Μάθημα 7ο

20/03/2015

Μερικά Θεωρήματα του Δ.Λ.

Εφαρμογές

Μερικές παράγωγοι ανωτέρας ή υψηλοτέρας τάξης / C^{κ} $\kappa \in \mathbb{N}$, $\kappa = \infty$.

$f: A (\subseteq \mathbb{R}^d) \longrightarrow \mathbb{R}$, A = ανοιχτό σύνολο

Έστω ότι $\exists\ \dfrac{\partial f}{\partial x_i}$ $i = 1, \dots, d$, στη $S(\vec{x}_0, \delta) \subseteq A$

Τότε $g_i(\vec{x}) = \dfrac{\partial f}{\partial x_i}(\vec{x})$, $\vec{x} \in S(\vec{x}_0, \delta)$

Εάν $\exists\ \dfrac{\partial g_i}{\partial x_j}(\vec{x}_0) = \dfrac{\partial}{\partial x_j}\left(\dfrac{\partial f}{\partial x_i}\right)\Big|_{\vec{x}_0} = \dfrac{\partial^2 f}{\partial x_j \partial x_i}(\vec{x}_0)$

2ης τάξης μερική παράγωγος, $i-j$

Εάν $i = j$, συμβολίζεται $\dfrac{\partial^2 f}{\partial x_i^2}(\vec{x}_0)$ / Άλλοι συμβ: f_{j-i}, $f_{x_j x_i}$

Έστω $\dfrac{\partial^2 f}{\partial x_j \partial x_i}$ στο $S(\vec{y}_0, \delta) \subseteq A$

Εάν $\exists\ \dfrac{\partial}{\partial x_\kappa}\left(\dfrac{\partial^2 f}{\partial x_j \partial x_i}\right)\Big|_{\vec{y}_0} = \dfrac{\partial^3 f(\vec{y}_0)}{\partial x_\kappa \partial x_j \partial x_i}$

κ.ο.κ...

C^k συναρτήσεις

$f = C^1 \iff \exists \ \dfrac{\partial f}{\partial x_i}(\vec{x}),\ \vec{x} \in A,\ i = 1, 2, \ldots, d$ και είναι συνεχείς στο A

$f = C^2 \iff$
- $f \in C^1$
- $\exists \ \dfrac{\partial^2 f}{\partial x_j \partial x_i}(\vec{x}),\ \vec{x} \in A \quad i, j = 1, \ldots, d$ και είναι συνεχείς στο A.

⋮

$f = C^\infty \iff \exists$ κάθε τάξης μ.π. και είναι συνεχείς στο A.

$\vec{F} = (F_1, \ldots, F_m) : A \, (\subseteq \mathbb{R}^d) \longrightarrow \mathbb{R}^m, \quad \vec{F} = C^k \iff$ οι $F_1, F_2, \ldots, F_m$ είναι C^k.

Ασκήσεις

1) $f(x,y,z) = xy^2 + e^{x^2+z} + \eta\mu(x+z^2)$

Να ευρεθούν οι $\dfrac{\partial^2 f}{\partial x_i \partial x_j}$ $\quad i, j = 1, 2, 3$.

$$\frac{\partial f}{\partial x} = y^2 + 2x e^{x^2+z} + \sigma\upsilon\nu(x+z^2), \quad \frac{\partial f}{\partial y} = 2xy, \quad \frac{\partial f}{\partial z} = e^{x^2+z} + 2z\,\sigma\upsilon\nu(x+z^2)$$

$$\frac{\partial^2 f}{\partial x^2} = 2e^{x^2+z} + 4x^2 e^{x^2+z} - \eta\mu(x+z^2)$$

$$\frac{\partial^2 f}{\partial y \partial x} = 2y$$

$$\frac{\partial^2 f}{\partial z \partial x} = 2x e^{x^2+z} - 2z\,\eta\mu(x+z^2).$$

$$\frac{\partial^2 f}{\partial y^2} = 2x \qquad \frac{\partial^2 f}{\partial z^2} = e^{x^2+z} + 2\sigma\upsilon\nu(x+z^2) - 4z^2\eta\mu(x+z^2)$$

$$\frac{\partial^2 f}{\partial x \partial y} = 2y \qquad \frac{\partial^2 f}{\partial x \partial z} = 2x\, e^{x^2+z} - 2z\,\eta\mu(x+z^2)$$

$$\frac{\partial^2 f}{\partial z \partial y} = 0 \qquad \frac{\partial^2 f}{\partial y \partial z} = 0$$

Παρατηρούμε ότι $\frac{\partial^2 f}{\partial z \partial y} = \frac{\partial^2 f}{\partial y \partial z}$, $\frac{\partial^2 f}{\partial x \partial z} = \frac{\partial^2 f}{\partial z \partial x}$, $\frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 f}{\partial z \partial y}$.

