Linear and Nonlinear Control Theory - CAD Lab 1
Dept of Mathematics, University of Athens

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Introduction

The purpose of this lab is to program and test an LQR optimisation algorithm taught in the module used to design a vibration-control scheme of a simple structure.

Vibration Control LQR design

Consider the structure shown in Figure 1 which is subjected to a disturbance force $f(t)$ applied at time $t = 0$. A linear actuator connected between masses M_0 and M_1 can produce an equal and opposite force $u(t)$ on the masses M_0 and M_1, respectively.

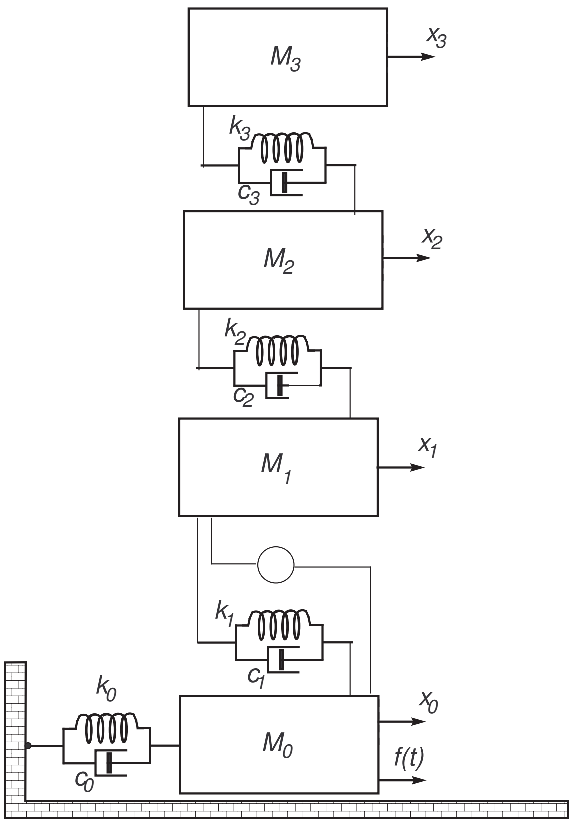

Figure 1: *Three-storey structure*

1. Obtain a state-space model of the system in the form

$$\dot{x}(t) = Ax(t) + Bu(t) + Ff(t)$$

where $x(t)$ represents the state-vector

$$x'(t) = \begin{pmatrix} x_0(t) & x_1(t) & x_2(t) & x_3(t) & \dot{x}_0(t) & \dot{x}_1(t) & \dot{x}_2(t) & \dot{x}_3(t) \end{pmatrix}$$

consisting of the displacements $x_i(t)$ (relative to the equilibrium) and the velocities $\dot{x}_i(t)$ of the four masses M_i.

2. Find also the corresponding C and D matrices of the output equation

$$y(t) = Cx(t) + Du(t)$$

if the two measurements are: (i) The acceleration of mass M_1, and (ii) The relative displacement $x_1(t) - x_0(t)$.

3. Investigate the damping of the poles of the system for the following values of the parameters: Base: $M_0 = 5$ Kg, $c_0 = 100$ N/m/s, $k_0 = 16000$ N/m; First floor: $M_1 = 1.72$ kg, $c_1 = 0.078$ N/m/s, $k_1 = 2600$ N/m; Second floor: $M_2 = 1.48$ kg, $c_2 = 0.078$ N/m/s, $k_2 = 2600$ N/m; Third floor: $M_3 = 2.34$ kg, $c_3 = 0.078$ N/m/s, $k_3 = 2600$ N/m. Obtain also (using Matlab) (i) the time-response of the system when $f(t)$ is a unit impulse (and $u(t) = 0$), (ii) The magnitude Bode plots between $f(t)$ and the two outputs. In each case explain the main characteristics of the response from the model equations.

4. We will use a slightly more general LQR problem from the one defined and solved in the course. We consider the same system as before:

$$x' = Ax + Bu, \ x(0) = x_0 \in \mathbb{R}^n$$

and cost function (to be minimized):

$$\begin{aligned} J[u, x_0] &= \int_0^\infty (x'(t)Qx(t) + 2x'(t)Nu(t)x + u'(t)Ru(t))\,dt \\ &= \int_0^\infty \begin{pmatrix} x'(t) & u'(t) \end{pmatrix} \begin{pmatrix} Q & N \\ N' & R \end{pmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} dt \end{aligned}$$

Let $Q = Q' \geq 0$ and $R = R' > 0$. Show that:

$$\begin{pmatrix} Q & N \\ N' & R \end{pmatrix} \geq 0$$

if and only if $Q - NR^{-1}N' \geq 0$. (Hint: Use Sylvester's law of inertia).

5. Show that

$$x'Qx + 2x'Nu + u'Ru = (u + R^{-1}N'x)'R(u + R^{-1}N'x) + x'(Q - NR^{-1}N')x$$

and hence that the cost function can be redefined as:

$$J[u, x_0] = \int_0^\infty \big(x'(t)(Q - NR^{-1}N')x(t) + \tilde{u}'(t)R\tilde{u}(t)\big)\, dt$$

where the system's dynamics are now:

$$x'(t) = (A - BR^{-1}N')x(t) + B\tilde{u}(t),\ x(0) = x_0 \in \mathbb{R}^n,\ \ \tilde{u}(t) = u(t) + R^{-1}N'x(t)$$

Show further that under the assumptions that: $(A - BR^{-1}B'N', B)$ is controllable (equivalently (A, B) is controllable) and $(A - BR^{-1}N', Q - NRN')$ is observable, the optimal solution is given as: $\tilde{u}(t) = -R^{-1}B'Px(t)$ where P is the (positive-definite) stabilizing solution of the ARE:

$$P(A - BR^{-1}N') + (A - BR^{-1}N')P - PBR^{-1}B'P + Q - NR^{-1}N' = 0$$

Conclude that Implementing $\tilde{u}$ on the redefined system is equivalent to implementing the control

$$u(t) = \tilde{u} - R^{-1}N'x(t) = -R^{-1}(B'P + N')x(t)$$

in the original system.

6. It is proposed to implement an LQR state-feedback active vibration control scheme of the form $u(t) = -Kx(t)$. The performance index which should be minimized is

$$J[u] = \frac{1}{2}\int_0^\infty \big(|\ddot{x}_1(t)|^2 + \rho_1|x_1(t) - x_0(t)|^2 + \rho_2 u^2(t)\big)\, dt$$

where ρ_1 and ρ_2 are non-negative scalar parameters. By finding appropriate matrices Q, R and N, show that this performance index can be put into standard form:

$$J[u] = \frac{1}{2}\int_0^\infty \big(x'(t)Qx(t) + 2x'(t)Nu(t) + Ru^2(t)\big)\, dt$$

and state what is the optimal control $u^\star(t)$.

7. Using Matlab, find the optimal control $u^\star(t)$ for $\rho_1 = 0$ $\rho_2 = 1$. Find also the closed-loop "A" matrix of the system, verify that it is asymptotically stable and calculate the damping factor of its eigenvalues. Show also that the solution of the Algebraic Riccati Equation associated with the optimal control is positive semi-definite.

8. With $\rho_1 = 0$, investigate the time and frequency-domain closed-loop responses of the system for different values of ρ_2. Do the results vary as you expect? What happens when ρ_2 is set to a large positive number?

9. By varying ρ_1 and ρ_2 try to obtain the "best" possible design, subject to the constraint that the peak force of the actuator does not exceed 20 N.

G.Halikias, 8-12-2018