Μάθημα 9ο   Ε1. Μέθοδοι Εφαρμοσμένων Μαθηματικών Ι (Αλικάκος)   20/12/2018

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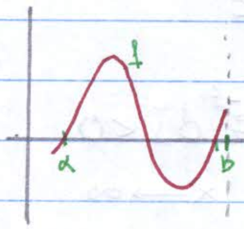
Θεωρούμε $\quad I(x) = \int_\alpha^b f(t)\, e^{ix\psi(t)}\, dt \quad , \quad \alpha, b, f, \psi \in \mathbb{R}$

$\psi(t) = t \quad$ Fourier

Riemann-Lebesgue Λήμμα: $f \in L^1(\alpha, b)$, $\lim\limits_{n\to\infty} \int_\alpha^b f(t)\sin(nt)\,dt = 0$

$\lim\limits_{n\to\infty} \int_\alpha^b f(t)\cos(nt)\,dt = 0$



Παράδειγμα 1: (Μέσω ολοκλήρωσης κατά μέρη)

$I(x) = \int_0^1 \frac{e^{ixt}}{1+t}\,dt \qquad I(x) \quad$ όπως $\quad x \to +\infty$

$$= -\frac{i}{x}\int_0^1 \left(\frac{1}{1+t}\right)\frac{d}{dt}\left(e^{ixt}\right) = -\frac{i}{x}\left[\frac{1}{1+t}e^{ixt}\Big|_0^1 + \int_0^1 \frac{1}{(1+t)^2}e^{ix}\,dt\right]$$

$$= -\frac{i}{x}\left[\frac{1}{2}e^{ix} - 1\right] + o\left(\frac{1}{x^2}\right)$$

$$I(x) \sim \frac{\alpha}{x} - \frac{b}{x^2} - \dots$$

Παράδειγμα 2: $I(x) = \int_0^1 \sqrt{t}\, e^{ixt}\,dt = -\frac{i}{x}\int_0^1 \sqrt{t}\,\frac{d}{dt}\left(e^{ixt}\right)dt$

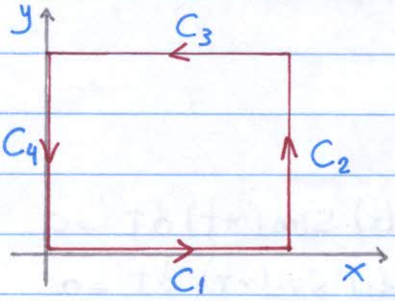
$$= -\frac{i}{x}\left[\sqrt{t}\,e^{ixt}\Big|_0^1 - \frac{1}{2}\int_0^1 \frac{e^{ixt}}{\sqrt{t}}\,dt\right] = -\frac{i}{x}e^{ix} + \frac{i}{2x}\int_0^1 \frac{e^{ixt}}{\sqrt{t}}\,dt$$

- $\frac{i}{2x}\int_0^1 \frac{e^{ixt}}{\sqrt{t}}\,dt \sim \frac{i}{2x^{3/2}}\sqrt{\pi}\,e^{i\pi/4} \qquad$ Prob 6.49

$s = xt \Rightarrow ds = x\,dt$

$$\frac{i}{2x}\int_0^1 \frac{e^{ixt}}{\sqrt{t}}\,dt = \frac{i}{2x}\int_0^x \frac{e^{is}}{\sqrt{tx}}\sqrt{x}\,\frac{1}{x}\,ds =$$

$$= \frac{i}{2x^{3/2}}\int_0^x \frac{e^{is}}{\sqrt{s}}\,ds \sim \frac{i}{2x^{3/2}}\int_0^\infty \frac{e^{is}}{\sqrt{s}}\,ds \qquad x\to\infty$$



Φορμαλιστικά

$s = i\tau$

$$\int_0^\infty \frac{e^{-\tau}}{\sqrt{i\tau}}\,i\,d\tau = \frac{1}{\sqrt{i}}\int_0^\infty \frac{e^{-\tau}}{\sqrt{\tau}}\,d\tau \qquad \Gamma(1/2)$$

$C = C_1 \cup C_2 \cup C_3 \cup C_4 \qquad 0 = \int_C \frac{e^{z}}{\sqrt{z}}\,dz$

$$\int_0^\infty e^{-\xi}\,d\xi < 0$$

$$\int_{C_2} = \int_0^y \frac{e^{i(x+i\xi)}}{\sqrt{x+i\xi}}\,d(i\xi) = \frac{e^{ix}}{i}\int_0^y \frac{e^{-\xi}}{\sqrt{x+i\xi}}\,d\xi \longrightarrow 0 \qquad x\to\infty$$

$$\int_{C_3} = -\int_0^x \frac{e^{i(\xi+iy)}}{\sqrt{\xi+iy}}\,d\xi = e^{-y}\int_0^x \frac{e^{i\xi}}{\sqrt{\xi+iy}}\,d\xi \longrightarrow 0 \qquad y\to\infty$$