- Ισχύει, όμως, αυτό γενικά; Όχι.

2) $$f(x,y) = \begin{cases} xy\,\dfrac{x^2-y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

ΝΔΟ. $\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0)$

- $$\frac{\partial f}{\partial x}(x,y) = \begin{cases} \dfrac{y(x^4-y^4+4x^2y^2)}{(x^2+y^2)^2}, & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

$$\frac{\partial^2 f}{\partial y \partial x}(x,y) = \begin{cases} \dfrac{x^6+9x^4y^2-9x^2y^4-y^6}{(x^2+y^2)^3}, & (x,y) \neq (0,0) \\ -1 & (x,y) = (0,0) \end{cases}$$

$$\frac{\partial f}{\partial x}(0,y) = -y, \quad y \in \mathbb{R}$$

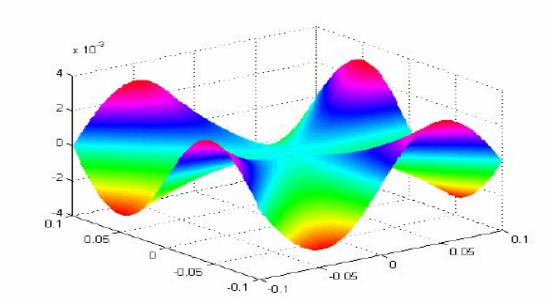
$$\frac{\partial f}{\partial y}(x,y)=\begin{cases}\dfrac{-x(y^4-x^4+4x^2y^2)}{(x^2+y^2)^2} & ,\ (x,y)\neq(0,0)\\ 0 & ,\ (x,y)=(0,0)\end{cases}$$

$$\frac{\partial^2 f}{\partial x\partial y}(x,y)=\begin{cases}\dfrac{x^6+9x^4y^2-9x^2y^4-y^6}{(x^2+y^2)^3} & ,\ (x,y)\neq(0,0)\\ 1 & ,\ (x,y)=(0,0)\end{cases}$$

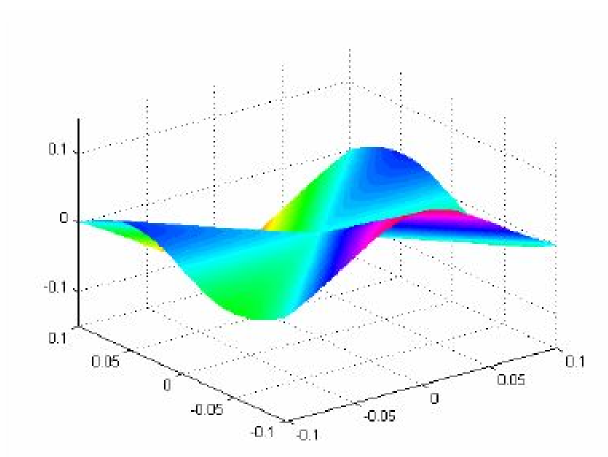
$$\frac{\partial f(x,0)}{\partial y}=x,\ x\in\mathbb{R}^2$$

Άρα $\dfrac{\partial^2 f}{\partial y\partial x}(0,0)=-1\neq 1=\dfrac{\partial^2 f}{\partial x\partial y}(0,0)$.

Οι $\dfrac{\partial^2 f}{\partial x\partial y}$, $\dfrac{\partial^2 f}{\partial y\partial x}$ συνεχείς $\mathbb{R}^2\setminus\underline{\underline{\{(0,0)\}}}$.



