 \to \infty$. Therefore, we get $-f(0) + z\tilde{f}(z)$, as asserted. ■

While $(df/dt)\tilde{}(z)$ exists for $\operatorname{Re} z > \rho$, its abscissa of convergence might be smaller than ρ.

If we apply the preceding proposition to d^2f/dt^2, we obtain

$$\left(\frac{d^2f}{dt^2}\right)\tilde{}(z) = z^2\tilde{f}(z) - zf(0) - \frac{df}{dt}(0).$$

The formula for $d\tilde{f}/dz$ in the convergence theorem (8.1.1) is related to the formula

$$\tilde{g}(z) = d\tilde{f}(z)/dz, \quad \text{where} \quad g(t) = -tf(t).$$

In Exercise 19 the student is asked to prove the next proposition, which contains a similar formula for integrals.

Proposition 8.1.4 *Let* $g(t) = \int_0^t f(\tau)d\tau$. *Then for* $\operatorname{Re} z > \max[0, \rho(f)]$,

$$\tilde{g}(z) = \frac{\tilde{f}(z)}{z}.$$

Shifting Theorems Table 8.1.1 at the end of this section lists some formulas that are useful for computing $\tilde{f}(z)$. The proofs of these formulas are straightforward and are included in the exercises and examples. However, three of the formulas are sufficiently important to be given separate explanation, which is done in the following three theorems.

Theorem 8.1.5 (First Shifting Theorem) *Fix* $a \in \mathbb{C}$ *and let* $g(t) = e^{-at}f(t)$. *Then for* $\operatorname{Re} z > \sigma(f) - \operatorname{Re} a$, *we have*

$$\tilde{g}(z) = \tilde{f}(z+a).$$

Proof By definition,

$$\tilde{g}(z) = \int_0^\infty e^{-zt}e^{-at}f(t)dt = \int_0^\infty e^{-(z+a)t}f(t)dt = \tilde{f}(z+a),$$

which is valid if $\mathrm{Re}(z+a) > \sigma$. ■

Theorem 8.1.6 (Second Shifting Theorem) *Let $H(t) = 0$ if $t < 0$ and $H(t) = 1$ if $t \geq 0$, which is called the* ***Heaviside****, or* ***unit step, function****. Also, let $a \geq 0$ and let $g(t) = f(t-a)H(t-a)$; that is, $g(t) = 0$ if $t < a$ while $g(t) = f(t-a)$ if $t \geq a$. (See Figure 8.1.2.) Then for $\mathrm{Re}\, z > \sigma$, we have*

$$\tilde{g}(z) = e^{-az}\tilde{f}(z).$$

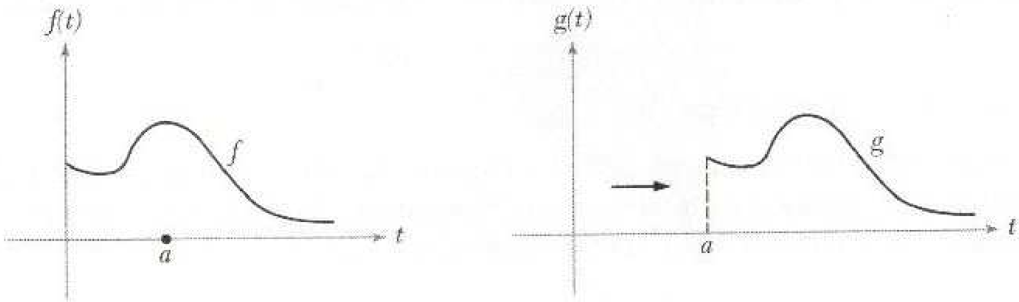

Figure 8.1.2: The function g in the second shifting theorem.

Proof By definition and because $g = 0$ for $0 \leq t < a$,

$$\tilde{g}(z) = \int_0^\infty e^{-zt}g(t)dt = \int_a^\infty e^{-zt}f(t-a)dt.$$

Letting $\tau = t - a$, we get

$$\tilde{g}(z) = \int_0^\infty e^{-z(\tau+a)}f(\tau)d\tau = e^{-za}\tilde{f}(z). \quad ■$$

From the second shifting theorem, we can deduce that if $a \geq 0$ and $g(t) = f(t)H(t-a)$, then $\tilde{g}(z) = e^{-az}\tilde{F}(z)$ where $F(t) = f(t+a), t \geq 0$ (see Figure 8.1.3).

Convolutions The ***convolution*** of two functions $f(t)$ and $g(t)$ is defined for $t \geq 0$ by

$$(f * g)(t) = \int_0^\infty f(t-\tau)\cdot g(\tau)d\tau$$

where we set $f(t) = 0$ if $t < 0$. Thus, the integration is really only from 0 to t. The convolution operation is related to Laplace transforms in the following way.

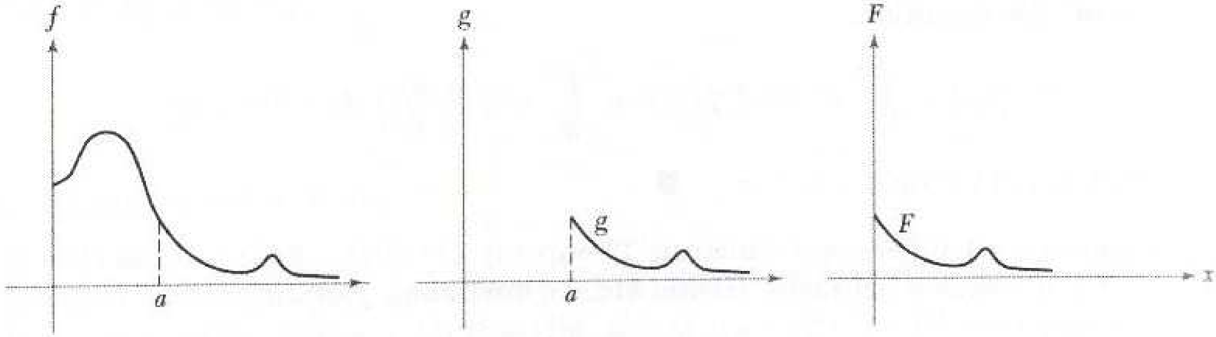

Figure 8.1.3: F is obtained from f by shifting and truncating.

Theorem 8.1.7 (Convolution Theorem) *The equalities* $f * g = g * f$ *and*

$$(f * g)\tilde{}(z) = \tilde{f}(z) \cdot \tilde{g}(z)$$

whenever $\operatorname{Re} z > \max[\rho(f), \rho(g)]$.

In brief, this theorem states that the *Laplace transform of a convolution of two functions is the product of their Laplace transforms*. It is precisely this property that makes the convolution an operation of interest to us.

