Μάθημα 19 (23/11/2020)

Χρήσιμες Επιφάνειες $\mathbb{R}^3$

Παραμετρημένη επιφάνεια

$\vec{r} : I \times J \longrightarrow \mathbb{R}^3$, $(u,v) \longrightarrow \vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$

Συνήθως $I, J \subseteq \mathbb{R}$ διαστήματα

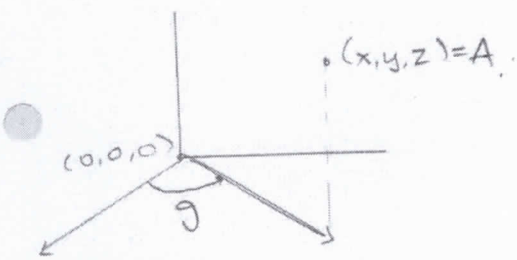
- $u = u_0$ / $\vec{r}(u_0, v)$ παραμετρημένη καμπύλη στον $\mathbb{R}^3$

  $v = v_0$ / $\vec{r}(u, v_0)$ " " " "

Κυλινδρικός Μετασχηματισμός

$\vec{T}(\vartheta, z) = (r\,συν\vartheta, r\,ημ\vartheta, z)$ $\quad (r, \vartheta, z) \in (0, +\infty) \times [0, 2\pi) \times \mathbb{R}$

1-1, επί του $\mathbb{R}^3 \setminus \{(0,0,z), z \in \mathbb{R}\}$



$r = \sqrt{x^2 + y^2}$

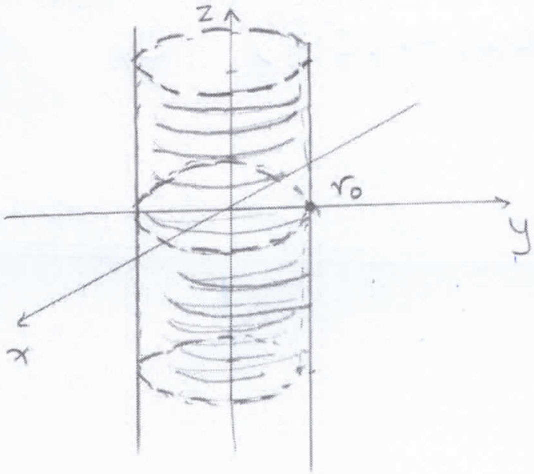
Παραδείγματα επιφανειών σε Κυλινδρικές, Καρτεσιανές εξισώσεις.

- Επιφάνεια του $\mathbb{R}^3$ $\quad r = r_0$ $(r_0 = σταθ, r_0 > 0)$

$\vec{r}(\vartheta, z) = (r_0\,συν\vartheta, r\,ημ\vartheta, z)$, $\vartheta \in [0, 2\pi)$, $z \in \mathbb{R}$

$\begin{cases} x = r_0\,συν\vartheta \\ y = r_0\,ημ\vartheta \end{cases} \Rightarrow \begin{cases} \sqrt{x^2 + y^2} = r_0 \\ z \in \mathbb{R} \end{cases}$ / $\{(x,y,z) : \sqrt{x^2+y^2} = r_0, z \in \mathbb{R}\}$

$= \{(x,y,z) : -r_0 \le x \le r_0, -\sqrt{r_0^2 - x^2} \le y \le \sqrt{r_0^2 - x^2}$



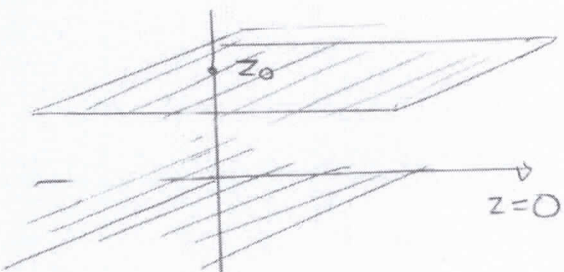
Εσωτερικό της επιφάνειας:

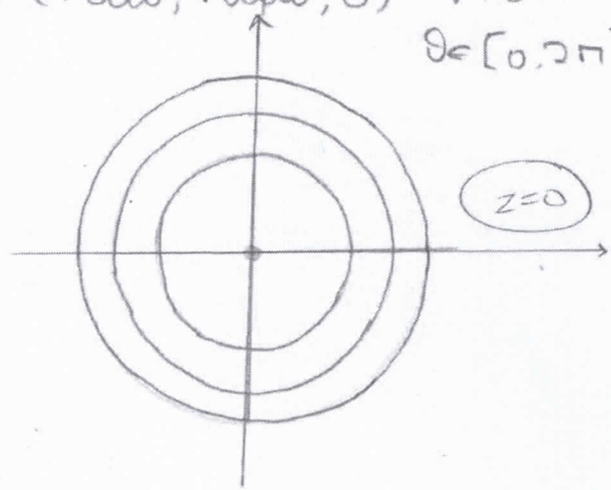
$\{(x,y,z): \sqrt{x^2+y^2} \leq r_0,\ z\in\mathbb{R}\}$

- Επιφάνεια $z=0$ στον $\mathbb{R}^3$

$\{(x,y,z)\in\mathbb{R}^3 : x,y\in\mathbb{R} : z=0\}$

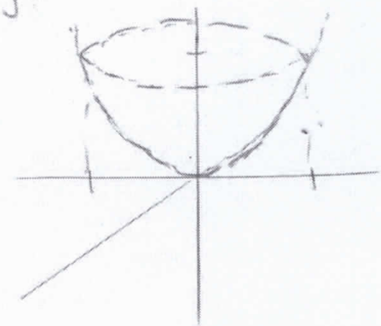
Διανυσματική εξίσωση $\vec{r}(r,\theta)=(r\sigma\upsilon\nu\theta, r\eta\mu\theta, 0)$ $r>0$
$\theta\in[0,2\pi)$





