Μάθημα 18 (20/11/2020)

Τριπλό Ολοκλήρωμα

Απλά Σύνολα στον $\mathbb{R}^3$

- xy-απλό, $B=\left\{(x,y,z)\in\mathbb{R}^3 : a\le x\le b,\ \varphi_1(x)\le y\le \varphi_2(x),\ h_1(x,y)\le z\le h_2(x,y)\right\}$

$\varphi_1,\varphi_2 : [a,b]\longrightarrow \mathbb{R}$ συνεχείς

$h_1,h_2 : D=\{(x,y) : a\le x\le b,\ \varphi_1(x)\le y\le \varphi_2(x)\}\longrightarrow\mathbb{R}$

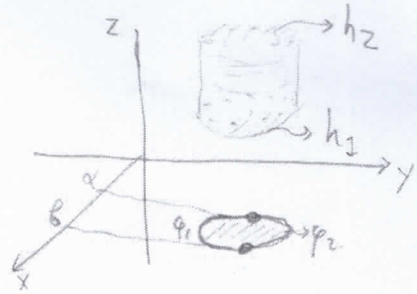
Δηλαδή, η προβολή του B στο επίπεδο xy είναι x-απλό σύνολο.

Ανάλογα, yx-απλό / xz, zx, yz, zy-απλά.

Θ. Fubini

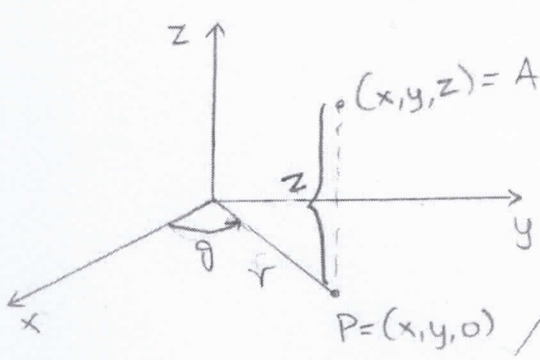
$f: B\longrightarrow\mathbb{R}$, B xy-απλό, f = συνεχής

Τότε: $\exists \iiint_B f = \int_a^b\left(\int_{\varphi_1(x)}^{\varphi_2(x)}\left(\int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z)\,dz\right)dy\right)dx$



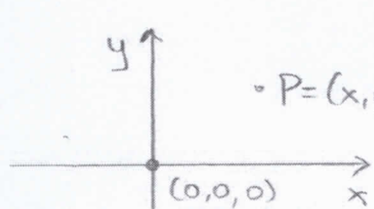
$V_3(B)=\iiint_B 1$.

Κυλινδρικές Συντεταγμένες



$\vec{T}:(0,+\infty)\times[0,2\pi)\times\mathbb{R}\longrightarrow\mathbb{R}^3\setminus\{(0,0,z) : z\in\mathbb{R}\}$

$\vec{T}(r,\theta,z)=(r\cos\theta, r\sin\theta, z)$
$=(x,y,z)$

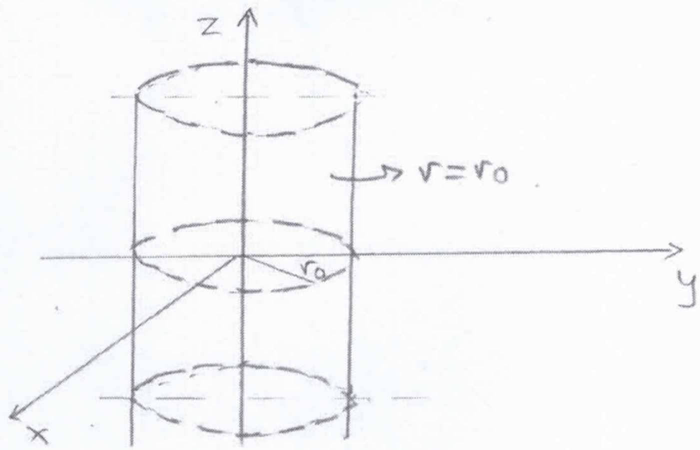


$x=r\cos\theta$, $y=r\sin\theta$ | $r\in(0,+\infty)$, $\theta\in[0,2\pi)$ / $r=\sqrt{x^2+y^2}>0$, $\theta=\sphericalangle(\vec{Ox},\vec{OP})$

Έστω (r_0, ϑ_0, z_0)

