Μάθημα 16ο   Ε14. Διακριτά Δυναμικά Συστήματα   10/12/2019

$x_{k+1} = A x_k + B u_k \quad , A \in \mathbb{R}^{n\times n} , B \in \mathbb{R}^{n\times m}$

$\Gamma_c = [B \;\; AB \; \dots \; A^{n-1}B] \in \mathbb{R}^{n\times mn}$

$\text{Rank}(\Gamma_c) = n \iff (A,B)$ π.ε

Παράδειγμα: $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \Gamma_c = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}, \det(\Gamma_c) = -1 \neq 0$

Παράδειγμα: $A = \begin{array}{cc} \begin{array}{cc} n_1 & n-n_1 \end{array} & \\ \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} & \begin{array}{l} n_1 \\ n-n_1 \end{array} \end{array} \quad B = \begin{array}{cc} m & \\ \begin{bmatrix} B_1 \\ 0 \end{bmatrix} & \begin{array}{l} n_1 \\ n-n_1 \end{array} \end{array}$

$$\begin{bmatrix} x_{k+1}^{(1)} \\ x_{k+1}^{(2)} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_k^{(1)} \\ x_k^{(2)} \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u_k$$

$$x_{k+1}^{(1)} = A_{11} x_k^{(1)} + A_{12} x_k^{(2)} + B_1 u_k$$
$$x_{k+2}^{(2)} = A_{22} x_k^{(2)} \implies x_k^{(2)} = A_{22}^k x^{(2)}(0)$$

$\Gamma_c = [B \;\; AB \dots A^{n-1}B]$

$$A^k B = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}^k \begin{bmatrix} B_1 \\ 0 \end{bmatrix} = \begin{bmatrix} A_{11}^k B_1 \\ 0 \end{bmatrix}$$

$$\implies \Gamma_c = \begin{bmatrix} B_1 & A_{11}B_1 & \dots & A_{11}^{n-1}B_1 \\ 0 & 0 & \dots & 0 \end{bmatrix} \begin{array}{l} \}n_1 \\ \}n-n_1 \end{array}$$

άρα $\text{Rank}(\Gamma_c) = \text{Rank}([B_1 \vdots A_{11}B_1 \vdots \dots \vdots A_{11}^{n_1-1}B_1 \vdots A_{11}^{n_1}B_1 \vdots \dots \vdots A_{11}^{n-1}B_1]) \overset{\text{απο C-H}}{=}$
$= \text{Rank}([B_1 \vdots A_{11}B_1 \vdots \dots \vdots A_{11}^{n_1-1}B_1]) \leq n_1$

Θεώρημα: Έστω (A,B) δεν είναι πλήρως ελέγξιμο. Τότε $\exists\, Q \in \mathbb{R}^{n\times n}$ $\det(Q) \neq 0$:

$$\tilde{A} = Q^{-1} A Q = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix} , \quad \tilde{B} = Q^{-1}B = \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix}$$

και $(\tilde{A}_{11}, \tilde{B}_1)$ πλήρως ελέγξιμο.

Οριζω $\mathcal{X}_c \subseteq \mathbb{R}^n$, $\mathcal{X}_c = \mathcal{R}(\Gamma_c)$ ελέγξιμος υποχωρος.

Θεώρημα: (A,B) είναι πλήρως ελέγξιμο $\Leftrightarrow \text{Rank}([sI_n - A \,\vdots\, B]) = n \;\; \forall s \in \sigma(A)$

Παράδειγμα: $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$, $B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$

Πίνακας pencil

$$[sI_n - A \,\vdots\, B] = \begin{bmatrix} sI_{n_1} - A_{11} & -A_{12} & \vdots & B_1 \\ 0 & sI_{n-n_1} - A_{22} & \vdots & 0 \end{bmatrix}$$

ιδιοδιάνυσμα του A_{22} $\qquad \xi_2^T A = s_0 \xi_2^T \Leftrightarrow A_{22}^T \xi_2 = s_0 \xi_2$

$$[\,0^T \;\; \underset{\neq 0}{\xi_2^T}\,]\,[sI_n - A \,\vdots\, B] = 0$$

Παράδειγμα: $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$$\Gamma_c = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 1 & 2 & 4 \end{bmatrix} \underset{\text{(ως προς rank)}}{\sim} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \text{Rank}(\Gamma_c) = 2 < 3$$