- Επιφάνεια $\vec{r}(r,\theta)=(r\sigma\upsilon\nu\theta, r\eta\mu\theta, r^2)$, $r>0$, $\theta\in[0,2\pi)$

$$\left.\begin{cases} \left.\begin{aligned} x&=r\sigma\upsilon\nu\theta \\ y&=r\eta\mu\theta \end{aligned}\right\} \Rightarrow x^2+y^2=r^2 \\ z=r^2 \end{cases}\right\} \Rightarrow x^2+y^2=z$$



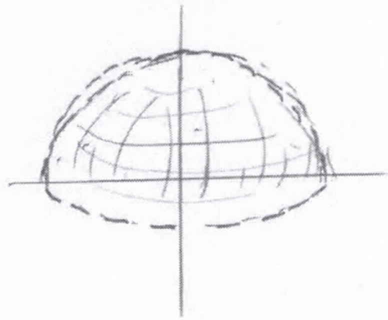
- $\{(x,y,z): x\in\mathbb{R}, y\in\mathbb{R}, z=x^2+y^2\}$

• Επιφάνεια $\vec{r}(r,\vartheta)=(r\sigma\upsilon\nu\vartheta, r\eta\mu\vartheta, \sqrt{a^2-r^2})$, $r\in(0,a]$, $\vartheta\in[0,2\pi)$ του $\mathbb{R}^3$

$(a>0)$.

$$\left.\begin{array}{l} \left.\begin{array}{l} x=r\sigma\upsilon\nu\vartheta \\ y=r\eta\mu\vartheta \end{array}\right\} \Longrightarrow x^2+y^2=r^2 \\ z=\sqrt{a^2-r^2} \end{array}\right\} \Longrightarrow z=\sqrt{a^2-(x^2+y^2)}$$

$$x^2+y^2+z^2=a^2,\ z\geq 0$$

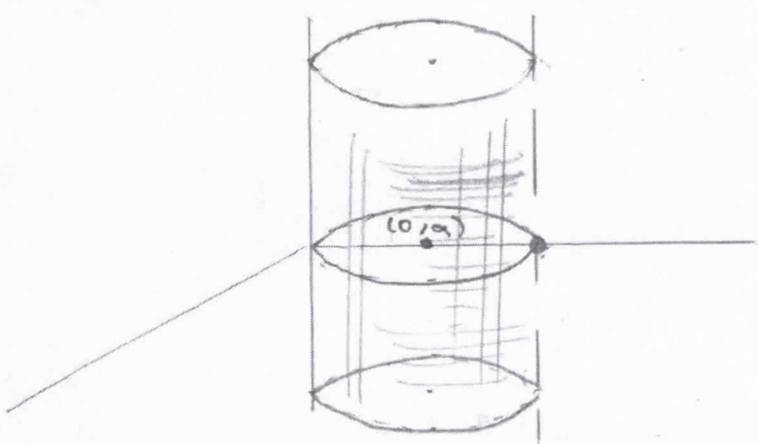


• Επιφάνεια $\{(x,y,z):\ x^2+y^2=2ay,\ z\in\mathbb{R}\}$ $\quad x^2+y^2-2ay=0$

$(a>0)$ $\quad x^2+(y-a)^2=a^2$

$$\left\{\begin{array}{l} x=r\sigma\upsilon\nu\vartheta \\ y=r\eta\mu\vartheta \\ z=z \end{array}\right\} x^2+y^2=r^2=2a(r\eta\mu\vartheta). \quad \begin{array}{|l} r=2a\eta\mu\vartheta \\ \hline z=z \end{array}$$

$$\vec{r}(\vartheta,z)=((2a\eta\mu\vartheta)\sigma\upsilon\nu\vartheta, (2a\eta\mu\vartheta)\eta\mu\vartheta, z)$$



Ασκήσεις (Παλιά Θέματα)

1) $I = \iiint_B \sqrt{x^2+y^2}\, dx\,dy\,dz$, $V(B)$?

$B = \{ (x,y,z) : -1 \le x \le 1, \underbrace{-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}}, 0 \le z \le 1-(x^2+y^2) \}$

$\hookrightarrow x^2+y^2 \le 1,\ r^2 \le 1,\ 0 < r \le 1$

$0 \le z \le 1-r^2$

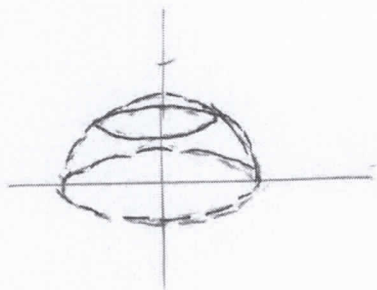
Λύση.
$$\begin{cases} x = r\,συνθ \\ y = r\,ημθ \\ z = z \end{cases}$$

$$I = \int_0^{2\pi}\left(\int_0^1\left(\int_0^{1-r^2} \sqrt{r^2}\cdot r\,dz\right)dr\right)d\theta = 2\pi\int_0^1 r^2(1-r^2)\,dr =$$
$$= 2\pi\left(\frac{1}{3}-\frac{1}{5}\right)$$

$$V(B) = \int_0^{2\pi}\int_0^1\int_0^{1-r^2} 1\cdot r\,dz\,dr\,d\theta$$

Με σχήμα:

$z = 1-(x^2+y^2)$



$\{ (r,\theta,z) : 0 \le r \le 1,\ 0 \le \theta \le 2\pi,\ 0 \le z \le 1-r^2 \}$

2) $V(B)$, B στην 1η στερεά γωνία $(x,y,z\geq 0)$ φράσσεται από τις επιφάνειες του $\mathbb{R}^3$, $x^2+y^2=1$, $x^2+y^2=4$
$z^2=x^2+y^2$, $z=0$

Λύση $x=r\sigma\upsilon\nu\vartheta$, $y=r\eta\mu\vartheta$, $z=z$

Επιφάνειες $x^2+y^2=1$ κ.σ $r=1$

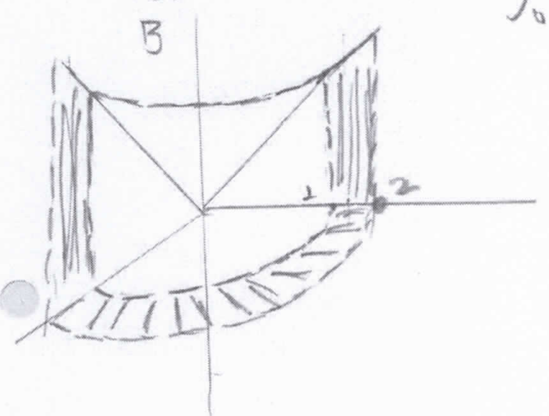
$x^2+y^2=4$ κ.σ $r=2$

$z^2=x^2+y^2$ κ.σ $z^2=r^2$, $z=r$

$z=0$

$$\left\{\begin{array}{ll} x\geq 0 & \sigma\upsilon\nu\vartheta\geq 0 \\ y\geq 0 & \eta\mu\vartheta\geq 0 \end{array}\right\} \Longleftrightarrow \vartheta\in[0,\pi/2].$$

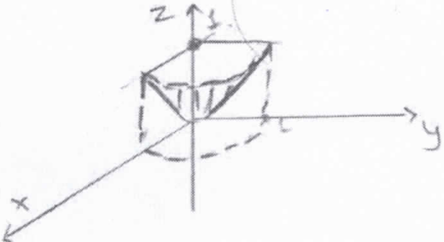
$$V(B)=\iiint 1\,dx\,dy\,dz=\int_0^{\frac{\pi}{2}}\int_1^2\left(\int_0^r r\,dz\right)dr\,d\vartheta=\frac{7\pi}{6}.$$



3) $$I=\int_0^1\int_0^{\sqrt{1-x^2}}\left(\int_{\sqrt{x^2+y^2}}^1 dz\right)dy\,dx$$

Λύση $\left\{(x,y,z):\ 0\leq x\leq 1,\ 0\leq y\leq\sqrt{1-x^2},\ \sqrt{x^2+y^2}\leq z\leq 1\right\}$

Με σχήμα.



$$\left\{\begin{array}{l} 0\leq\vartheta\leq\frac{\pi}{2} \\ 0\leq r\leq 1 \\ r\leq z\leq 1. \end{array}\right.$$

$x = r\,συνθ$
$y = r\,ημθ$
$z = z$

$x, y \geq 0 \Longrightarrow συνθ, ημθ \geq 0 \Rightarrow θ \in [0, \frac{\pi}{2}]$

$0 \leq y \leq \sqrt{1-x^2}, \quad y^2 \leq 1 - x^2, \quad x^2+y^2 \leq 1, \quad r^2 \leq 1, \quad r \leq 1$

$\sqrt{r^2} \leq z \leq 1, \quad r \leq z \leq 1$

$$I = \int_0^{\frac{\pi}{2}} \int_0^1 \left(\int_r^1 r\,dz \right) dr\,dθ$$

4) i) Όγκος Σφαίρας $x^2+y^2+z^2 \leq a^2 \ (α > 0)$

ii) $V(B)$, B φράσσεται από τις επιφάνειες

$x^2+y^2+z^2 = 4, \quad x^2+y^2 = 1, \quad z = 0, \quad x, y, z \geq 0$

Λύση

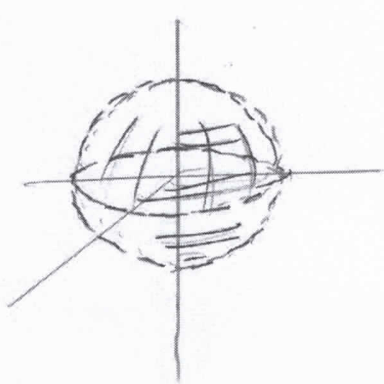
i) $x^2+y^2+z^2 = a^2$

$x = r \cdot συνθ$
$y = r \cdot ημθ$
$z = z$

$r^2 + z^2 = a^2 \ / \ -\sqrt{a^2+r^2} \leq z \leq \sqrt{a^2+r^2}, \ r \in (0, α]$

$$V(B) = \int_0^{2\pi} \left(\int_0^a \left(\int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r \cdot dz \right) dr \right) dθ =$$

$$2\pi \left(\int_0^a 2r\sqrt{a^2-r^2} \right) dr = 2\pi \left[-\frac{2}{3} (a^2-r^2)^{3/2} \Big|_0^a \right] = \frac{4\pi}{3} a^3$$



ii)