Proof We have

$$\begin{aligned}(f * g)\tilde{}(z) &= \int_0^\infty e^{-zt} \left[\int_0^\infty f(t-\tau) \cdot g(\tau) d\tau\right] dt \\ &= \int_0^\infty \left[\int_0^\infty e^{-z\tau} e^{-z(t-\tau)} f(t-\tau) g(\tau) d\tau\right] dt.\end{aligned}$$

For $\operatorname{Re} z > \max[\rho(f), \rho(g)]$ the integrals for $\tilde{f}(z)$ and $\tilde{g}(z)$ converge absolutely, so we can interchange the order of integration[2] to obtain

$$\int_0^\infty e^{-z\tau} \left[\int_0^\infty e^{-z(t-\tau)} f(t-\tau) dt\right] g(\tau) d\tau.$$

Letting $s = t - \tau$ and remembering that $f(s) = 0$ if $s < 0$, we get

$$\int_0^\infty e^{-z\tau} \tilde{f}(z) g(\tau) d\tau = \tilde{f}(z) \cdot \tilde{g}(z). \quad \blacksquare$$

By changing variables, it is not difficult to verify that $f * g = g * f$, but such verification also follows from what we have done if f and g are continuous. We have

$$(f * g)\tilde{} = \tilde{f} \cdot \tilde{g} = \tilde{g} \cdot \tilde{f} = (g * f)\tilde{}.$$

Thus, $(f * g - g * f)\tilde{} = 0$, so by uniqueness theorem (8.1.2), $f * g - g * f = 0$.

[2]This is a theorem concerning integration theory from advanced calculus. See, for instance, J. Marsden and M. Hoffman, *Elementary Classical Analysis*, Second Edition (New York: W. H. Freeman and Company, 1993), Chapter 9.

Technical Proofs of Theorems To prove the convergence theorem (8.1.1), we shall use the following important result.

Lemma 8.1.8 *Suppose that $\tilde{f}(z) = \int_0^\infty e^{-zt} f(t)dt$ converges for $z = z_0$. Assume that $0 \le \theta < \pi/2$ and define the set*

$$S_\theta = \{z \text{ such that } |\arg(z - z_0)| \le \theta\}$$

(see Figure 8.1.4). *Then $\tilde{f}$ converges uniformly on S_θ.*

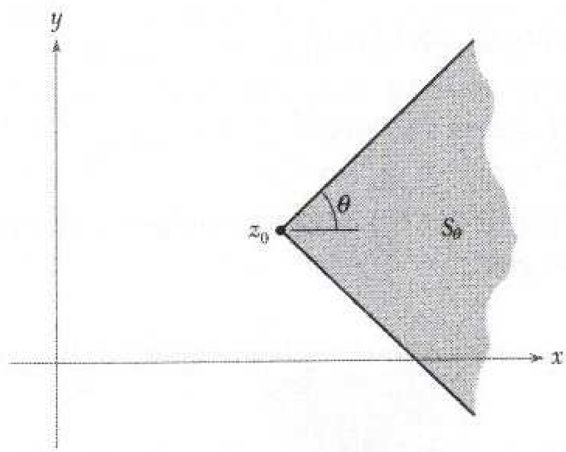

Figure 8.1.4: Sector of uniform convergence.

Proof Let

$$h(x) = \int_0^x e^{-z_0 t} f(t)dt - \int_0^\infty e^{-z_0 t} f(t)dt$$

so that $h \to 0$ as $x \to \infty$. We must show that for every $\epsilon > 0$, there is a t_0 such that $t_1, t_2 \ge t_0$ implies that

$$\left| \int_{t_1}^{t_2} e^{-zt} f(t)dt \right| < \epsilon$$

for all $z \in S_\theta$. It follows that $\int_0^x e^{-zt} f(t)dt$ converges uniformly on S_θ as $x \to \infty$, by the Cauchy Criterion. We will make use of the function $h(x)$ as follows. Write

$$\int_{t_1}^{t_2} e^{-zt} f(t)dt = \int_{t_1}^{t_2} e^{-(z-z_0)t}[e^{-z_0 t} f(t)]dt.$$

Integrating by parts, we get

$$e^{-(z-z_0)t_2} h(t_2) - e^{-(z-z_0)t_1} h(t_1) + (z - z_0) \int_{t_1}^{t_2} e^{-(z-z_0)t} h(t)dt.$$

Given $\epsilon > 0$, choose t_0 such that $|h(t)| < \epsilon/3$ and $|h(t)| < \epsilon' = \epsilon/(6\sec\theta)$ if $t \geq t_0$. Then for $t_2 > t_0$,

$$|e^{-(z-z_0)t_2}h(t_2)| \leq |h(t_2)| < \frac{\epsilon}{3},$$

since $|e^{-(z-z_0)t_2}| = e^{-(\operatorname{Re} z - \operatorname{Re} z_0)t_2} \leq 1$ because $\operatorname{Re} z > \operatorname{Re} z_0$. Similarly, for $t_1 > t_0$,

$$|e^{-(z-z_0)t_1}h(t_1)| < \frac{\epsilon}{3}.$$

We must still estimate the last term:

$$\left|(z - z_0)\int_{t_1}^{t_2} e^{-(z-z_0)t}h(t)dt\right| \leq |z - z_0|\epsilon' \int_{t_1}^{t_2} e^{-(x-x_0)t}dt,$$

where $x = \operatorname{Re} z$ and $x_0 = \operatorname{Re} z_0$. If $z = z_0$, this term is zero. If $z \neq z_0$, then $x \neq x_0$ (see the figure), and we get

$$\epsilon' \frac{|z - z_0|}{x - x_0}\left(e^{-(x-x_0)t_1} - e^{-(x-x_0)t_2}\right) < 2\epsilon' \frac{|z - z_0|}{x - x_0} \leq 2\epsilon' \sec\theta = \frac{\epsilon}{3}$$

(see **Figure** 8.1.5). Note that the restriction $0 \leq \theta < \pi/2$ is necessary for $\sec\theta = 1/\cos\theta$ to be finite.

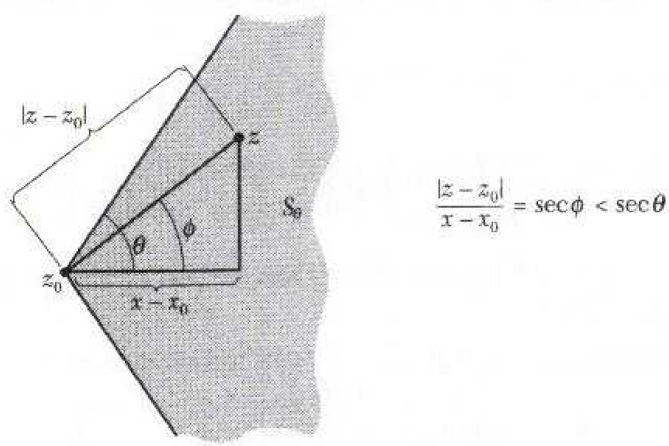

Figure 8.1.5: Some geometry in the region S_θ.

Combining the preceding inequalities, we get

$$\left|\int_{t_1}^{t_2} e^{-zt}f(t)dt\right| < \epsilon$$

if $t_1, t_2 \geq t_0$ for all $z \in S_\theta$, thus completing the proof of the lemma. ▼

Proof of the Convergence Theorem 8.1.1 Let

$$\sigma = \inf\left\{x \in \mathbb{R} \,\middle|\, \int_0^\infty e^{-\tau} f(t)dt \text{ converges}\right\},$$

where inf stands for "greatest lower bound." Note from Lemma 8.1.8 that if $\tilde{f}(z_0)$ converges, then, for $\operatorname{Re} z > \operatorname{Re} z_0, \tilde{f}(z)$ converges because z lies in some S_θ for z_0 (Why?).