άρα όχι πλήρως ελέγξιμο

$$[sI_n - A \,\vdots\, B] = \begin{bmatrix} s-1 & 0 & -1 & \vdots & 1 \\ 0 & s-1 & -1 & \vdots & 0 \\ -1 & 0 & s-1 & \vdots & 1 \end{bmatrix} \overset{\text{Θα πρέπει}}{=} [\,0 \;\; 0 \;\; 0\,]$$

$$\varphi(s) = \begin{vmatrix} s-1 & 0 & -1 \\ 0 & s-1 & -1 \\ -1 & 0 & s-1 \end{vmatrix} = (s-1)^3 - (s-1) = (s-1)^2 s \longrightarrow s_1 = 0, \; s_{2,3} = 1$$

Για $s_1 = 0$

$$[\,1 \;\; 0 \;\; -1\,] \begin{bmatrix} -1 & 0 & -1 & \vdots & 1 \\ 0 & -1 & -1 & \vdots & 0 \\ -1 & 0 & -1 & \vdots & 1 \end{bmatrix} = [\,0 \;\; 0 \;\; 0\,]$$

Ε14. Διακριτά Δυναμικά Συστήματα 10/12/2019

Παρατηρησιμότητα

$$x_{k+1} = A x_k \quad , \quad x_0 \in \mathbb{R}^n$$
$$y_k = C x_k$$

Το σύστημα είναι πλήρως παρατηρήσιμο αν από την ακολουθία $\{y_k\}_{k=0}^{N}$ (N πεπερασμένο), το $x_0 \in \mathbb{R}^n$ καθορίζεται μονοσήμαντα

Θεώρημα: (A, C) είναι πλήρως παρατηρήσιμο αν και μόνο αν $\mathrm{Rank}(\Gamma_o) = n$ όπου

$$\Gamma_o = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \in \mathbb{R}^{np \times n}$$

Απόδειξη:

$(\Leftarrow)$ Έστω $\mathrm{Rank}(\Gamma_o) = n$ Ισχύει ότι $x_k = A^k x_0 \Rightarrow y_k = CA^k x_0 \quad k \in \mathbb{N}_0$

$$\underbrace{\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{bmatrix}}_{Y_n} = \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}}_{\Gamma_o} x_0 \Longrightarrow \Gamma_o^T Y_n = \Gamma_o^T \Gamma_o x \Rightarrow x_0 = (\Gamma_o^T \Gamma_o)^{-1} \Gamma_o^T Y_n$$

$(\Rightarrow)$ Έστω (A, C) π.π. αλλά $\mathrm{Rank}(\Gamma_o) < n$ τότε $\exists\, \xi \neq 0 : \Gamma_o \xi = 0$

$$\Rightarrow \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \xi = 0 \Longrightarrow \begin{array}{c} C\xi = 0 \\ CA\xi = 0 \\ \vdots \\ \underline{CA^{n-1}\xi = 0} \\ CA^{n}\xi = 0 \\ \vdots \end{array}$$

← μπορώ να το επεκτείνω

και γενικά $CA^k \xi = 0 \quad k \in \mathbb{N}_0$

Αν $x_k = \xi$ τότε $y_k = CA^k \xi = 0 \Longrightarrow (A, C)$ δεν είναι π.π.

Θεώρημα: (A,B) είναι Π.Ε. $\Longleftrightarrow$ (A^T,B^T) είναι Π.Π

Δηλαδή αν $x_{k+1} = Ax_k + Bu_k$ Π.Ε $\Leftrightarrow$ $x_{k+1} = A^T x_k$ ΠΠ

$$y_k = B^T x_k$$

Απόδειξη:

$$(A,B) \text{ ΠΕ} \Leftrightarrow \text{Rank}([B \,\vdots\, AB \,\vdots\, \dots \,\vdots\, A^{n-1}B]) = n$$

$$\Longleftrightarrow \text{Rank}\left(\begin{bmatrix} B^T \\ B^T A^T \\ \vdots \\ B^T (A^T)^{n-1} \end{bmatrix}\right) = n \Longleftrightarrow (A^T, B^T) \text{ ΠΠ}$$

Θεώρημα: (A,C) ΠΠ αν και μόνο αν $\text{Rank}\left(\begin{bmatrix} sI_n - A \\ C \end{bmatrix}\right) = n \quad \forall s \in \sigma(A)$

Παρατήρηση: $\text{Ker}(\Gamma_0) = \mathcal{X}_{\bar{o}} \subseteq \mathbb{R}^n$ (μη παρατηρήσιμος υπόχωρος)

$$\mathcal{X}_{\bar{o}} = \left\{ x_0 : \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} x = 0 \right\} = \bigcap_{k=0}^{n-1} \text{Ker}(CA^k)$$