Η γραφική παράσταση της $f_x(x,y)=\begin{cases}\dfrac{x^4y+4x^2y^3-y^5}{(x^2+y^2)^2}, & (x,y)\neq(0,0)\\ 0 & ,(x,y)=(0,0)\end{cases}$ και της $f_x(0,y)=-y$ φαίνονται στο παρακάτω σχήμα



Θεώρημα Clairaut, Schwarz, Μεικτών Παραγώγων

Έστω $f : A(\subseteq \mathbb{R}^2) \longrightarrow \mathbb{R}$

$\vec{x_0} \in A$ τ.ω.

i) $\exists\ \dfrac{\partial^2 f}{\partial x \partial y}, \dfrac{\partial^2 f}{\partial y \partial x}$ σε $S(\vec{x_0}, \delta) \subseteq A$

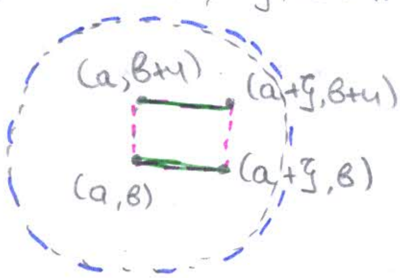
ii) Συνεχείς στο $\vec{x_0}$

Τότε $\dfrac{\partial^2 f}{\partial x \partial y}(\vec{x_0}) = \dfrac{\partial^2 f}{\partial y \partial x}(\vec{x_0})$

- Ιδιαιτέρως Εάν $f = C^2$, τότε $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$ στο A.

ΑΠΟΔ.

$\vec{x_0} = (\alpha, \beta)$, $\xi, \eta \in \mathbb{R}$ $\xi \cdot \eta \neq 0$



$\in S((a,\beta),\delta) \subseteq A$

- $F(\xi,\eta) = [f(a+\xi, \beta+\eta) - f(a+\xi,\beta)] - [f(a,\beta+\eta) - f(a,\beta)]$ (1)

$||$

$F(\xi,\eta) = [f(\alpha+\xi, \beta+\eta) - f(a,\beta+\eta)] - [f(\alpha+\xi,\beta) - f(a,\beta)]$ (2)

Ορίζουμε $g(x) = f(x, \beta+\eta) - f(x,\beta)$, $|x-\alpha| < \delta$, Παραγωγίσιμη ως προς x

$F(\xi,\eta) = g(a+\xi) - g(\alpha)$

Εφαρμόζουμε Θ.Μ.Τ $\exists\ \vartheta_1 \in (0,1)$:

$g'(a+\vartheta_1 \xi) \cdot \xi = \left[\dfrac{\partial f}{\partial x}(\alpha+\vartheta_1\xi, \beta+\eta) - \dfrac{\partial f}{\partial x}(\alpha+\vartheta_1\xi, \beta)\right]\xi =$

$\underset{\exists \vartheta_2 \in (0,1)}{\overset{ΘΜΤ}{=}} \dfrac{\partial^2 f}{\partial y \partial x}(\alpha+\vartheta_1\xi, \beta+\vartheta_2\eta) \cdot \eta\xi$.

$$\lim_{(\xi,\eta)\to(0,0)} \frac{F(\xi,\eta)}{\xi\eta} = \lim_{(\xi,\eta)\to(0,0)} \frac{\partial^2 f}{\partial y\partial x}(\alpha+\vartheta_1\xi,\, \beta+\vartheta_2\eta) \overset{\text{Υπόθ.}}{\underset{(ii)}{=\!=}} \frac{\partial^2 f}{\partial y\partial x}(\alpha,\beta) \quad (*)$$

(2) $\varphi(y) = f(\alpha+\xi,\, y) - f(\alpha,y)$, $|y-\beta|<r$.