Let $\operatorname{Re} z > \sigma$. By the definition of σ, there is an $x_0 < \operatorname{Re} z$ such that the integral $\int_0^\infty e^{-x_0 t} f(t)dt$ converges. Hence $\tilde{f}(z)$ converges by Lemma 8.1.8. Conversely, assume $\operatorname{Re} z < \sigma$ and $\operatorname{Re} z < x < \sigma$. If $\tilde{f}(z)$ converges, then so does $\tilde{f}(x)$, and therefore $\sigma \leq x$ gives a contradiction. Thus $\tilde{f}(z)$ does not converges if $\operatorname{Re} z < \sigma$.

We use the Analytic Convergence Theorem 3.1.8, to show that $\tilde{f}$ is analytic on the set $\{z \mid \operatorname{Re} z > \sigma\}$. Let $g_n(z) = \int_0^n e^{-zt} f(t)dt$. Then $g_n(z) \to \tilde{f}(z)$. By Worked Example 2.4.15, g_n is analytic with $g_n'(z) = -\int_0^n te^{-zt} f(t)dt$. We must show that $g_n \to \tilde{f}$ uniformly on closed disks in $\{z \mid \operatorname{Re} z > \sigma\}$. But each disk lies in some S_θ relative to some z_0 with $\operatorname{Re} z_0 > \sigma$ (Figure 8.1.6).

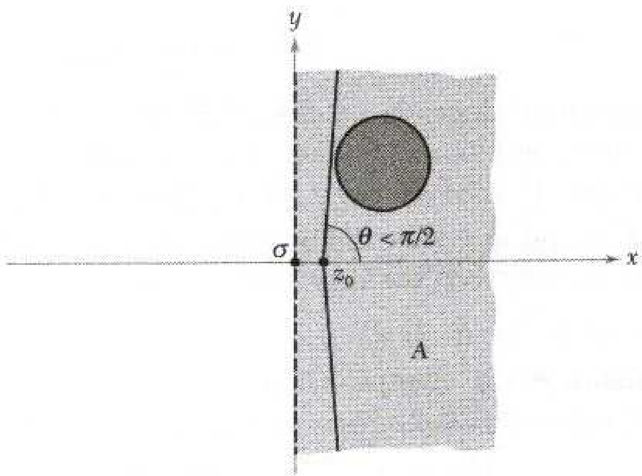

Figure 8.1.6: Each disk lies in S_θ for some $\theta, 0 \leq \theta < \pi/2$.

Thus, by the Analytic Convergence Theorem 3.1.8, $\tilde{f}$ is analytic on the set $\{z \mid \operatorname{Re} z > \sigma\}$ and

$$(\tilde{f})'(z) = -\int_0^\infty te^{-zt} f(t)dt.$$

It follows that this integral representation for the derivative of $\tilde{f}$ converges for $\operatorname{Re} z > \sigma$, as do all the iterated derivatives.

It remains to be shown that $\sigma \leq \rho$. To prove this we need to show that $\sigma \leq B$ if $|f(t)| \leq Ae^{Bt}$. This will hold, by what we have proven, if $\tilde{f}(z)$ converges for $\operatorname{Re} z > B$. Indeed, we show absolute convergence. Note that

$$|e^{-zt} f(t)| = |e^{-(z-B)t} e^{-Bt} f(t)| \leq e^{-(\operatorname{Re} z - B)t} A.$$

Since the integral $\int_0^\infty e^{-\alpha t}dt = 1/\alpha$ converges for $\alpha > 0$, it follows that the integral $\int_0^\infty e^{-zt}f(t)dt$ converges absolutely. ■

To prove that $\tilde{f} = \tilde{h}$ implies that $f = h$ for continuous functions f and h, it suffices, by considering $f - h$, to prove the following special case of Theorem 8.1.2.

Proposition 8.1.9 *Suppose that f is continuous and that for some real y_0, $\tilde{f}(z) = 0$ whenever $\operatorname{Re} z > y_0$. Then $f(t) = 0$ for all $t \in [0, \infty[$.*

The crucial lemma we use to prove this is the following.

Lemma 8.1.10 *Let f be continuous on $[0, 1]$ and suppose that $\int_0^1 t^n f(t)dt = 0$ for all $n = 0, 1, 2, \ldots$. Then $f = 0$.*

This assertion is reasonable since it follows that $\int_0^1 P(t)f(t)dt = 0$ for any polynomial P.

Proof The precise proof depends on the *Weierstrass approximation theorem*, which states that any continuous function is the uniform limit of polynomials.[3] By this theorem we get $\int_0^1 g(t)f(t)dt = 0$ for any continuous g. The result follows by taking $g(t) = \overline{f(t)}$ and applying the fact that if the integral of a nonnegative continuous function is zero, then the function is zero. ▼

Proof of Proposition 8.1.9 Suppose that

$$\tilde{f}(z) = \int_0^\infty e^{-zt}f(t)dt = 0$$

whenever $\operatorname{Re} z > \sigma$. Fix $x_0 > y_0$ real and let $s = e^{-t}$. By changing variables to express the integrals in terms of s and letting $z = x_0 + n$ for $n = 0, 1, 2, \ldots$, we get

$$0 = \int_0^\infty e^{-nt}e^{-x_0 t}f(t)dt = \int_1^0 s^n s^{x_0} f(-\log s)\left(-\frac{1}{s}\right) ds = \int_0^1 s^n h(s)ds = 0,$$

where $h(s) = s^{x_0 - 1}f(-\log s) = e^{-x_0 t + t}f(t)$. By the Lemma, h must be identically zero, and f must be also since the exponential function is never zero. ■

It is useful to note that $\tilde{f}(z) \to 0$ as $\operatorname{Re} z \to \infty$. This follows from the argument used to prove Theorem 8.1.1 (see Review Exercise 10).

[3]See, for example, J. Marsden and M. Hoffman, *Elementary Classical Analysis*, Second Edition (New York: W. H. Freeman and Company, 1993), Chapter 5.