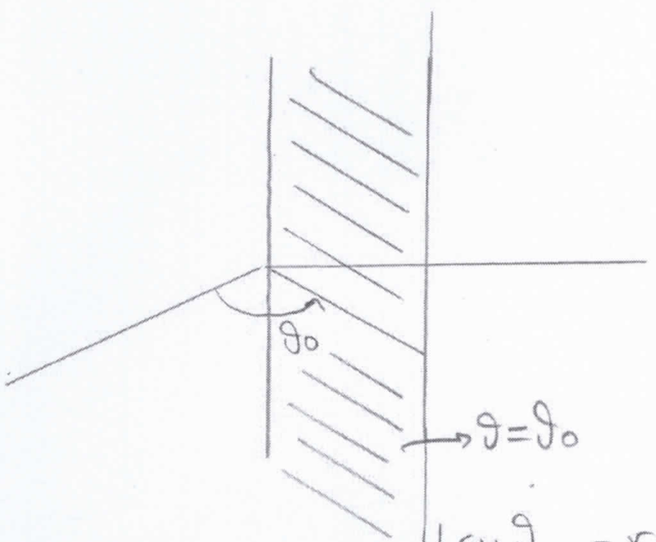
• $(x,y,z) \in \mathbb{R}^3 : r = r_0 \quad (\vartheta, z)$

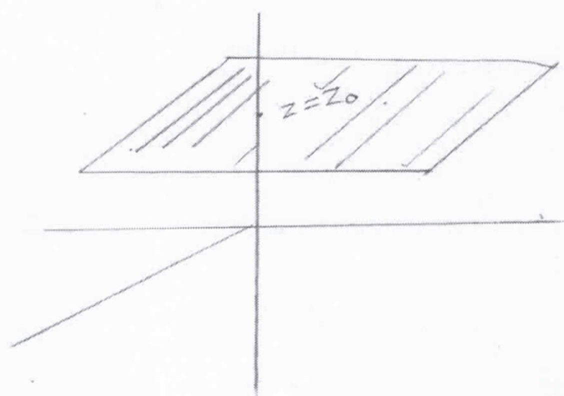
$: \sqrt{x^2+y^2} = r_0, \ z \in \mathbb{R}$



$\vartheta = \vartheta_0 \ (r>0, \ z \in \mathbb{R})$

$(x,y,z) : \vartheta = \vartheta_0 \ (r>0, z \in \mathbb{R})$



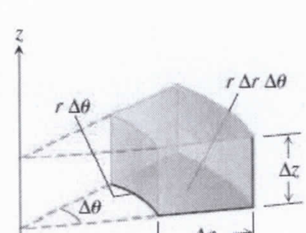


$$\det \vec{T}(r,\vartheta,z) = \begin{vmatrix} \sigma\upsilon\nu\vartheta & -r\eta\mu\vartheta & 0 \\ \eta\mu\vartheta & r\sigma\upsilon\nu\vartheta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\sigma\upsilon\nu^2\vartheta + r\eta\mu^2\vartheta = r > 0$$

$\vec{T}(r,\vartheta,z) = (r\sigma\upsilon\nu\vartheta, r\eta\mu\vartheta, z)$

$f: B \longrightarrow \mathbb{R}, \ \vec{T}(B^*) = B \ / \ \iiint\limits_{B} f(x,y,z)\, \underline{dx\,dy\,dz} =$

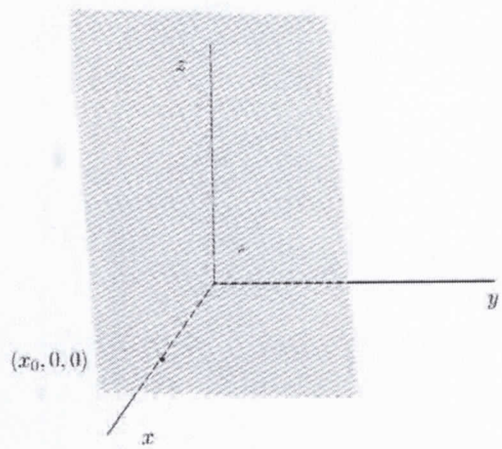
$= \iiint\limits_{B^*} f(r\sigma\upsilon\nu\vartheta, r\eta\mu\vartheta, z) \cdot \underline{r\, dr\, d\vartheta\, dz}.$