$$F(\xi,\eta) = \varphi(\beta+\eta) - \varphi(\beta)$$

$$\overset{\exists\vartheta_3\in(0,1)}{=\!=} \varphi'(\beta+\vartheta_3\eta)\cdot\eta$$

$$= \left[\frac{\partial f}{\partial y}(\alpha+\xi,\, \beta+\vartheta_3\eta) - \frac{\partial f}{\partial y}(\alpha,\, \beta+\vartheta_3\eta)\right]\cdot\eta$$

$$\overset{\text{ΘΜΤ}}{\underset{\exists\vartheta_4\in(0,1)}{=\!=}} \frac{\partial^2 f}{\partial x\partial y}(\alpha+\vartheta_4\xi,\, \beta+\vartheta_3\eta)\cdot\xi\cdot\eta$$

$$\lim_{(\xi,\eta)\to(0,0)} \frac{F(\xi,\eta)}{\xi\eta} \overset{\text{Υπόθ}}{\underset{ii)}{=\!=}} \frac{\partial^2 f}{\partial x\partial y}(\alpha,\beta) \quad (**)$$

$$(*) = (**)$$

$$\frac{\partial^2 f}{\partial y\partial x}(\alpha,\beta) = \frac{\partial^2 f}{\partial x\partial y}(\alpha,\beta).$$

Στροβιλισμός, Απόκλιση, Τελεστής Laplace

Αστρόβιλο Διανυσματικού Πεδίου, Διανυσματικό Πεδίο Κλίσεων / Συντηρητικό Διαν. Πεδίο, Αρμονικές Συναρτήσεις

Ορισμοί.

1) $\vec{F}=(P,Q,R): A \,(\mathbb{R}^3) \longrightarrow \mathbb{R}^3$ $\exists$ μ.π. των P, Q, R

$$\operatorname{curl}\vec{F} = \operatorname{rot}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$=: \left(\frac{\partial R}{\partial y} - \frac{\partial R}{\partial z}\right)\vec{i} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$ Στροβιλισμός

• $\operatorname{curl}\vec{F}(x_0,y_0,z_0) = (0,0,0)$, Ο στροβιλισμός του $\vec{F}$ στο (x_0,y_0,z_0) είναι $\vec{0}$.

Εάν $\operatorname{curl}\vec{F}(x,y,z)=\vec{0}$ $(x,y,z)\in A$, $\vec{F}=$ αστρόβιλο Διανυσματικό Πεδίο.

• Εάν $\vec{F}(x,y,z) = (P(x,y), Q(x,y), 0)$

$$\operatorname{curl}\vec{F}(x_0,y_0,z_0) = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

2) $\vec{F}: A \longrightarrow \mathbb{R}^3$ Δ.Π.

Εάν $\exists\, f: A \longrightarrow \mathbb{R}$: $\vec{F}=\nabla f$ στο A

Τότε το $\vec{F}$ Δ.Π. κλίσεων / Συντηρητικό Διανυσματικό Πεδίο.

$V=-f$ Δυναμικό του $\vec{F}$

3) $\vec{F} = (P, Q, R)$ $\operatorname{div}\vec{F} = \nabla \cdot \vec{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (R, Q, R)$

$$= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$\operatorname{div}\vec{F} : A \to \mathbb{R}$, βαθμωτό πεδίο.

Τελεστής Laplace, Αρμονική Συνάρτηση

$f : A (\subseteq \mathbb{R}^d) \to \mathbb{R}$ $\quad f \left(\exists \frac{\partial^2 f}{\partial x^2}\right)$

$$\Delta f = \nabla \cdot \nabla f = \nabla^2 f = \frac{\partial^2 f}{\partial x_1^2} + \dots + \frac{\partial^2 f}{\partial x_d^2}$$

Εάν $\Delta f = 0$ στο A, τότε f = αρμονική συνάρτηση

Ενημερωτικά:

∇ Ανάδελτα, Del, Nabla, Gradient

$\nabla \to f : A (\subseteq \mathbb{R}^d) \to \mathbb{R}$

$\nabla \times \to \nabla \times \vec{F} = \operatorname{curl} \vec{F}$

$\nabla \cdot \to \nabla \cdot \vec{F}$ = απόκλιση

$\nabla^2 = \nabla \cdot \nabla$ Τελεστής Laplace.

Ασκήσεις

1) $\vec{F}=(P,Q,R)$, C^2 Δ.Π.

$\operatorname{div}(\operatorname{curl}\vec{F}) = \nabla\cdot(\nabla\times\vec{F}) = 0$

Λύση $\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ R & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}; -\frac{\partial R}{\partial x} + \frac{\partial P}{\partial z}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$

$$\operatorname{div}(\operatorname{curl}\vec{F}) = \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y}\right) - \frac{\partial}{\partial x}\left(\frac{\partial Q}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial R}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x}\right) - \frac{\partial}{\partial z}\left(\frac{\partial P}{\partial y}\right) \underset{=}{\overset{\text{Clairaut}}{=}} 0$$

2) (α) $\vec{F} = C^1$ Δ.Π.