Table 8.1.1 Some Common Laplace Transforms

Definition

$$\tilde{f}(z) = \int_0^\infty e^{-zt} f(t)dt$$

Properties

1. $\tilde{g}(z) = -\frac{d}{dz}\tilde{f}(z)$ where $g(t) = tf(t)$.
2. $(af + bg)\tilde{} = a\tilde{f} + b\tilde{g}$.
3. $\left(\frac{df}{dt}\right)\tilde{}(z) = z\tilde{f}(z) - f(0)$. (Assume that f is piecewise C^1.)
4. $\tilde{g}(z) = \frac{1}{z}\tilde{f}(z)$ where $g(t) = \int_0^t f(\tau)d\tau$.
5. $\tilde{g}(z) = \tilde{f}(z+a)$ where $g(t) = e^{-at}f(t)$.
6. $\tilde{g}(z) = e^{-az}\tilde{f}(z)$, where $a > 0$, and
$$g(t) = f(t-a) \quad \text{for} \quad t \geq a \quad \text{and} \quad 0 \quad \text{if} \quad t < a.$$
7. $\tilde{g}(z) = e^{-az}\tilde{F}(z)$, where $a \geq 0, F(t) = f(t+a)$, and
$$g(t) = f(t) \quad \text{if} \quad t \geq a \quad \text{and} \quad 0 \quad \text{if} \quad 0 \leq t < a.$$
8. $(f * g)\tilde{}(z) = \tilde{f}(z) \cdot \tilde{g}(z)$, where the ***convolution*** is defined by
$$(f * g)(t) = \int_0^\infty f(t-\tau)g(\tau)d\tau.$$
9. If $f(t) = e^{-at}$, then $\tilde{f}(z) = \frac{1}{z+a}$ and $\sigma(f) = -\operatorname{Re} a$.
10. For $f(t) = \cos at, \tilde{f}(z) = \frac{z}{z^2+a^2}$ and $\sigma(f) = |\operatorname{Im} a|$.
11. If $f(t) = \sin at, \tilde{f}(z) = \frac{a}{z^2+a^2}$ and $\sigma(f) = |\operatorname{Im} a|$.
12. If $f(t) = t^a, a > -1, \tilde{f}(z) = \frac{\Gamma(a+1)}{z^{a+1}}$ and $\sigma(f) = 0$.
13. If $f(t) = 1, \tilde{f}(z) = \frac{1}{z}$ and $\sigma(f) = 0$.

Worked Examples

Example 8.1.11 *Prove formula 9 in Table 8.1.1 and find $\sigma(f)$ in that case.*

Solution By definition,

$$\tilde{f}(z) = \int_0^\infty e^{-at}e^{-zt}dt = \int_0^\infty e^{-(a+z)t}dt = -\left.\frac{e^{-(a+z)t}}{a+z}\right|_0^\infty = \frac{1}{z+a}.$$

The evaluation at $t = \infty$ is justified by noting that $\lim_{t\to\infty} e^{-(a+z)t} = 0$ provided $\operatorname{Re}(a+z) > 0$, since $|e^{-(a+z)t}| = e^{-\operatorname{Re}(a+z)t} \to 0$ as $t \to \infty$. Thus, the formula is valid if $\operatorname{Re} z > -\operatorname{Re} a$.

Note that the formula for $\tilde{f}$ is valid only for $\operatorname{Re} z > -\operatorname{Re} a$, although $\tilde{f}$ coincides there with a function that is analytic except at $z = -a$. This situation is similar to that for the gamma function (see formula 12 of Table 7.1.1).

Finally, we show that for $f(t) = e^{-at}, \sigma(f) = -\operatorname{Re} a$. We have already shown that $\sigma(f) \le -\operatorname{Re} a$. But the integral diverges at $z = a$, so $\sigma(f) \ge -\operatorname{Re} a$, and thus $\sigma(f) = -\operatorname{Re} a$. If $a = 0$, this example specializes to formula 13 of Table 8.1.1.

Example 8.1.12 *Suppose that we have computed $\tilde{f}(z)$ and found it to converge for $\operatorname{Re} z > \gamma$. Suppose also that $\tilde{f}$ coincides with an analytic function that has a pole on the line $\operatorname{Re} z = \gamma$. Show that $\sigma(f) = \gamma$.*

Solution We know that $\sigma(f) \le \gamma$ by the basic property of σ in the convergence theorem. Also, since $\tilde{f}$ is analytic for $\operatorname{Re} z > \sigma$, there can be no poles in the region $\{z \mid \operatorname{Re} z > \sigma\}$. If $\sigma(f)$ were $< \gamma$, there would be a pole in this region. Hence $\sigma(f) = \gamma$ (see Figure 8.1.7).

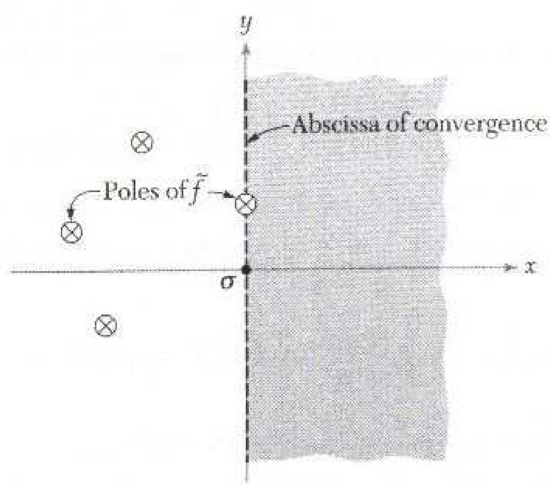

Figure 8.1.7: Location of poles of $\tilde{f}$.

Example 8.1.13 *Let $f(t) = \cosh t$. Compute $\tilde{f}$ and $\sigma(f)$.*

Solution $f(t) = \cosh t = (e^t + e^{-t})/2$. Thus, by formulas 2 and 9 of Table 8.1.1,

$$\tilde{f}(z) = \frac{1}{2}\left(\frac{1}{z-1} + \frac{1}{z+1}\right) = \frac{z}{z^2-1}.$$

Here $\sigma(f) = 1$ by Worked Example 8.1.12; $\sigma(e^t) = 1$ and $\sigma(e^{-t}) = -1$, so $\sigma(f) \leq 1$ but it cannot be < 1 since $\tilde{f}$ has a pole at $z = 1$.

Exercises

In Exercises 1 through 9, compute the Laplace transform of $f(t)$ and find the abscissa of convergence.

1. $f(t) = t^2 + 2$
2. $f(t) = \sinh t$
3. $f(t) = t + e^{-t} + \sin t$
4.• $f(t) = \begin{cases} 0 & 0 \leq t \leq 1 \\ 1 & 1 < t < 2 \\ 0 & t \geq 2 \end{cases}$
5. $f(t) = (t+1)^n$, n a positive integer
6. $f(t) = \sin t$ if $0 \leq t \leq \pi$ and 0 if $t > \pi$
7. $f(t) = t \sin at$
8. $f(t) = t \sinh at$
9. $f(t) = t \cos at$
10. Use the shifting theorems to show the following:

    (a) If $f(t) = e^{-at} \cos bt$, then

    $$\tilde{f}(z) = \frac{z+a}{(z+a)^2 + b^2}.$$

    (b) If $f(t) = e^{-at} t^n$, then

    $$\tilde{f}(z) = \frac{\Gamma(n+1)}{(z+a)^{n+1}}.$$

    What is $\sigma(f)$ in each case?

11. Prove formula 10 of Table 8.1.1.
12. Prove formula 11 of Table 8.1.1.

13.• Prove formula 12 of Table 8.1.1.

14. Prove formula 13 of Table 8.1.1.

15.• Suppose that f is periodic with period p (that is, $f(t+p) = f(t)$ for all $t \geq 0$). Prove that

$$\tilde{f}(z) = \frac{\int_0^p e^{-zt} f(t)dt}{1 - e^{-pz}}$$

is valid if $\operatorname{Re} z > 0$. *Hint*: Write out $\tilde{f}(z)$ as an infinite sum.

16. Use Exercise 15 to prove that

$$\tilde{f}(z) = \frac{1}{z} \cdot \frac{1 - e^{-z}}{1 - e^{-2z}}$$

where $f(t)$ is the ***pulse function*** illustrated in Figure 8.1.8.