$\psi(t) \neq$ σταθερό σε διαστήματα.

$$I(x) = \int_\alpha^b f(t)\,e^{ix\psi(t)}\,dt = \int_\alpha^b \frac{f(t)}{ix\psi'(t)}\,\frac{d}{dt}\left(e^{ix\psi(t)}\right)dt$$

υποθέτουμε $\psi \in C^1[\alpha, b]$, $f \in C^1[\alpha, b]$

$$= \frac{f(t)}{ix\,\psi'(t)}\,e^{ix\psi(t)}\Big|_\alpha^b - \frac{1}{ix}\int_\alpha^b \left(\frac{f(t)}{\psi'(t)}\right)' e^{ix\psi(t)}\,dt$$

Αν $\psi'(t) \neq 0 \Rightarrow$ ο κυρίαρχος όρος είναι ο πρώτος

$$I(x) \sim \frac{f(t)}{ix\psi'(t)}\,e^{ix\psi(t)}\Big|_{t=\alpha}^{t=b} \qquad x\to\infty$$

αφού ο άλλος από το λήμμα Riemann-Lebesgue $\longrightarrow 0$

Ε1. Μέθοδοι Εφαρμοσμένων Μαθηματικών Ι (Αλικάκος) 20/12/2018

Θεωρούμε τώρα ότι έχουμε κρίσιμο σημείο

$\psi'(\alpha)=0$, $\psi'\neq 0$ στο (α, b)

$$I(x) \simeq \int_\alpha^{\alpha+\varepsilon} f(t) e^{ix\psi(t)} dt$$

υποθέτουμε $f(\alpha)\neq 0$

$\psi^{(p)}(\alpha)\neq 0$

$s = t\alpha$ >0

$$\approx f(\alpha)\int_0^\infty e^{ix\left[\psi(\alpha)+\frac{1}{p!}\psi^{(p)}(\alpha)(t-\alpha)^p\right]} dt$$

$$s = e^{\frac{i\pi}{2p}}\left[\frac{p!\,u}{x\psi^{(p)}(\alpha)}\right]^{1/p}$$

$\psi^{(p)}(\alpha) < 0$, >0

$$I(x) \sim f(\alpha)\,\exp\{ix\psi(\alpha) \pm i\pi/2p\}\left[\frac{p!}{x\,|\psi^{(p)}(\alpha)|}\right]^{1/p} \quad \frac{\Gamma(1/p)}{1/p} \quad , x\to\infty$$

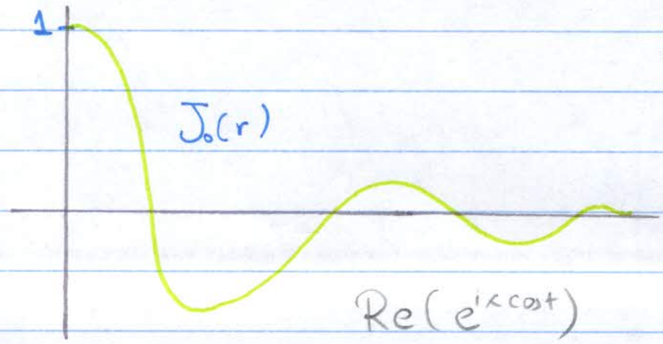
Παράδειγμα: $\int_0^\infty \cos(xt^2-t)\,dt$, $x\to\infty$ $f(t)$

$$= \operatorname{Re}\int_0^\infty e^{i(xt^2-t)}dt = \operatorname{Re}\int_0^\infty e^{ixt^2}dt \quad , \quad \psi(t)=t^2$$

$p=2$, $\psi''(0)=2$

$$\int_0^\infty \cos(xt^2-t)\,dt \sim \operatorname{Re}\left(\frac{1}{2}\sqrt{\frac{\pi}{x}}\,e^{i\pi/4}\right) = \frac{1}{2}\sqrt{\pi/2x}$$

Παράδειγμα: $r^2y''+ry'+(r^2-v^2)y=0$ για $v=0 \to J_0(r)$



$$J_0(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{(k!)^2}\left(\frac{x}{2}\right)^{2k} \quad x>0$$

Μπορεί να γραφεί $\longrightarrow J_0(x) = \frac{2}{\pi}\int_0^{\pi/2}\cos(x\cos t)\,dt$

$\operatorname{Re}(e^{ix\cos t})$ $\psi(t)=\cos t$

$$J_0(x) \sim \sqrt{\frac{2}{\pi x}}\cos(x-\pi/4)$$