$x^2+y^2+z^2=4$ — $r^2+z^2=4$

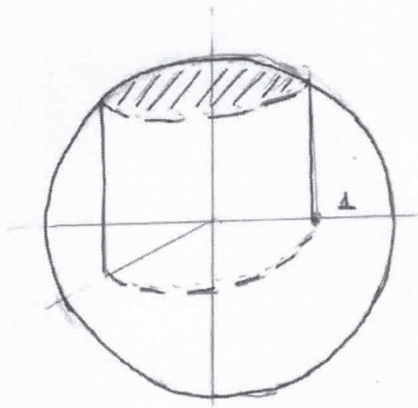
$x^2+y^2=1$ — $r^2=1$, $r=1$

$z=0$

$x,y,z \geq 0$, $x,y\geq 0$, $z\geq 0$

$\Rightarrow$ $0\leq z\leq \sqrt{4-r^2}$, $0\leq r\leq 1$, $\theta\in[0,\frac{\pi}{2}]$

$$V(B)=\int_0^{\frac{\pi}{2}}\int_0^1\left(\int_0^{\sqrt{4-r^2}} r\,dz\right)dr\,d\theta=\frac{\pi}{2}$$



5) $V(B)$

B εντός της επιφάνειας $x^2+y^2=2x$

φράσσεται από $z=x^2+y^2$

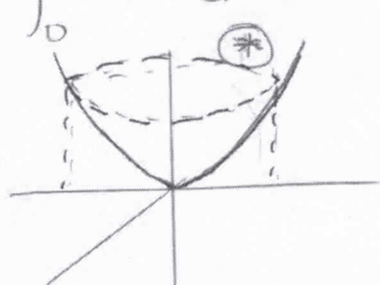
$z=0$

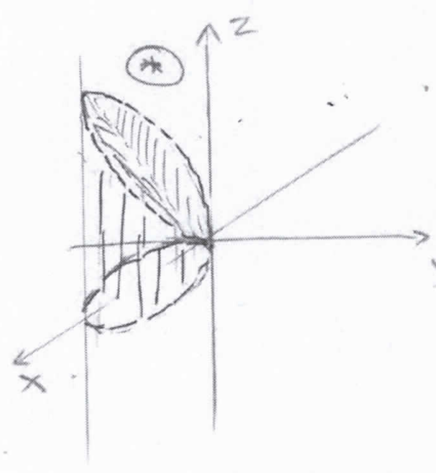
Λύση

$x^2+y^2=2x$, $r^2=2r\sigma\upsilon\nu\theta$, $r=2\sigma\upsilon\nu\theta$, $\sigma\upsilon\nu\theta\geq 0$.

$z=x^2+y^2$, $z=r^2$

$$V(B)=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\sigma\upsilon\nu\theta}\int_0^{r^2} r\,dz\,dr\,d\theta=\dots$$





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$$V(B) = 2\int_0^{\pi/2}\int_0^{2\sigma\upsilon\nu\vartheta}\int_0^{r^2} r\,dz\,dr\,d\vartheta$$

6) $I = \iiint_{\Omega} \sqrt{x^2+y^2}\,dx\,dy\,dz$

Ω φράσσεται από $x^2+y^2=2y$, $z=x^2+y^2$, $z\geq 0$

Λύση

$x^2+y^2=2y$ $\quad r^2=2r\eta\mu\vartheta$, $r=2\eta\mu\vartheta$, $\eta\mu\vartheta\geq 0$, $\vartheta\in[0,\pi]$

$(x^2+(y-1)^2=1)$

$z=x^2+y^2$, $z=r^2$

$$I=\int_0^{\pi}\int_0^{2\eta\mu\vartheta}\left(\int_0^{r^2} r\cdot r\,dz\right)dr\,d\vartheta$$

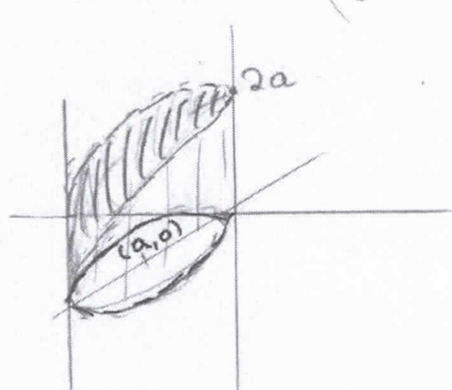
7) $V(B)$, $B=\{(x,y,z) : x^2+y^2\leq 2ax,\ x^2+y^2+z^2\leq 4a^2,\ z\geq 0\}$

$(\alpha>0)$

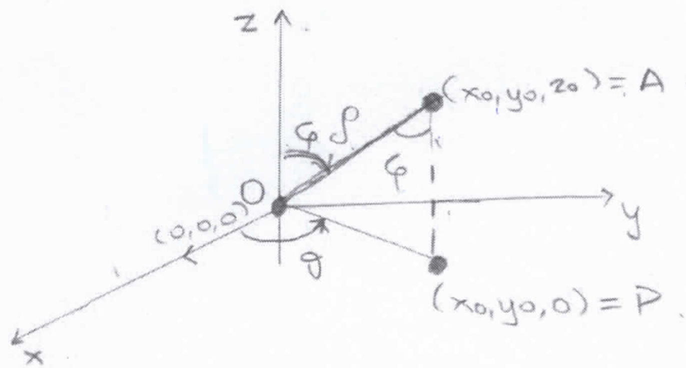
Λύση $\quad x^2+y^2=2ax$, $r^2=2ar\sigma\upsilon\nu\vartheta$, $r=2a\sigma\upsilon\nu\vartheta$, $\sigma\upsilon\nu\vartheta\geq 0$

$x^2+y^2+z^2=4a^2$, $r^2+z^2=4a^2$, $z=\sqrt{4a^2-r^2}$

$$V(B)=2\int_0^{\pi/2}\int_0^{2a\sigma\upsilon\nu\vartheta}\left(\int_0^{\sqrt{4a^2-r^2}} r\,dz\right)dr\,d\vartheta=\frac{8a^3}{3}\left(\pi-\frac{4}{3}\right)$$