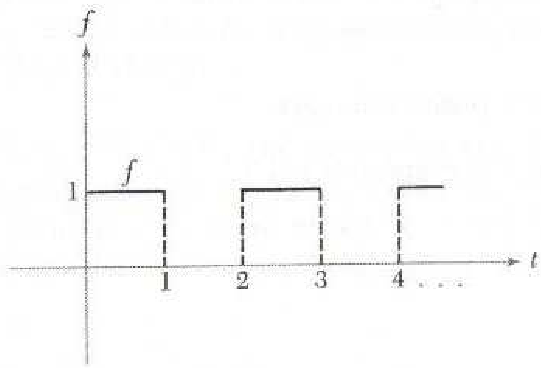

Figure 8.1.8: The unit pulse function.

17. Let $g(t) = \int_0^t e^{-s} \sin s\, ds$. Compute $\tilde{g}(z)$. Compute $\tilde{f}(z)$ if $f(t) = tg(t)$.

18. Let $f(t) = (\sin at)/t$. Show that $\tilde{f}(z) = \tan^{-1}(a/z)$.

19.• Prove Proposition 8.1.4. First establish that $\rho(g) \leq \max[0, \rho(f)]$.

20. Give a direct proof that $f * g = g * f$ (see the Convolution Theorem 8.1.7).

21.• Let $f(t) = e^{-e^t}, t \geq 0$. Show that $\sigma(f) = -\infty$.

22. Referring to the Convergence Theorem 8.1.1, show that, in general, $\sigma \neq \rho$. *Hint*: Consider $f(t) = e^t \sin e^t$ and show that $\sigma = 0, \rho = 1$.

8.2 Complex Inversion Formula

To be able to recover a function from its Laplace transform, it is important to be able to compute $f(t)$ when $\tilde{f}(z)$ is known. One technique for such a computation, using the *complex inversion formula*, will be established in this section. Using the formulas of Table 8.1.1 in reverse gives useful alternative techniques. (See Worked Examples 8.2.4 and 8.2.5.)

Main Inversion Formula The complex inversion formula, one of the key results for the Laplace transform, draws on many of the main points developed in the first four chapters of this book.

Theorem 8.2.1 *Suppose that $F(z)$ is analytic on $\mathbb{C}$ except for a finite number of isolated singularities and that for some real number σ, F is analytic on the half plane $\{z \mid \operatorname{Re} z > \sigma\}$. Suppose also that there are positive constants M, R, and β such that $|F(z)| \le M/|z|^{\beta}$ whenever $|z| \ge R$ (this is true, for example, if $F(z) = P(z)/Q(z)$ for polynomials P and Q with $\deg(Q) \ge 1 + \deg(P)$). For $t \ge 0$, let*

$$f(t) = \sum \{\text{residues of } e^{zt}F(z) \text{ at each of its singularities in } \mathbb{C}\}.$$

Then $\tilde{f}(z) = F(z)$ for $\operatorname{Re} z > \sigma$. We call this the ***complex inversion formula****.*

Proof Let $\alpha > \sigma$ and consider a large rectangle Γ with sides along the lines $\operatorname{Re} z = -x_1, \operatorname{Re} z = x_2, \operatorname{Im} z = y_2$, and $\operatorname{Im} z = -y_1$ selected large enough so that all the singularities of F are inside Γ and $|z| > R$ everywhere on Γ. Split Γ into a sum of two rectangular paths γ and $\tilde{\gamma}$ by a vertical line through $\operatorname{Re} z = \alpha$. (See Figure 8.2.1.)

The proof of the complex inversion formula could just as well be carried out using a large circle instead of the rectangle Γ. In fact, in the last paragraph of the proof, Γ is briefly deformed to such a circle. However, the rectangular path will be useful in Corollary 8.2.2, in which it plays a role like that of the rectangular path in the proof of Proposition 4.3.9 concerning the evaluation of Fourier transforms.

Since all singularities of F are inside γ, the definition of f gives

$$\int_{\gamma} e^{zt}F(z)dz = 2\pi i \sum \{\text{residues of } e^{zt}F(z)\} = 2\pi i f(t),$$

so

$$2\pi i \tilde{f}(z) = \lim_{r\to\infty} \int_0^r e^{-zt} \left[\int_{\gamma} e^{\zeta t}F(\zeta)d\zeta\right] dt = \lim_{r\to\infty} \int_{\gamma}\int_0^r e^{(\zeta - z)t}F(\zeta)dt\, d\zeta.$$

We may interchange the order of integration, because both integrals are over finite intervals. Therefore,

$$2\pi i \tilde{f}(z) = \lim_{r\to\infty} \int_{\gamma} \left(e^{(\zeta-z)r} - 1\right) \frac{F(\zeta)}{\zeta - z} d\zeta.$$

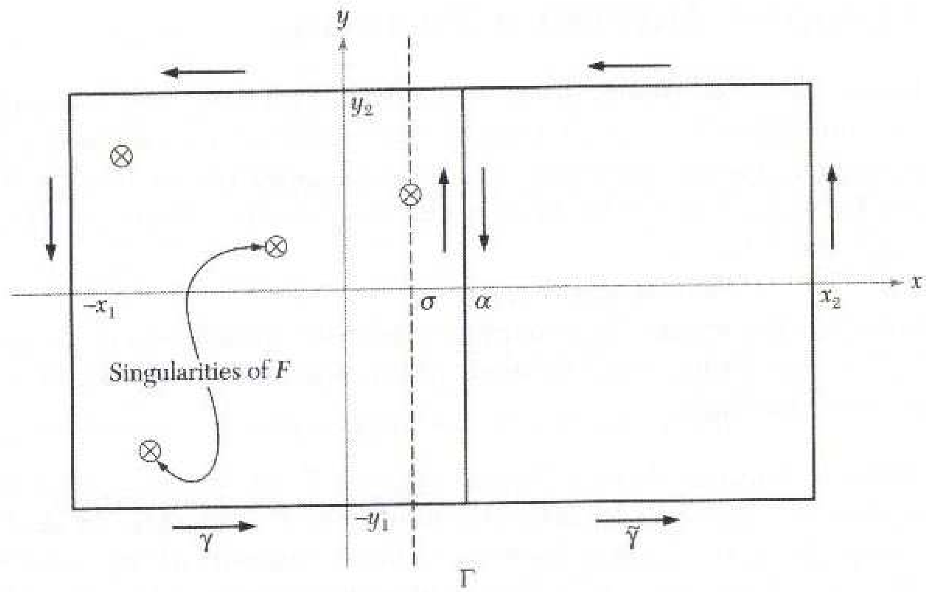

Figure 8.2.1: The large contour is the sum of the smaller two: $\Gamma = \gamma + \tilde{\gamma}$.

