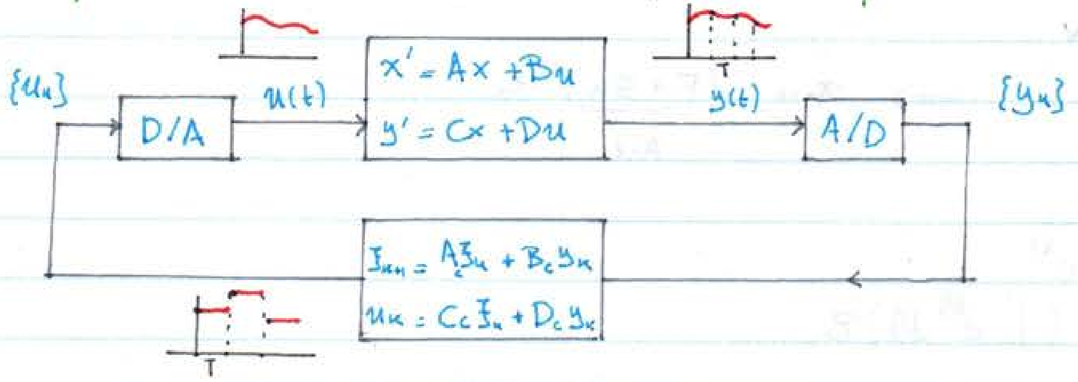
Μάθημα 12ο   Ε14. Διακριτά Δυναμικά Συστήματα   26/11/2019



$$x_{k+1} = F x_k + G u_k \quad \text{το ισοδύναμο διακριτό}$$
$$y_k = C x_k + D u_k$$

$$y_k = C x_k + D(C_c \xi_k + D_c y_k) \iff (I - D D_c) y_k = C x_k + D C_c \xi_k$$
$$\Longrightarrow y_k = \underbrace{(I - D D_c)^{-1}}_{L_1} C x_k + \underbrace{(I - D D_c)^{-1}}_{L_1} D C_c \xi_k$$

$$u_k = C_c \xi_k + D_c (C x_k + D u_k) \iff (I - D_c D) u_k = D_c C x_k + C_c \xi_k$$
$$\Longrightarrow u_k = \underbrace{(I - D_c D)^{-1}}_{L_2} D_c C x_k + \underbrace{(I - D_c D)^{-1}}_{L_2} C_c \xi_k$$

$$x_{k+1} = F x_k + G [L_2 D_c C x_k + L_2 C_c \xi_k]$$
$$\xi_{k+1} = A_c \xi_k + B_c [L_1 C x_k + L_1 D C_c \xi_k]$$

$$\underbrace{\begin{bmatrix} x_{k+1} \\ \xi_{k+1} \end{bmatrix}}_{W_{k+1}} = \underbrace{\begin{bmatrix} F + G L_2 D_c C & G L_2 C_c \\ B_c L_1 C & A_c + B_c L_1 D C_c \end{bmatrix}}_{A_{cl}} \underbrace{\begin{bmatrix} x_k \\ \xi_k \end{bmatrix}}_{W_k}$$

$$W_k = A_{cl}^k W_0 \qquad W_0 = \begin{pmatrix} x_0 \\ \xi_0 \end{pmatrix}$$

Αν $D = 0$ και διαλέξω $D_c = 0 \Longrightarrow L_1, L_2 = I$ τότε

$$W_{k+1} = \underbrace{\begin{bmatrix} F & G C_c \\ B_c C & A_c \end{bmatrix}}_{A_{cl}} W_k \qquad A_{cl} = \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & G \\ I & 0 \end{bmatrix} \begin{bmatrix} A_c & B_c \\ C_c & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ C & 0 \end{bmatrix}$$

(πίνακας κλειστού βρόχου με ανάδραση εξόδου)

Ανάδραση Καταστάσεων

$$x_{k+1} = F x_k + G u_k \implies x_{k+1} = \underbrace{(F + GK)}_{A_{cl}} x_k$$
$$u_k = K x_k$$

Παράδειγμα:

$$F = e^{AT}$$
$$G = \left(\int_0^T e^{A\lambda} d\lambda\right) B$$

(σύστημα) (αντισταθμιστής)

$$\left.\begin{aligned} x_{k+1} &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} x_k + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_k \\ y_k &= \begin{pmatrix} 1 & 0 \end{pmatrix} x_k \end{aligned}\right\} \qquad \left.\begin{aligned} \xi_{k+1} &= \xi_k + y_k \\ u_k &= \xi_k \end{aligned}\right)$$

$$A_{ce} = \begin{bmatrix} F & G C_c \\ B_c C & A_c \end{bmatrix} = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 1 \\ \hline 1 & 0 & 1 \end{array}\right]$$

$$\sigma(A_c) = \{\lambda : \det(\lambda I_3 - A_{ce}) = 0\} = (\lambda-1)^3 - 1 = (\lambda-2)(\lambda^2+\lambda+1)$$

$$\hat{G}_1(z) = \left[1 \;\vdots\; 0\right] \begin{bmatrix} z-1 & -1 \\ 0 & z-1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\hat{G}_2(z) = \frac{1}{z-1}$$

$$\hat{y}(z) = \hat{G}_1(z)\left[r(z) + \hat{G}_2(z)\hat{y}(z)\right]$$
$$(1 - \hat{G}_1(z)\hat{G}_2(z))\hat{y}(z) = \hat{G}_1(z) r(z)$$

$$\frac{\hat{y}(z)}{r(z)} = \frac{\hat{G}_1(z)}{1 - \hat{G}_1(z)\hat{G}_2(z)} = \frac{\frac{1}{(z-1)^2}}{1 - \frac{1}{(z-1)^3}} = \frac{z-1}{(z-1)^3 - 1} = \frac{z-1}{(z-2)(z^2-z+1)}$$