$\vec{F} = \nabla f$ συντηρητικό

Τότε $\vec{F}$ είναι αστρόβιλο, $\operatorname{curl}\vec{F} = \vec{0}$

(β) $\vec{F}(x,y,z) = \left(\frac{y}{x^2+y^2}, \frac{-x}{x^2+y^2}, 0\right)$

ΝΔΟ i) $\vec{F}$ = αστρόβιλο

ii) $\vec{F}$ δεν είναι συντηρητικό.

$(x,y,z) \in \mathbb{R}^3 \setminus \{(0,0,z), z\in\mathbb{R}\}$

Λύση

(α) $\vec{F} = \nabla f$, $P = \frac{\partial f}{\partial x}$, $Q = \frac{\partial f}{\partial y}$, $R = \frac{\partial f}{\partial z}$.

$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x}$ Από Clairaut $(f = C^2)$: $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. Ανάλογα

και $\frac{\partial Q}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}$ $\qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}$, $\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$

(β) i) $\vec{F}(x,y,z) = \Big(\underset{\underset{P}{\shortparallel}}{\frac{y}{x^2+y^2}}, \underset{\underset{Q}{\shortparallel}}{-\frac{x}{x^2+y^2}}, \underset{\underset{R}{\shortparallel}}{0}\Big)$

$$\frac{\partial P}{\partial y} = \frac{(x^2+y^2) - y \cdot 2y}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\frac{\partial Q}{\partial x} = -\frac{(x^2+y^2) - x \cdot 2x}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\frac{\partial P}{\partial z} = 0 = \frac{\partial R}{\partial z}$$

Άρα $\text{curl}\, \vec{F}(x,y,z) = (0,0,0)$

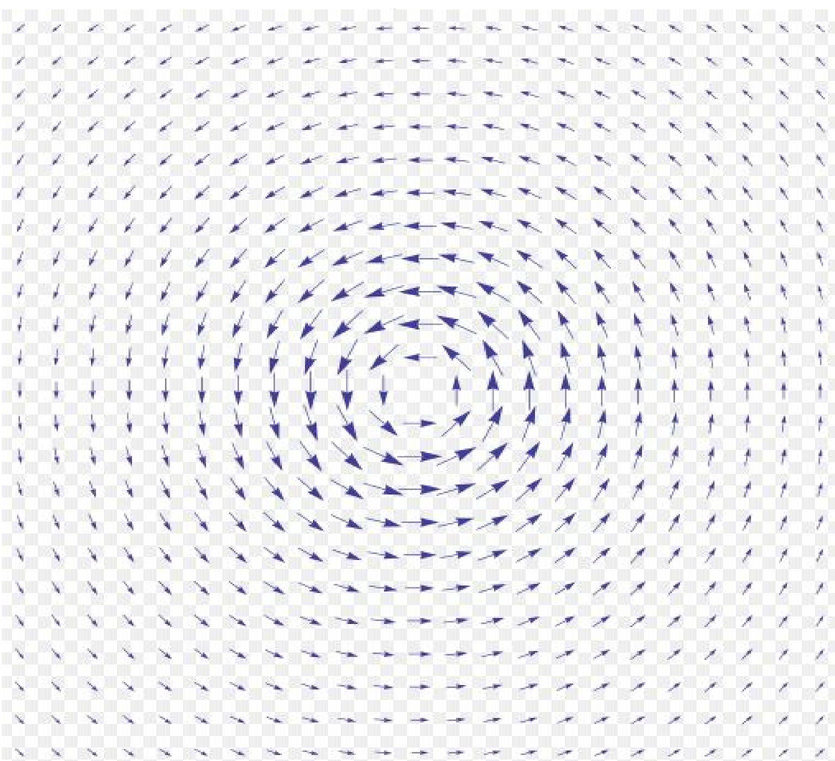
$\mathbb{R}^3 \setminus \{(0,0,z) : z \in \mathbb{R}\}$

$$\frac{\partial Q}{\partial z} = 0 = \frac{\partial R}{\partial y}$$

ii) $\vec{F}$ δεν είναι συντηρητικό, δηλαδή $\nexists f : \nabla f = \vec{F}$

→ Μάθημα 8 (ΘΜΤ Δ.Λ)