With z fixed in the half plane $\operatorname{Re} z > \alpha$, the term $e^{(\zeta - z)r}$ approaches 0 and the integrand converges uniformly to $-F(\zeta)/(\zeta - z)$ on γ. We obtain

$$\begin{aligned} 2\pi i \tilde{f}(z) &= -\int_\gamma \frac{F(\zeta)}{\zeta - z} d\zeta = \int_{\tilde{\gamma}} \frac{F(\zeta)}{\zeta - z} d\zeta - \int_\Gamma \frac{F(\zeta)}{\zeta - z} d\zeta \\ &= 2\pi i F(z) - \int_\Gamma \frac{F(\zeta)}{\zeta - z} d\zeta \end{aligned}$$

provided Γ is large enough so that z is inside $\tilde{\gamma}$. Finally,

$$\left| \int_\Gamma \frac{F(\zeta)}{\zeta - z} d\zeta \right| \le \int_\Gamma \frac{M}{|z|^\beta |\zeta - z|} |d\zeta| \le \frac{2\pi M \rho}{\rho^\beta (\rho - R)},$$

which is obtained by choosing Γ large enough so that it lies outside the circle $|\zeta| = \rho > R$ with all the singularities of $F(\zeta)/(\zeta - z)$ inside this circle, and then deforming Γ to this circle. This last expression goes to 0 as $\rho \to \infty$. Thus, letting Γ expand outward toward ∞, we obtain $\tilde{f}(z) = F(z)$. Since $\alpha > \sigma$ is arbitrary, the complex inversion formula holds for any z in the half plane $\operatorname{Re} z > \sigma$. ■

Corollary 8.2.2 *Let the conditions of the complex inversion formula hold. If $F(z)$ is analytic for $\operatorname{Re} z > \sigma$ and has a singularity on the line $\operatorname{Re} z = \sigma$, then* (i) *the abscissa of convergence of f is σ, and* (ii)

$$f(t) = \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} e^{zt} F(z) dz = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{(\alpha + iy)t} F(\alpha + iy) dy$$

for any constant $\alpha > \sigma$. The first integral is taken along the vertical line $\operatorname{Re} z = \alpha$ and converges as an improper Riemann integral; the second integral is used as alternative notation for the first.

Proof

(i) The complex inversion formula shows that $\sigma(f) \leq \sigma$ since $\tilde{f}(z)$ converges for $\operatorname{Re} z > \sigma$. If $\sigma(f)$ were $< \sigma$, then $\tilde{f}(z)$ would be analytic for $\operatorname{Re} z > \sigma(f)$ by the Convergence Theorem 8.1.1. But F has a singularity at a point z_0 on the line $\operatorname{Re} z = \sigma$, so there is a sequence of points $z_1, z_2, z_3, \ldots$ converging to z_0 with $F(z_n) \to \infty$. Since $\tilde{f}(z) = F(z)$ for $\operatorname{Re} z > \sigma$, and since both are analytic in a deleted neighborhood of z_0, they would be equal in that deleted neighborhood by the principle of analytic continuation. This would mean that $\tilde{f}(z_n) \to \infty$. But that is impossible, since $\tilde{f}(z)$ is analytic on $\operatorname{Re} z > \sigma(f)$. Thus, $\sigma(f) < \sigma$ is not possible, so $\sigma(f) = \sigma$.

(ii) From the complex inversion formula, $2\pi i f(t) = \int_\gamma e^{zt} F(z) dz$. This integral converges to the integral in the statement, exactly as in the proof of Proposition 4.3.9, as x_1, y_1, and $y_2 \to \infty$. Since y_1 and y_2 go independently to ∞, this establishes convergence of the improper integral. (The situation here is rotated by $90°$ from that of Proposition 4.3.9.) ■

In working examples, all conditions of the theorem must be checked. If they do not hold, these formulas for $f(t)$ may not be valid (see Example 8.2.5). The complex inversion formula is sometimes more convenient than Table 8.1.1 for computing inverse Laplace transforms since it is systematic and requires no guesswork as to which formula is appropriate. However, the table may be useful in cases in which hypotheses of the theorem do not apply or are inconvenient to check.

Heaviside Expansion Theorem Now we apply the complex inversion formula to the case in which $F(z) = P(z)/Q(z)$ where P and Q are polynomials. We give a simple case here.

Theorem 8.2.3 *Let $P(z)$ and $Q(z)$ be polynomials with* $\deg Q \geq \deg P + 1$. *Suppose that the zeros of Q are located at the points $z_1, \ldots, z_m$ and are simple zeros. Then the inverse Laplace transform of $F(z) = P(z)/Q(z)$ is given by the* ***Heaviside expansion formula:***

$$f(t) = \sum_{i=1}^{m} e^{z_i t} \frac{P(z_i)}{Q'(z_i)}.$$

Furthermore, $\sigma(f) = \max\{\operatorname{Re} z_i \mid i = 1, 2, \ldots, m\}$.

Proof Since $\deg Q \geq \deg P + 1$, the conditions of the complex inversion formula (8.2.1) are met (compare Proposition 4.3.9). Thus,

$$f(t) = \sum \left\{ \text{residues of } e^{zt} \frac{P(z)}{Q(z)} \right\}.$$

But the poles are all simple and so, by formula 4 of Table 4.1.1, we have

$$\text{Res}\left(e^{zt}\frac{P(z)}{Q(z)}; z_i\right) = e^{z_i t}\frac{P(z_i)}{Q'(z_i)}.$$

The formula for $\sigma(f)$ is a consequence of Corollary 8.2.2. ■

Worked Examples

Example 8.2.4 *If* $\tilde{f}(z) = 1/(z-3)$, *find* $f(t)$.

Solution Refer to formula 9 of Table 8.1.1. Let $a = -3$; then we get $f(t) = e^{3t}$. Alternatively, we could get the same result by using the Heaviside expansion formula. In this example, $\sigma(f) = 3$.

Example 8.2.5 *If* $\tilde{f}(z) = \log(z^2 + z)$, *what is* $f(t)$*?*

Solution If f were such a function and $g(t) = tf(t)$, then by formula 1 of Table 8.1.1, we would have

$$\tilde{g}(z) = -\frac{d}{dz}\tilde{f}(z) = -\frac{d}{dz}\log(z^2+z) = -\frac{2z+1}{z^2+z}.$$

To find $g(t)$ we could use partial fractions.

$$\tilde{g}(z) = -\frac{2z+1}{z^2+z} = -\frac{1}{z} - \frac{1}{z+1}.$$

Therefore $g(t) = -1 - e^{-t}$, and so

$$f(t) = -\frac{1}{t}(1 + e^{-t}).$$

Although this argument seems satisfactory, it is deceptive because *there is in fact no* $f(t)$ *whose Laplace transform is* $\log(z^2+z)$. If there were, then this procedure would show that $f(t) = -(1+e^{-t})/t$ is the only possibility. However, the integral

$$\int_0^\infty e^{-xt} f(t)\, dt$$

cannot converge for any real x because e^{-xt} is larger than $1/2$ near 0 and $|f(t)| \geq 1/t$. But $1/t$ is not integrable. Thus f does not exist in any sense we have studied. The argument above does not actually find such an f. It assumes that there is one and shows that there is only one possibility. But that one does not work. See also the remark at the end of §8.1.

Example 8.2.6 *Compute the inverse Laplace transform of*

$$F(z) = \frac{z}{(z+1)^2(z^2+3z-10)}.$$

Then compute $\sigma(f)$, *the abscissa of convergence of* f.