Σφαιρικός Μετασχηματισμός



$\rho = \sqrt{x^2+y^2+z^2} > 0$

$P = (x_0, y_0, 0)$, $\vartheta = \angle(O\vec{x}, O\vec{P})$

$\varphi = \angle(O\vec{z}, O\vec{A})$

$\eta\mu\varphi = \frac{(OP)}{\rho}$

$x = (OP)\,\sigma\upsilon\nu\vartheta = (\rho\,\eta\mu\varphi)\,\sigma\upsilon\nu\vartheta$

$y = (OP)\,\eta\mu\vartheta = (\rho\,\eta\mu\varphi)\,\eta\mu\vartheta$

$z = \rho\,\sigma\upsilon\nu\varphi$, $\varphi \in [0, \pi]$

$\vartheta \in [0, 2\pi)$

$\rho > 0$.

$\vec{T}(\rho, \vartheta, \varphi) = (\rho\,\eta\mu\varphi\,\sigma\upsilon\nu\vartheta, \rho\,\eta\mu\varphi\,\eta\mu\vartheta, \rho\,\sigma\upsilon\nu\varphi)$

1-1, επί του $\mathbb{R}^3 \setminus \{(0,0,z)\}$

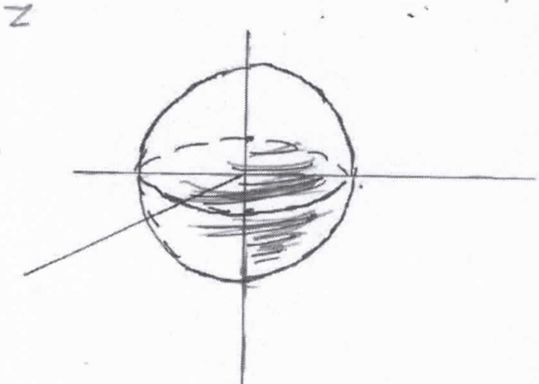
$(\rho, \vartheta, \varphi) \in (0, +\infty) \cup [0, 2\pi) \cup (0, \pi)$

Παραδείγματα Επιφανειών σε Σφαιρικές, Καρτεσιανές συντετ.

- $\rho = \rho_0$ ($\rho_0 > 0$, σταθερός)

$\vec{r}(\vartheta, \varphi) = (\underbrace{\rho_0\,\eta\mu\varphi\,\sigma\upsilon\nu\vartheta}_{x''}, \underbrace{\rho_0\,\eta\mu\varphi\,\eta\mu\vartheta}_{y''}, \underbrace{\rho_0\,\sigma\upsilon\nu\varphi}_{z''})$ (Παραμετρημένη Επιφάνεια)

$$\left.\begin{aligned} x^2+y^2 &= \rho_0^2\,\eta\mu^2\varphi \\ z^2 &= \rho_0^2\,\sigma\upsilon\nu^2\varphi \end{aligned}\right\} \Longrightarrow x^2+y^2+z^2 = \rho_0^2$$



• $\varphi = \varphi_0 \in [0, \pi]$.

$\vec{r}(\rho,\theta) = (\rho\,\sigma\upsilon\nu\theta\,\eta\mu\varphi_0,\ \rho\,\eta\mu\theta\,\eta\mu\varphi_0,\ \rho\,\sigma\upsilon\nu\varphi_0)$

(Παραμετρημένη εξίσωση επιφάνειας)

$$\begin{cases} x = \rho\,\sigma\upsilon\nu\theta\,\eta\mu\varphi_0 \\ y = \rho\,\eta\mu\theta\,\eta\mu\varphi_0 \\ z = \rho\,\sigma\upsilon\nu\theta \end{cases} \qquad \begin{array}{l} \rho > 0 \\ \theta \in [0, 2\pi) \end{array}$$

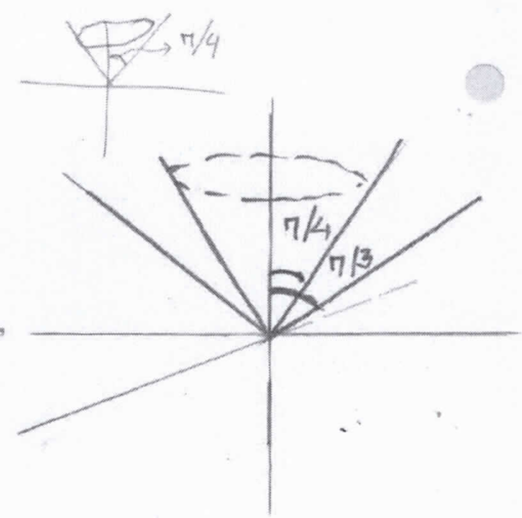
• $\varphi_0 \in (0, \frac{\pi}{2})$, $z > 0$

$$\left.\begin{array}{l} x^2+y^2 = \rho^2\,\eta\mu^2\varphi_0 \\ z = \rho \cdot \sigma\upsilon\nu\varphi_0 \end{array}\right| \ \rho = \frac{z}{\sigma\upsilon\nu\varphi_0} \ \Big\} \Longrightarrow x^2+y^2 = (\varepsilon\varphi^2\varphi_0)\,z^2$$

$$z = \frac{1}{(\varepsilon\varphi\varphi_0)}\sqrt{(x^2+y^2)}$$

(Κώνος $z \geq 0$)

• $\varphi_0 \in (-\frac{\pi}{2}, 0)$: $z = +\frac{1}{(\varepsilon\varphi\varphi_0)}\sqrt{x^2+y^2}$ $\left(\begin{array}{c}\text{Κώνος} \\ z \leq 0\end{array}\right)$