→ Θεώρημα Green



3) Έστω i) $\vec{F_1}(x,y,z) = (y, x, 0)$

ii) $\vec{F_2}(x,y,z) = (3x^2y, x^3+y, 0)$

iii) $\vec{F_3}(x,y,z) = (2xy+z-1, x^2+2yz^2, x+2y^2z-2z)$

$(x,y,z) \in \mathbb{R}^3$

ΝΔΟ α) Τα $\vec{F_\kappa}$, $\kappa=1,2,3$ είναι αστρόβιλα

β) Τα $\vec{F_\kappa}$ $\kappa=1,2,3$ είναι Δ.Π. Κλίσης / Συντηρητικά

Λύση

α) Εύκολα

β) i) $\vec{F_1}(x,y,z) = (y,x,0)$, $f: \mathbb{R}^3 \to \mathbb{R}$: $\nabla f = \vec{F}$

$\frac{\partial f}{\partial x} = y \xrightarrow[\text{ως προς } x]{\text{ολοκλήρωση}} f(x,y,z) = xy + G_1(y,z)$

$\Downarrow$

$\frac{\partial f}{\partial y} = x \longleftrightarrow \frac{\partial f}{\partial y}(x,y,z) = x + \frac{\partial}{\partial y}G_1(y,z)$, $\frac{\partial G_1(y,z)}{\partial y} = 0$

$\frac{\partial f}{\partial z} = 0$

$G_1(y,z) = G_2(z)$

Άρα $f(x,y,z) = xy + G_2(z)$

$\Downarrow$

$\frac{\partial f}{\partial z} = 0$ $\frac{\partial f}{\partial z} = G_2'(z) \Rightarrow G_2(z) = c$

$\Big\}$ $f(x,y,z) = xy + c$

ii) $\vec{F_2}(x,y,z) = (3x^2y, x^3+y, 0)$ $\vec{F_2} = \nabla f$

$\frac{\partial f}{\partial x} = 3x^2y \Longrightarrow f(x,y,z) = x^3y + G_1(y,z)$

$\Downarrow \frac{\partial}{\partial y}$

$\frac{\partial f}{\partial y} = x^3+y$ $= \frac{\partial f}{\partial y} = x^3 + \frac{\partial}{\partial y}G_1(y,z)$

$\frac{\partial f}{\partial z} = 0$

$$\frac{\partial G_1}{\partial y}(y,z) = y \xRightarrow{\int y} G_1(y,z) = \frac{1}{2}y^2 + G_2(z)$$

$$f(x,y,z) = x^3 y + \frac{1}{2}y^2 + G_2(z)$$

$$\Downarrow \frac{\partial}{\partial z}$$

$$\frac{\partial f}{\partial z} = G_2'(z) = 0 \Rightarrow G_2(z) = c$$

Άρα $f(x,y,z) = x^3 y + \frac{1}{2}y^2 + c$

iii) $\vec{F}_3(x,y,z) = (2xy + z - 1,\ x^2 + 2yz^2,\ x^2 + 2y^2 z - 2z)$

$$\frac{\partial f}{\partial x} = 2xy + z - 1 \xRightarrow{\int x} f(x,y,z) = x^2$$

$$\Downarrow \frac{\partial}{\partial y}$$

$$\frac{\partial f}{\partial x} = x^2 + 2yz^2$$

$$\frac{\partial f}{\partial y} = x^2 + \frac{\partial G_1(y,z)}{\partial y}$$

$$\frac{\partial f}{\partial z} = x + 2y^2 z - 2z$$

$$\frac{\partial G_1(y,z)}{\partial y} = 2yz^2$$

$$\Downarrow \int y$$

$$G_1(y,z) = y^2 z^2 + G_2(z)$$

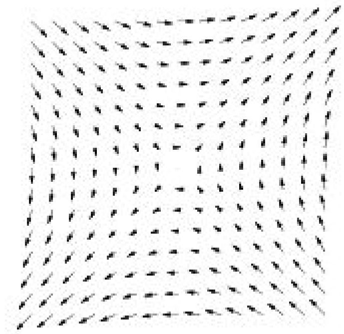
$$f(x,y,z) = x^2 y + zx - x + y^2 z^2 + G_2(z)$$

$$\Downarrow \frac{\partial}{\partial z}$$

$$\frac{\partial f}{\partial z} = x + 2y^2 z + G_2'(z) \Rightarrow G_2'(z) = -2z \Rightarrow G_2(z) = -z^2 + c$$