Solution In this case the hypotheses of the complex inversion formula clearly hold. Thus

$$f(t) = \sum \left\{ \text{residues of } \frac{e^{zt}z}{(z+1)^2(z^2+3z-10)} = \frac{e^{zt}z}{(z+1)^2(z+5)(z-2)} \right\}.$$

The poles are at $z=-1$, $z=-5$, and $z=2$. The pole at -1 is double, whereas the others are simple. By formula 7 of Table 4.1.1, the residue at -1 is $g'(-1)$, where

$$g(z) = \frac{e^{zt}z}{z^2+3z-10}.$$

Thus, we obtain

$$\frac{-te^{-t}}{-12} + \frac{e^{-t}}{-12} - \frac{(-e^{-t}) \cdot [2 \cdot (-1) + 3]}{144} = \frac{1}{12}\left(te^{-t} - e^{-t} + \frac{e^{-t}}{12}\right).$$

The residue at -5 is $e^{-5t} \cdot 5/16 \cdot 7$; the residue at 2 is $e^{2t} \cdot 2/9 \cdot 7$. Thus,

$$f(t) = \frac{1}{12}\left(te^{-t} - e^{-t} + \frac{e^{-t}}{12}\right) + \frac{5e^{-5t}}{16 \cdot 7} + \frac{2e^{2t}}{63}.$$

By Corollary 8.2.2, $\sigma(f) = 2$.

Exercises

1. Compute the inverse Laplace transform of each of the following.

   (a) $F(z) = \dfrac{z}{z^2+1}$.

   (b) $F(z) = \dfrac{1}{(z+1)^2}$.

   (c) $F(z) = \dfrac{z^2}{z^3-1}$.

2. Check formulas 10 and 11 of Table 8.1.1 using Theorem 8.2.1.

3. Explain what is wrong with the following reasoning. Let $g(t)=0$ on $[0,1[$ and be 1 on $[1,\infty)$. Then, by formulas 6 and 13 of Table 8.1.1, $\tilde{g}(z) = e^{-z}/z$. By the complex inversion formula, $g(t) = \text{Res}(e^{z(t-1)}/z; 0) = 1$. Therefore, $1=0$.

4.• Prove a Heaviside expansion formula for P/Q when Q has double zeros.

5. Compute the inverse Laplace transform of each of the following:

   (a) $\dfrac{z}{(z+1)(z+2)}$

   (b) $\sinh z$

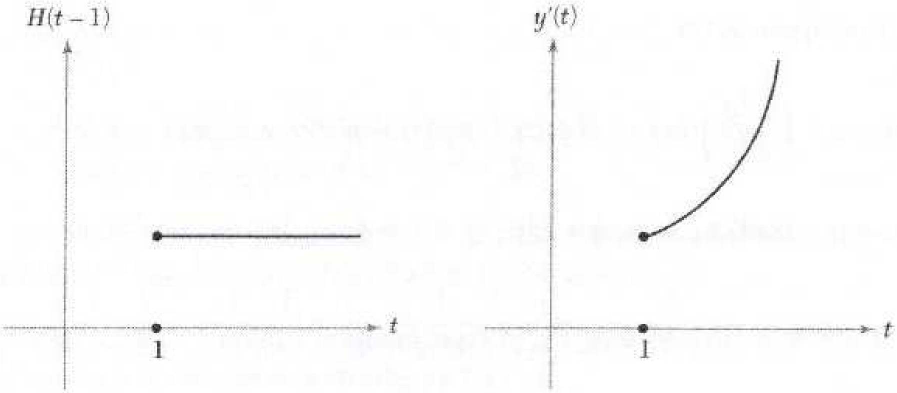

Figure 8.3.1: At $t = 1, y$ receives an impulse.

Solution Let us find the solution with $y(0) = 0, y'(0) = 0$. Taking Laplace transforms,

$$z^2\tilde{y}(z) + 2z\tilde{y}(z) + 2\tilde{y}(z) = \tilde{f}(z),$$

so $\tilde{y}(z) = \tilde{f}(z)/(z^2 + 2z + 2)$. The inverse Laplace transform of $1/(z^2 + 2z + 2)$ is

$$g(t) = \frac{e^{z_1 t}}{2(z_1 + 1)} + \frac{e^{z_2 t}}{2(z_2 + 1)},$$

where z_1, z_2 are the two roots of $z^2 + 2z + 2$, namely, $-1 \pm i$. Simplifying, $g(t) = e^{-t}\sin t$. Thus, by formula 8 of Table 8.1.1,

$$y(t) = (g * f)(t) = \int_0^\infty f(t - \tau)g(\tau)d\tau = \int_0^\infty f(t - \tau)e^{-\tau}\sin\tau\, d\tau.$$

This is the particular solution we sought. ♦

Generally such particular solutions to differential equations of the form

$$a_n y^{(n)} + \ldots + a_1 y = f,$$

where $a_1, \ldots, a_n$ are constants, may be expressed in the form of a convolution. To obtain a solution with the values $y(0), y'(0), \ldots, y^{(n-1)}(0)$ prescribed, we can add a particular solution y_p satisfying

$$y_p(0) = 0, y_p'(0) = 0, \ldots, y_p^{(n-1)}(0) = 0$$

to a solution y_c of the homogeneous equation in which f is set equal to zero and with $y_c(0), y_c'(0), \ldots, y_c^{(n-1)}(0)$ prescribed. The sum $y_p + y_c$ is the solution sought. (These statements are easily checked.)

The method of Laplace transforms is a systematic method for handling constant coefficient differential equations. (Of course, these equations can be handled by other means as well.) If the coefficients are not constant, the method fails, because transformation of a product then involves a convolution, and then solving for $\tilde{y}(z)$ becomes difficult.

Applying this again gives

$$\left(\frac{d^2y}{dt^2}\right)^{\tilde{}}(z) = z^2\tilde{y}(z) - zy(0) - y'(0) = z^2\tilde{y}(z) - 1.$$

Therefore, our equation becomes $z^2\tilde{y}(z) - 1 + 4z\tilde{y}(z) + 3\tilde{y}(z) = 0$, so

$$\tilde{y}(z) = \frac{1}{z^2 + 4z + 3} = \frac{1}{(z+1)(z+3)} = \frac{1}{2}\frac{1}{z+1} - \frac{1}{2}\frac{1}{z+3}.$$

By the inversion formula, the inverse Laplace transform of this function is

$$y(t) = \sum \left\{ \text{residues of } \frac{e^{zt}}{(z+1)(z+3)} \text{ at } -1, -3 \right\}.$$

Thus,

$$y(t) = \frac{e^{-t} - e^{-3t}}{2}.$$

(We could also apply line 9 of Table 8.1.1 to the partial fraction expansion.) This is the desired solution, as can be checked directly by substitution into the differential equation. ♦

Example 8.3.2 *Solve the equation* $y'(t) - y(t) = H(t-1), t \geq 0, y(0) = 0$, *where* H *is the Heaviside function.*

Solution Take the Laplace transforms of both sides of the equation. We get

$$z\tilde{y}(z) - y(0) - \tilde{y}(z) = e^{-z}/z.$$

Therefore, $\tilde{y}(z) = e^{-z}/z(z-1)$. The inverse Laplace transform of $1/[z(z-1)]$ is $1 - e^{-t}$, so that of $e^{-z}/[z(z-1)]$ is, by formula 6 of Table 8.1.1,

$$y(t) = \begin{cases} 0 & 0 \leq t < 1 \\ -1 + e^{t-1} & t \geq 1 \end{cases}.$$

Note that the complex inversion formula does not apply as stated. This solution (see Figure 8.3.1) is not differentiable and thus cannot be considered a solution in the strict sense. However, it is a solution in a generalized sense, as previously explained. In Figure 8.3.1, the discontinuity in $H(t-1)$ causes the sudden jump in $y'(t)$. We say that $y(t)$ receives an "impulse" at $t = 1$. ♦

Example 8.3.3 *Find a particular solution of* $y''(t) + 2y'(t) + 2y(t) = f(t)$.