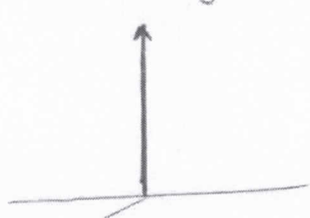
πχ. $\varphi = \frac{\pi}{4}$: $z = \sqrt{x^2+y^2}$

$\varphi = \pi - \frac{\pi}{4}$: $z = -\sqrt{x^2+y^2}$

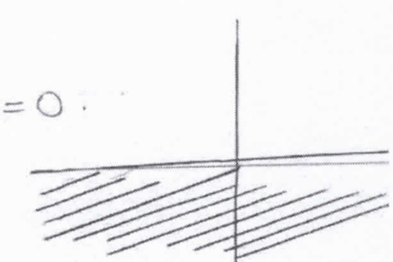
$\varphi = \frac{\pi}{6}$, $\varepsilon\varphi\frac{\pi}{6} = \frac{1}{\sqrt{3}}$: $z = \sqrt{3(x^2+y^2)}$

$\varphi = \frac{\pi}{3}$, $\varepsilon\varphi\frac{\pi}{3} = \sqrt{3}$, $z = \sqrt{\frac{x^2+y^2}{3}}$

$\varphi = 0$, $z = \rho > 0$



$\varphi = \frac{\pi}{2}$, $z = 0$.

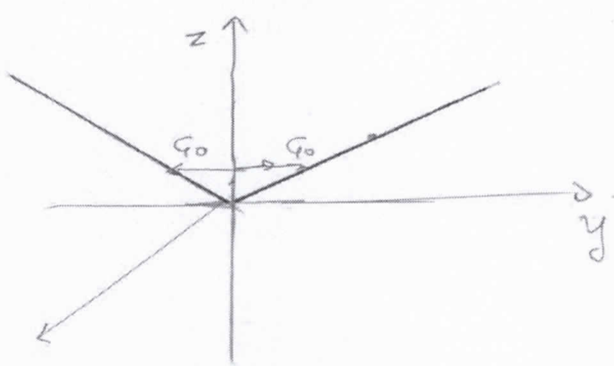


• $c z = \sqrt{x^2+y^2} \quad (c>0)$

$$\begin{array}{l|l} x = \rho\,\sigma\upsilon\nu\theta\,\eta\mu\varphi & c\rho\,\sigma\upsilon\nu\varphi = \sqrt{\rho^2\eta\mu^2\varphi(\sigma\upsilon\nu^2\theta+\eta\mu^2\theta)} = \\ y = \rho\,\eta\mu\theta\,\eta\mu\varphi & \\ z = \rho\,\sigma\upsilon\nu\varphi & \quad = \sqrt{\rho^2\eta\mu^2\varphi} = \rho\,\eta\mu\varphi \end{array}$$

$\Longrightarrow \varepsilon\varphi\varphi = c, \quad \varphi = \tau o\xi\varepsilon\varphi(c) (= \varphi_0)$

$\varphi = \varphi_0$



Ορίζουσα του Σφαιρικού Μετασχηματισμού.

$(\Delta x \Delta y \Delta z = |\det J_{\vec{T}}| \Delta\rho\, \Delta\theta\, \Delta\varphi$

$\vec{T}(\rho,\theta,\varphi) = (\rho\,\sigma\upsilon\nu\theta\,\eta\mu\varphi,\ \rho\,\eta\mu\theta\,\eta\mu\varphi,\ \rho\,\sigma\upsilon\nu\varphi)$

$$\det J_{\vec{T}}(\rho,\theta,\varphi) = \begin{vmatrix} \sigma\upsilon\nu\theta\,\eta\mu\varphi & -\rho\,\eta\mu\theta\,\eta\mu\varphi & \rho\,\sigma\upsilon\nu\theta\,\sigma\upsilon\nu\varphi \\ \eta\mu\theta\,\eta\mu\varphi & \rho\,\sigma\upsilon\nu\theta\,\eta\mu\varphi & \rho\,\eta\mu\theta\,\sigma\upsilon\nu\varphi \\ \sigma\upsilon\nu\varphi & 0 & -\rho\,\eta\mu\varphi \end{vmatrix}$$

$= \sigma\upsilon\nu\theta\,\eta\mu\varphi(-\rho^2\sigma\upsilon\nu\theta\,\eta\mu^2\vartheta) - \eta\mu\theta\,\eta\mu\varphi(+\rho^2\eta\mu\theta\,\eta\mu^2\varphi) +$

$+ (-\rho^2\eta\mu^2\theta\,\eta\mu\varphi\,\sigma\upsilon\nu\varphi - \rho^2\sigma\upsilon\nu^2\theta\,\eta\mu\varphi\,\sigma\upsilon\nu\varphi) =$

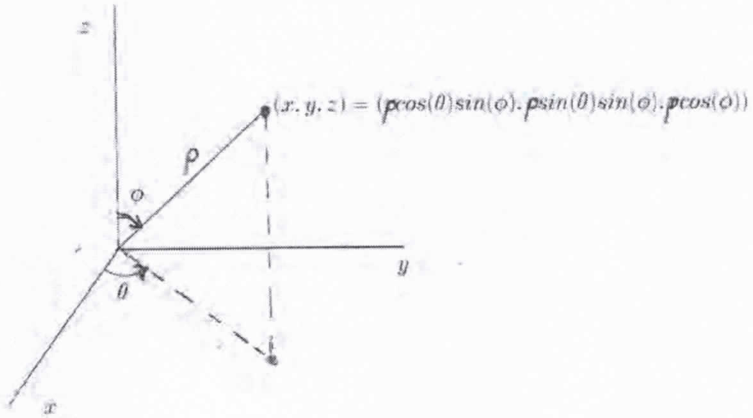
$= -\rho^2[\sigma\upsilon\nu^2\theta\,\eta\mu^3\varphi + \eta\mu^2\theta\,\eta\mu^3\varphi + \sigma\upsilon\nu\varphi(\eta\mu\varphi\,\sigma\upsilon\nu\varphi)] =$