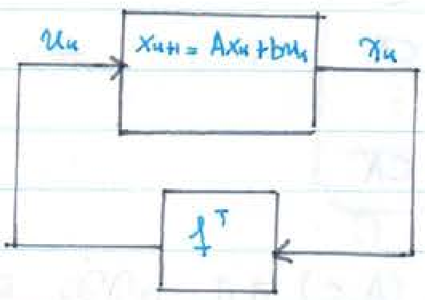
Έστω το σύστημα

$$x_{n+1} = Ax_k + \underline{b} u_k$$

$$u_k = f^T x_k$$

$$\Downarrow$$

$$x_{k+1} = (\underbrace{A + bf^T}_{A_c}) x_k$$



Μάθημα 17ο   Ε14. Διακριτά Δυναμικά Συστήματα   11/12/2019

Ανάδραση Καταστάσεων

$$\left.\begin{aligned} x_{k+1} &= Ax_k + bu_k \\ u_k &= f^T x \end{aligned}\right\}$$

$$x_{k+1} = (\underbrace{A + bf^T}_{A_{cl}})\, x_k \qquad \rho(A_{cl}) < 1$$

Έστω
$$A = \begin{bmatrix} \overbrace{0 & 1 & 0 & \dots & 0}^{\text{"Companion"}} \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \dots & & 1 \\ -\alpha_0 & -\alpha_1 & & & -\alpha_{n-1} \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \qquad \varphi_A(\lambda) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \dots + \alpha_0$$

$$\varphi_{A+bf^T}(\lambda) = \lambda^n + d_{n-1}\lambda^{n-1} + \dots + d_0$$

Έστω $f^T = [f_0 \;\; f_1 \;\; \dots \;\; f_{n-1}]$ τότε

$$A + bf^T = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & \dots & & & 1 \\ \underbrace{f_0 - \alpha_0}_{-d_0} & \underbrace{f_1 - \alpha_1}_{-d_1} & \dots & & \underbrace{f_{n-1} - \alpha_{n-1}}_{-d_{n-1}} \end{bmatrix} \qquad f_i = \alpha_i - d_i, \quad i = 0, 1, \dots, n-1$$

Παράδειγμα: $\varphi(\lambda) = (\lambda - 2)(\lambda + 2) = \lambda^2 - 4$

$$A = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\varphi_{A+bf^T}(\lambda) = \lambda^2 \qquad A_{cl} = A + bf^T = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} -4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Θεώρημα: Το $(A,B) \xrightarrow[\det(Q)\neq 0]{Q} \overbrace{(Q^{-1}AQ,\ Q^{-1}b)}^{\text{κανονική μορφή ελεγξιμότητας}} \iff (A,b)$ ΠΕ

Παράδειγμα:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Βρίσκω πίνακα ελεγξιμότητας $\Gamma_c = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ 0 & -1 & -3 \end{bmatrix}$

$\mathrm{Rank}(\Gamma_c) = 3 \Rightarrow (A,b)$ ΠΕ

$$\varphi(\lambda) = (\lambda-1)^3 = \lambda^3 - 3\lambda^2 + 3\lambda - 1 \longrightarrow A_c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}, \ b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$Q^{-1}AQ = A_c \qquad Q^{-1}b = b_c \iff Qb_c = b$$

$$Q = [q_1\ q_2\ q_3]$$

$$A[q_1\ q_2\ q_3] = [q_1\ q_2\ q_3]\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & 3 \end{bmatrix}$$

$$[q_1\ q_2\ q_3]\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = q_3$$

$$Aq_3 = q_2 + 3q_3 \Rightarrow q_2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - 3\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$$

$$Aq_2 = q_1 - 3q_3 \Rightarrow q_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Άρα $\Rightarrow Q = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$

Ε14. Διακριτά Δυναμικά Συστήματα

11/12/2019

$$\varphi_{A_c+b_c\tilde{f}^T}(\lambda) = \lambda^3$$

$$A_c + b_c\tilde{f}^T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\begin{bmatrix} -1 & 3 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A_c + b_c\tilde{f}^T = Q^{-1}AQ + Q^{-1}b\tilde{f}^TQ^{-1}Q = Q^{-1}(A + b\underbrace{\tilde{f}^TQ^{-1}}_{f^T = \tilde{f}^TQ^{-1}})Q$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right] \implies f^T = \tilde{f}^TQ^{-1} = \begin{bmatrix} -1 & \vdots & -2 & \vdots & -2 \end{bmatrix}$$

$$\underbrace{\qquad\qquad}_{Q^{-1}}$$

$$\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} + \begin{pmatrix} b_1 \\ 0 \end{pmatrix}\begin{pmatrix} f_1^T & f_2^T \end{pmatrix} = \begin{pmatrix} \overbrace{A_{11} + b_1 f_1^T}^{\varphi_1(\lambda)\ \text{αυθαίρετο}} & A_{12} + b_1 f_2^T \\ 0 & \underbrace{A_{22}}_{\varphi_2(\lambda)} \end{pmatrix},\ (A_1, b_1)\ \text{πε}$$