Ε14. Διακριτά Δυναμικά Συστήματα

26/11/2019

Απόκριση συστημάτων (Modal analysis)

$$\Sigma\begin{cases} x_{k+1} = Ax_k + Bu_k & k\geq 0\\ y_k = Cx_k + Du_k \end{cases} \qquad \left. \begin{aligned} x_k &= A^k x_0 + \sum_{j=0}^{k-1} A^{k-j-1} B u_j \\ y_k &= CA^k x_0 + \sum_{j=0}^{k-1} CA^{k-j-1} B u_j + D u_k \end{aligned}\right\}$$

Έστω $A \in \mathbb{R}^{n\times n}$

$$A\underline{u} = \lambda \underline{u} \quad \underline{u}\neq 0 \implies (\lambda I - A)\underline{u} = 0 \implies \varphi(\lambda) = \det(\lambda I - A) = 0$$

Έστω $\varphi(\lambda) = (\lambda-\lambda_1)^{\tau_1}(\lambda-\lambda_2)^{\tau_2}\cdots(\lambda-\lambda_p)^{\tau_p}$ $\quad \lambda_i\neq\lambda_j \quad i\neq j$

τ_i = αλγεβρική πολλαπλότητα της λ_i $\quad (i=1,2,\dots,p)$

$\dim(N_r(\lambda_i I - A)) = d_i$, γεωμετρική πολλαπλότητα της λ_i $\qquad 1\leq d_i\leq \tau_i$

$= n - \mathrm{Rank}[\lambda_i I - A] = n - r_i$

$N_r(A) = \{ \underline{x}\in\mathbb{C}^n : A\underline{x} = 0\}$ (πυρήνας)

Ορισμός. A απλής δομής αν $d_i = \tau_i \quad \forall i = 1,2,\dots,p$

Έστω A απλής δομής (u_i ιδιοδιάνυσμα)

$$A\underbrace{[u_1\ u_2\ \dots\ u_n]}_{P} = \underbrace{[u_1\ u_2\ \dots\ u_n]}_{P}\underbrace{\begin{bmatrix}\lambda_1 & & \emptyset\\ & \lambda_2 & \\ & & \ddots \\ \emptyset & & & \lambda_n\end{bmatrix}}_{\Lambda} \implies A = P\Lambda P^{-1}$$

$$\implies A^2 = P\Lambda P^{-1}P\Lambda P^{-1} = P\Lambda^2 P^{-1} \implies A^k = P\Lambda^k P^{-1}$$

$$x_k = A^k x_0 + \sum_{j=0}^{k-1} A^{k-j-1} B u_j = P\Lambda^k P^{-1} x_0 + \sum_{j=0}^{k-1} P\Lambda^{k-j-1}P^{-1} B u_j$$

$$\left.\begin{aligned} P &= [u_1 \dots u_n] \\ P^{-1} &= \begin{bmatrix}\tilde{u}_1^T\\ \vdots \\ \tilde{u}_n^T\end{bmatrix}\end{aligned}\right\} \begin{bmatrix}\tilde{u}_1^T\\ \vdots \\ \tilde{u}_n^T\end{bmatrix}[u_1 \dots u_n] = I_n$$

$$x_k = [u_1\dots u_n]\begin{bmatrix}\lambda_1^k & & \emptyset\\ & \ddots & \\ \emptyset & & \lambda_n^k\end{bmatrix}\begin{bmatrix}\tilde{u}_1^T\\ \vdots \\ \tilde{u}_n^T\end{bmatrix} x_0 + \sum_{j=0}^{k-1}[u_1\dots u_n]\begin{bmatrix}\lambda_1^{k-j-1} & & \emptyset\\ & \ddots & \\ \emptyset & & \lambda_n^{k-j-1}\end{bmatrix}\begin{bmatrix}\tilde{u}_1^T\\ \vdots \\ \tilde{u}_n^T\end{bmatrix} B u_j$$

$$x_k = \sum_{i=1}^{n} \langle \tilde{u}_i, x_0 \rangle \lambda_i^{k} \underline{u}_i + \sum_{j=1}^{k-1} \sum_{i=1}^{n} \langle \tilde{u}_i, Bu_j \rangle \lambda_i^{k-j-1} u_i$$

Ορισμός: Το σύστημα $x_{k+1} = A x_k$ λέγεται ασυμπτωτικά ευσταθές αν και μόνο αν

$$\|x_k\| \longrightarrow 0 \qquad \forall\, x_0 \in \mathbb{R}$$

(ισοδύναμα $\|A^k x_0\| \longrightarrow 0$)

Θεώρημα: $x_{k+1} = A x_k$ ασυμπτωτικά ευσταθές $\iff \underset{\max\limits_{i\in\{1,\dots,n\}} |\lambda_i(A)|}{\underset{\parallel}{\rho(A)}} < 1$ (φασματική)