$= -\rho^2[\eta\mu^3\varphi + \eta\mu\varphi\,\sigma\upsilon\nu^2\varphi] = -\rho^2\eta\mu\varphi$. Άρα $|\det J_{\vec{T}}(\rho,\theta,\varphi)| = \rho^2\eta\mu\varphi > 0$

II. Σφαιρικές συντεταγμένες (ρ, θ, φ)

Ο $\vec{T}: [0,\infty) \times \boldsymbol{R} \times \boldsymbol{R} \to \boldsymbol{R}^3$ (επί), με $\vec{T}(\rho,\theta,\varphi) = (\rho\sigma\upsilon\nu\theta\ \eta\mu\varphi, \rho\eta\mu\theta\ \eta\mu\varphi, \rho\sigma\upsilon\nu\varphi)$ καλείται σφαιρικός μετασχηματισμός και τα ρ, θ, φ σφαιρικές συντεταγμένες.

Ο περιορισμός του $\vec{T}: (0,\infty) \times [0,2\pi) \times (0,\pi) \to \boldsymbol{R}^3 \backslash \{(0,0,z), z \in \boldsymbol{R}\}$ είναι 1-1 και επί.

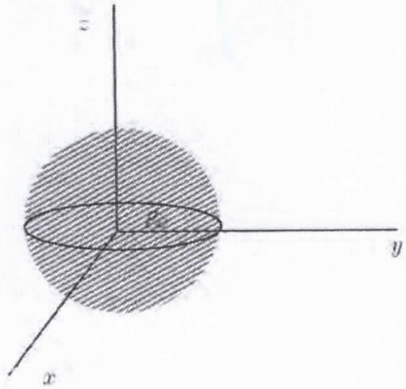


Σχέση Καρτεσιανών Σφαιρικών συντεταγμένων

$$\begin{cases} x = \rho\sigma\upsilon\nu\theta\eta\mu\varphi \\ y = \rho\eta\mu\theta\eta\mu\varphi \\ z = \rho\sigma\upsilon\nu\varphi \end{cases}, (\rho,\theta,\varphi) \in [0,\infty) \times \boldsymbol{R} \times \boldsymbol{R}$$

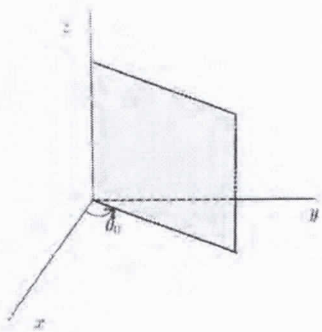
$$\begin{cases} \rho = \sqrt{x^2+y^2+z^2} \\ \varepsilon\varphi\theta = \dfrac{y}{x}\ \alpha\nu\ x \neq 0.\ A\nu\ x = 0: \theta = \dfrac{\pi}{2}\ \gamma\iota\alpha\ y > 0, \theta = \dfrac{3\pi}{2}\ \gamma\iota\alpha\ y < 0 \qquad (x,y,z) \in \boldsymbol{R}^3 \backslash \{(0,0,z): z \in \boldsymbol{R}\} \\ \sigma\upsilon\nu\varphi = \dfrac{z}{\sqrt{x^2+y^2+z^2}} \end{cases}$$

Σφαιρικές επιφάνειες $\rho=\rho_0>0, \theta=\theta_0, \varphi=\varphi_0\in(0,\pi)$ (στο καρτεσιανό σύστημα)



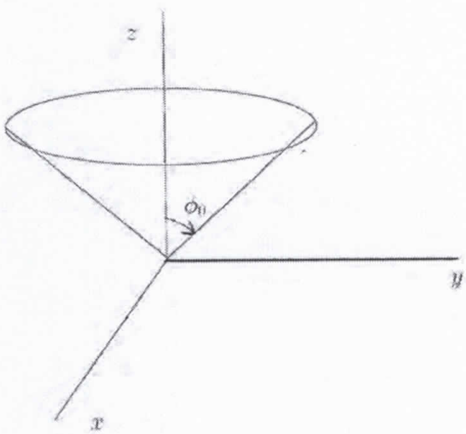
S_1: $\rho=\rho_0$, σφαίρα $\{(x,y,z)\in \boldsymbol{R}^3: x^2+y^2+z^2=\rho_0^2\}$

$$\left\{\begin{matrix}\vec{r}_1(\theta,\varphi)=(\rho_0\,συνθ\,ημφ, \rho_0\,ημθ\,ημφ, \rho_0\,συνφ),\\ \theta\in[0,2\pi], \varphi\in[0,\pi]\end{matrix}\right\}$$



S_2: $\theta=\theta_0$, ημιεπίπεδο

$$\left\{\begin{matrix}\vec{r}_2(\rho,\varphi)=(\rho συν\theta_0\,ημφ, \rho ημ\theta_0 ημφ, \rho συνφ),\\ \rho\geq\ 0, \varphi\in[0,\pi]\end{matrix}\right\}$$