$$\varphi(\lambda) = \varphi_1(\lambda)\varphi_2(\lambda)$$

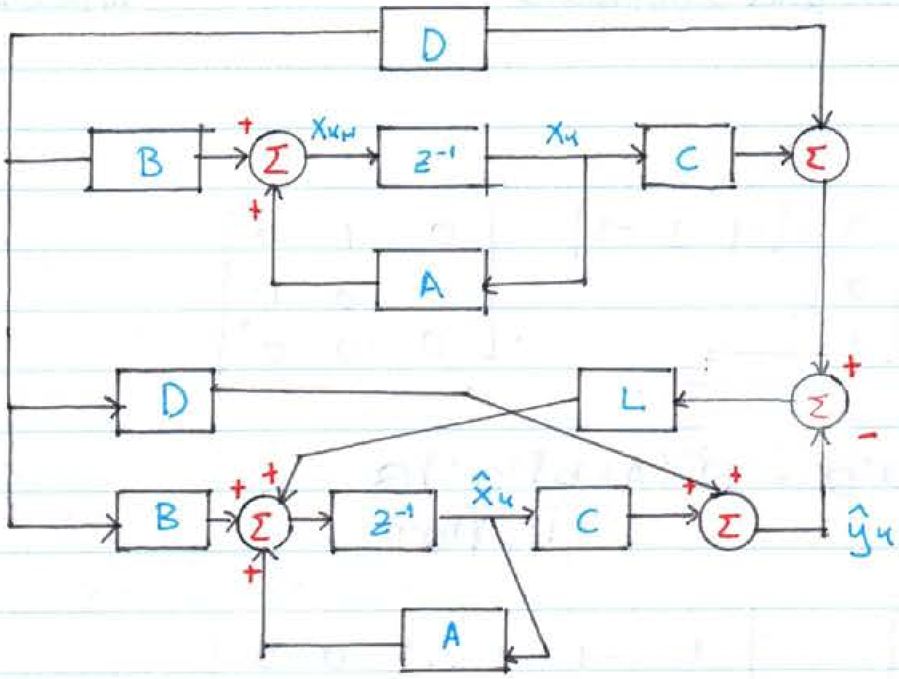
Παρατηρητές

$x_{k+1} = Ax_k + Bu_k$ $\quad A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m}$ $\quad (A,B)$ π.ε

$y_k = Cx_k + Du_k$ $\quad C \in \mathbb{R}^{p\times n},\ D \in \mathbb{R}^{p\times m}$ $\quad (A,C)$ π.π.

$$\hat{x}_{k+1} = \hat{A}x_k + Bu_k - L(y_k - \hat{y}_k) \qquad L \in \mathbb{R}^{n\times p}$$

$$\hat{y}_k = C\hat{x}_k + D\hat{u}_k$$



$e_k := x_k - \hat{x}_k$

$$e_{k+1} = x_{k+1} - \hat{x}_{k+1} = Ax_k + Bu_k - A\hat{x}_k - Bu_k + LC\overbrace{(x_k + Du_k - \hat{x}_k - Du_k)}^{e_k}$$

$$= Ae_k + LCe_k = (A+LC)e_k \Rightarrow e_k = (A+LC)^k x_0 \quad , e_0 = x_0$$

Θέλουμε $e_k \to 0$

Άρα $\rho(A+LC) < 1 \iff \rho(\underbrace{A^T}_{\tilde{A}} + \underbrace{C^T}_{\tilde{B}}\underbrace{L^T}_{\bar{F}}) < 1$

$$x_{k+1} = Ax_k + Bu_k = Ax_k + BK(x_k - e_k) = (A+BK)x_k - Bk\,e_k \quad \circledast$$

$u_k = K\hat{x}_k$

επω $\quad \hat{x}_{k+1} = A\hat{x}_k + Bu_k - L(Cx_k - C\hat{x}_k)$

$$e_{k+1} = x_{k+1} - \hat{x}_{k+1} = Ax_k + Bx_k - A\hat{x}_k - Bx_k + LCe_k \Rightarrow e_{k+1} = (A+LC)e_k \ (*)$$

$$\begin{bmatrix} x_{k+1} \\ e_{k+1} \end{bmatrix} = \overbrace{\begin{bmatrix} A+BK & -BK \\ 0 & A+LC \end{bmatrix}}^{A_c} \begin{bmatrix} x_k \\ e_k \end{bmatrix} \qquad \sigma(A_c) = \sigma(A+BK) \cup \sigma(A+LC)$$