Άρα $f(x,y,z) = x^2 y + zx - x + y^2 z^2 - z^2 + c$

{y, x}



4) $\vec{F} = -\frac{GMm}{r^3}\vec{r}$ (Πεδίο Βαρύτητας).

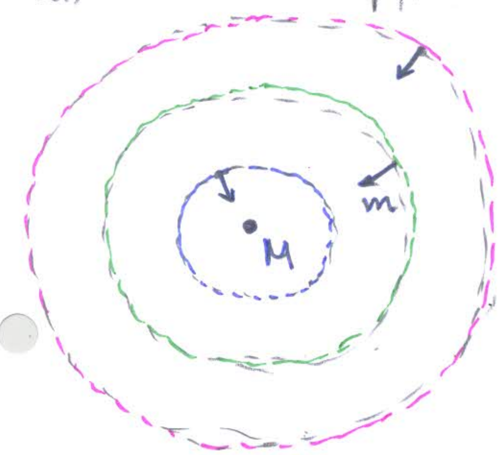
$\vec{r}=(x,y,z)$, $r=\|\vec{r}\| = \sqrt{x^2+y^2+z^2}$, $(x,y,z)\in \mathbb{R}^3\setminus\{(0,0,0)\}$

i) Για Δυναμικό του Νεύτωνα. $V = -\frac{GMm}{r}$

Ισχύει $\vec{F} = -\nabla V$. Άρα το Δ.Π. είναι συντηρητικό

ii) $\vec{F}$ αστρόβιλο

iii) V είναι αρμονική συνάρτηση, $\operatorname{div}\vec{F} = 0$



Λύση

$$\vec{F}(x,y,z) = -GMm\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}}\right)$$

$$V(x,y,z) = \frac{GMm}{(x^2+y^2+z^2)^{1/2}} \quad \frac{\partial V}{\partial x} = GMm\frac{x}{(x^2+y^2+z^2)^{3/2}}$$

Τελικά, $\vec{F} = -\nabla V$ / ii) $\vec{F}$ = συντηρητικό $\xRightarrow{\text{Ασκ. 1}}$ $\vec{F}$ αστρόβιλο.

iii) $$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 .$$

$$\varphi(x,y,z) = \frac{V(x,y,z)}{-GMm} = \frac{1}{(x^2+y^2+z^2)^{1/2}} .$$

$$\frac{\partial \varphi}{\partial x} = \frac{x}{(x^2+y^2+z^2)^{3/2}} .$$

$$\frac{\partial^2 \varphi}{\partial x^2} = \frac{(x^2+y^2+z^2)(x^2+y^2+z^2)^{1/2} - x(2x)\,3/2\,(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3} .$$

$$\Longrightarrow \quad \frac{\partial^2 \varphi}{\partial x^2} = \frac{(x^2+y^2+z^2) - 3x^2}{(x^2+y^2+z^2)^{5/2}}$$

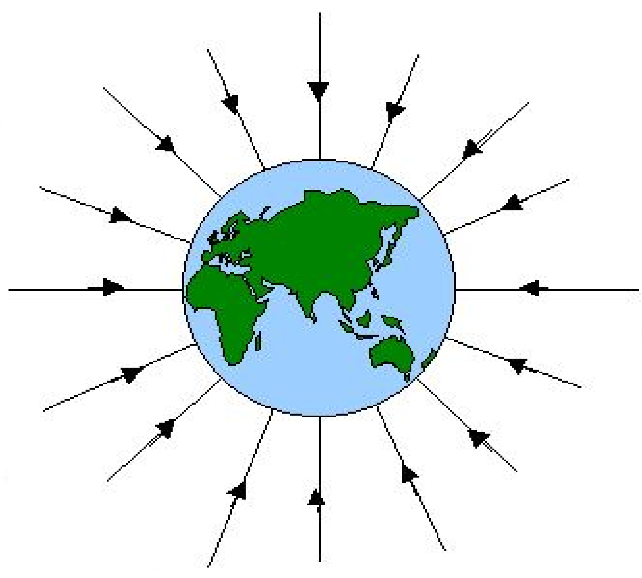
$$\frac{\partial^2 \varphi}{\partial y^2} = \frac{(x^2+y^2+z^2) - 3y^2}{(x^2+y^2+z^2)^{5/2}}$$