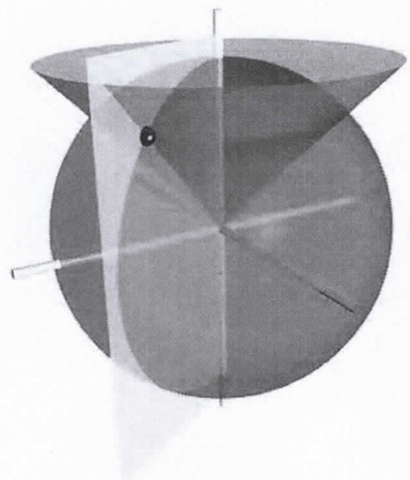


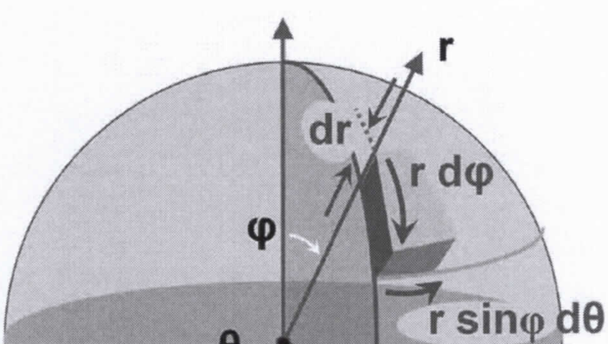
S_3: $\varphi=\varphi_0$, κώνος

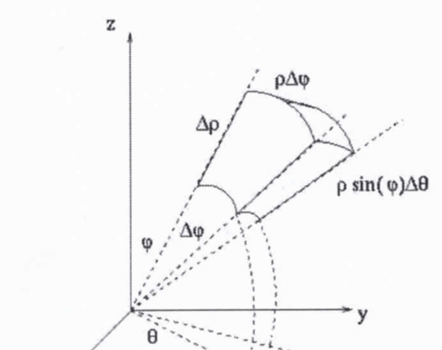
$$\left\{\begin{matrix}\vec{r}_3(\rho,\theta)=(\rho συνθ\,ημ\varphi_0, \rho ημθ\,ημ\varphi_0, \rho συν\varphi_0),\\ \rho\geq\ 0, \theta\in[0,2\pi]\end{matrix}\right\}$$

Το $(x_0, y_0, z_0) = (\rho_0 συνθ_0\, ημφ_0, \rho_0\, ημθ_0\, ημφ_0, \rho_0\, συνφ_0)$ είναι το σημείο τομής των επιφανειών

$$S_1 : \rho = \rho_0, S_2 : \theta = \theta_0, S_3 : \varphi = \varphi_0$$







Στοιχειώδεις Όγκοι
ΔxΔyΔz≈ρ²ημφdρdθdφ

Ασκήσεις

1. Να υπολογιστεί ο όγκος της σφαίρας $B: x^2+y^2 \le a^2 \ (a>0)$

Λύση

$x = \rho\,\text{συν}\theta\,\text{ημ}\varphi$

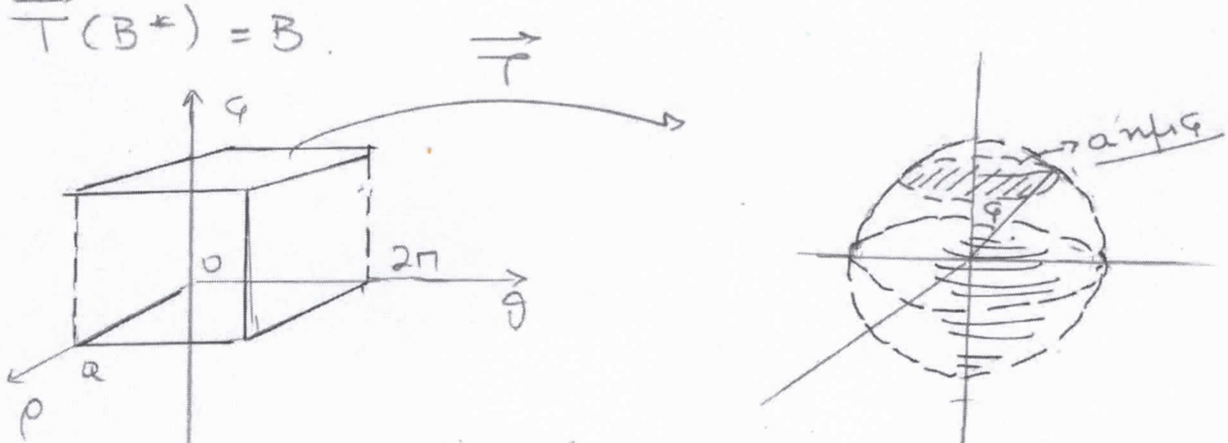
$y = \rho\,\text{ημ}\theta\,\text{ημ}\varphi$

$z = \rho\,\text{συν}\varphi$

$x^2+y^2+z^2 = \rho^2 \le a^2$

$B^* = \{(\rho,\theta,\varphi) : 0 < \rho \le a^2,\ 0 \le \theta < 2\pi,\ 0 \le \varphi \le 2\pi\}$

$\vec{T}(B^*) = B.$



$$V(B) = \int_0^{2\pi}\int_0^{\pi}\left(\int_0^{a} \rho^2\,\text{ημ}\varphi\, d\rho\right) d\varphi\, d\theta =$$

$$= 2\pi\int_0^{\pi}\left(\frac{a^3}{3}\,\text{ημ}\varphi\right) d\varphi = \frac{2\pi a^3}{3}\left[-\text{συν}\varphi\Big|_0^{\pi}\right] =$$

$$= \frac{2\pi a^3}{3}\left(-(-1)+1\right) = \frac{4\pi a^3}{3}$$