4. Let $f(t)$ be a bounded function of t. Show that $\sigma(f) \leq 0$.

5. Compute the Laplace transform and the abscissa of convergence for
$$f(t) = \frac{e^t - 1}{t}.$$

6. If $f(t) = 0$ for $t < 0$, then $\hat{f}(y) = (1/\sqrt{2\pi})\tilde{f}(iy)$ is called the ***Fourier transform*** of f. Using Corollary 8.2.2, show that, under suitable conditions,
$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(y) e^{ixy} dy.$$
(This result is called the ***Inversion Theorem for Fourier transforms***.)

7. Compute the inverse Laplace transform and the abscissa of convergence for
$$F(z) = \frac{e^{-z}}{z^2 + 1}.$$

8. • Compute the inverse Laplace transform and the abscissa of convergence for
$$F(z) = \frac{1}{(z+1)^2}.$$

9. Compute the inverse Laplace transform and the abscissa of convergence for
$$F(z) = \frac{z}{(z+1)^2} + \frac{e^{-z}}{z}.$$

10. (a) Let $\tilde{f}(z)$ be the Laplace transform of $f(t)$. Show that $\tilde{f}(z) \to 0$ as $\operatorname{Re} z \to \infty$.
    (b) Use (a) to show that, under suitable conditions, $z\tilde{f}(z) \to f(0)$ as $\operatorname{Re} z \to \infty$.
    (c) Can a nonzero polynomial be the Laplace transform of any $f(t)$?
    (d) Can a nonzero entire function F be the Laplace transform of a function $f(t)$?

11. Solve the following differential equations using Laplace transforms:
    (a) $y'' + 8y + 15 = 0, y(0) = 1, y'(0) = 0$
    (b) $y' + y = 3, y(0) = 0$

12. • Suppose that $f(t) \geq 0$ and is infinitely differentiable. Prove that $(-1)^k \tilde{f}^{(k)}(z) \geq 0, k = 0, 1, 2, \ldots,$ for $z \geq 0$. (The converse, called ***Bernstein's Theorem***, is also true but is more difficult to prove.)

13. Solve the following differential equations using Laplace transforms:
    (a) $y'' + y = H(t-1), y(0) = 0, y'(0) = 0$
    (b) $y'' + 2y' + y = 0, y(0) = 1, y'(0) = 1$

Exercises

Solve the differential equations in Exercises 1 through 6 using Laplace transforms.

1. $y'' - 4y = 0, y(0) = 2, y'(0) = 1$
2. $y'' + 6y - 7 = 0, y(0) = 1, y'(0) = 0$
3. • $y'' + 9y = H(t-1), y(0) = y'(0) = 0$
4. $y' + y = e^t, y(0) = 0$
5. $y' + y + \int_0^t y(\tau)d\tau = f(t)$ where $y(0) = 1$ and where $f(t) = 0$ for $0 \le t < 1$ or $t \ge 2$ and $f(t) = 1$ if $1 \le t < 2$.
6. $y'' + 9y = H(t), y(0) = y'(0) = 0$.
7. Solve the following systems of equations for $y_1(t), y_2(t)$ by using Laplace transforms.

   (a)

$$\begin{cases} y_1' + y_2 = 0 \\ y_2' + y_1 = 0 \end{cases} \quad \text{where } y_1(0) = 1, y_2(0) = 0.$$

   (b)

$$\begin{cases} y_1' + y_2' + y_1 = 0 \\ y_2' + y_1 = 3 \end{cases} \quad \text{where } y_1(0) = 0, y_2(0) = 0.$$

8. Solve: $y' + y = \cos t, y(0) = 1$.
9. • Solve: $y'' + y = t \sin t, y(0) = 0, y'(0) = 1$.
10. Study the solution of $y'' + \omega_0^2 y = \sin \omega t, y(0) = y'(0) = 0$, and examine the behavior of solutions for various ω, especially those near $\omega = \omega_0$. Interpret these solutions in terms of forced oscillations.

Review Exercises for Chapter 8

1. Compute the Laplace transform and the abscissa of convergence for $f(t) = H(t-1)\sin(t-1)$.
2. • Compute the Laplace transform and the abscissa of convergence for $f(t) = H(t-1) + 3e^{-(t+6)}$.
3. Compute the Laplace transform and the abscissa of convergence for

$$f(t) = \begin{cases} t & 0 \le t \le 1 \\ 1 & t > 1 \end{cases}.$$

4. Let $f(t)$ be a bounded function of t. Show that $\sigma(f) \leq 0$.

5. Compute the Laplace transform and the abscissa of convergence for

$$f(t) = \frac{e^t - 1}{t}.$$

6. If $f(t) = 0$ for $t < 0$, then $\hat{f}(y) = (1/\sqrt{2\pi})\tilde{f}(iy)$ is called the ***Fourier transform*** of f. Using Corollary 8.2.2, show that, under suitable conditions,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(y) e^{ixy} dy.$$

(This result is called the ***Inversion Theorem for Fourier transforms.***)

7. Compute the inverse Laplace transform and the abscissa of convergence for

$$F(z) = \frac{e^{-z}}{z^2 + 1}.$$

8.• Compute the inverse Laplace transform and the abscissa of convergence for

$$F(z) = \frac{1}{(z+1)^2}.$$

9. Compute the inverse Laplace transform and the abscissa of convergence for

$$F(z) = \frac{z}{(z+1)^2} + \frac{e^{-z}}{z}.$$

10. (a) Let $\tilde{f}(z)$ be the Laplace transform of $f(t)$. Show that $\tilde{f}(z) \to 0$ as $\operatorname{Re} z \to \infty$.
    (b) Use (a) to show that, under suitable conditions, $z\tilde{f}(z) \to f(0)$ as $\operatorname{Re} z \to \infty$.
    (c) Can a nonzero polynomial be the Laplace transform of any $f(t)$?
    (d) Can a nonzero entire function F be the Laplace transform of a function $f(t)$?

11. Solve the following differential equations using Laplace transforms:
    (a) $y'' + 8y + 15 = 0, y(0) = 1, y'(0) = 0$
    (b) $y' + y = 3, y(0) = 0$

12.• Suppose that $f(t) \geq 0$ and is infinitely differentiable. Prove that $(-1)^k \tilde{f}^{(k)}(z) \geq 0, k = 0, 1, 2, \ldots,$ for $z \geq 0$. (The converse, called ***Bernstein's Theorem***, is also true but is more difficult to prove.)

13. Solve the following differential equations using Laplace transforms:
    (a) $y'' + y = H(t-1), y(0) = 0, y'(0) = 0$
    (b) $y'' + 2y' + y = 0, y(0) = 1, y'(0) = 1$