$$\frac{\partial^2 \varphi}{\partial z^2} = \frac{(x^2+y^2+z^2) - 3z^2}{(x^2+y^2+z^2)^{5/2}}$$

Αρα $$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = \frac{3(x^2+y^2+z^2) - 3(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{5/2}} = 0$$

$$\operatorname{div}\vec{F} = \operatorname{div}(-\nabla V) = -\frac{\partial^2 V}{\partial x^2} - \frac{\partial^2 V}{\partial y^2} - \frac{\partial^2 V}{\partial z^2} = 0 .$$

$(\operatorname{div}(\nabla f) = \Delta f)$



Αρα, το Δ.Π. Βαρύτητας είναι αστρόβιλο, συντηρητικό.
Η δυναμική ενέργεια V είναι Αρμονική Συνάρτηση και είναι ασυμπίεστο ($\operatorname{div}\vec{F} = 0$)

Σημαντική Παρατήρηση !

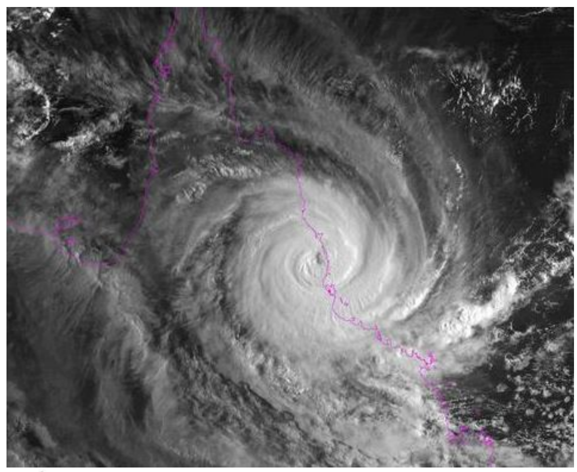
- $\vec{F}(x,y,z) = \left(\frac{y}{x^2+y^2}, \frac{-x}{x^2+y^2}, 0 \right)$ $(x,y,z) \in \mathbb{R}^3 \setminus \{(0,0,z), z \in \mathbb{R}\}$

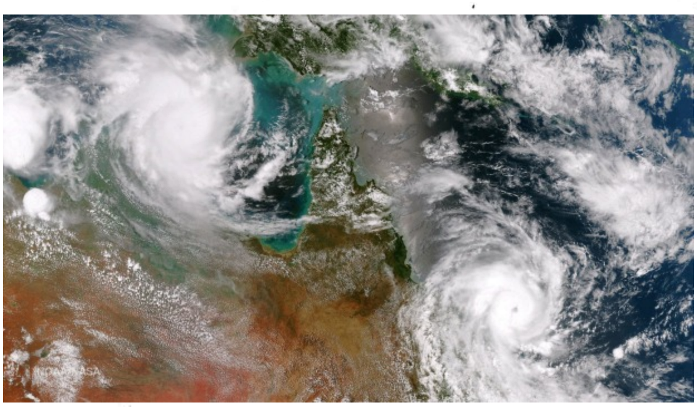
είναι C^∞, αστρόβιλο, αλλά όχι συντηρητικό.

- $\vec{F}(x,y,z) = \left(\frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}} \right)$ $(x,y,z) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$

είναι C^∞, αστρόβιλο + συντηρητικό.

Συμπέρασμα. Σημασία έχει ΚΑΙ το είδος του Πεδίου Ορισμού, όχι η συνάρτηση ΜΟΝΟΝ.





Κυκλώνες στην Αυστραλία Μάρτιος 2015