Συστήματα συντεταγμένων στον R^3

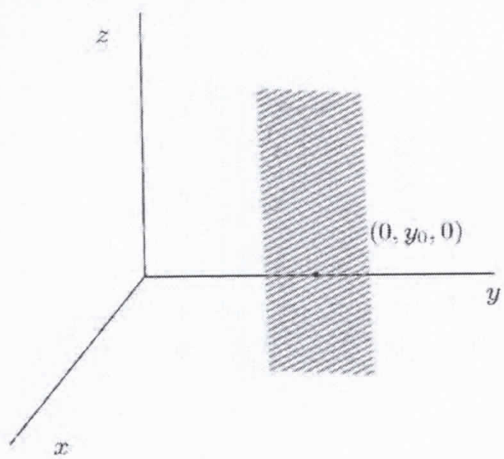
I. Καρτεσιανές συντεταγμένες (x, y, z)

Καρτεσιανές επιφάνειες (επίπεδα)) $x = x_0, y = y_0, z = z_0$.



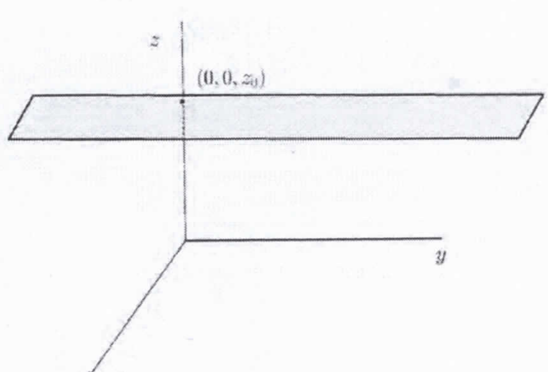
$$S_1: x = x_0, \{(x, y, z) \in \boldsymbol{R}^3: x = x_0, y, z \in \boldsymbol{R}\}$$

$$\{\vec{r}_1(y, z) = (x_0, y, z), y, z \in \boldsymbol{R}\}$$



$$S_2: y = y_0, \{(x, y, z) \in \boldsymbol{R}^3: y = y_0, x, z \in \boldsymbol{R}\}$$

$$\{\vec{r}_2(x, z) = (x, y_0, z), x, z \in \boldsymbol{R}\}$$

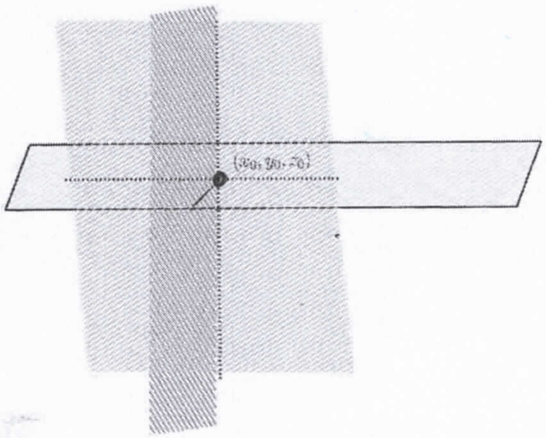


$$S_3: z = z_0, \{(x, y, z) \in \boldsymbol{R}^3: z = z_0, x, y \in \boldsymbol{R}\}$$

$$\{\vec{r}_3(x, y) = (x, y, z_0), x, y \in \boldsymbol{R}\}$$

Το (x_0, y_0, z_0) είναι το σημείο τομής των επιφανειών

$$S_1 : x = x_0,\ S_2 : y = y_0,\ S_3 : z = z_0$$



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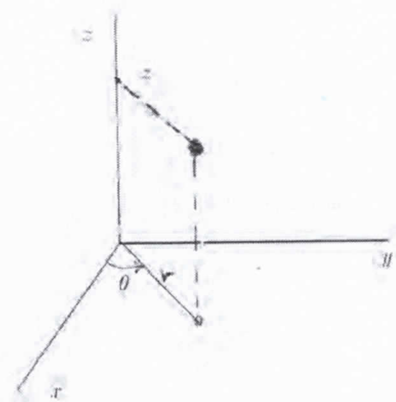
II. Κυλινδρικές συντεταγμένες (r, θ, z)

Ο $\vec{T}: [0, \infty) \times \boldsymbol{R} \times \boldsymbol{R} \to \boldsymbol{R}^3$ (επί), με $\vec{T}(r, \theta, z) = (r\sigma\upsilon\nu\theta, r\eta\mu\theta, z)$ καλείται κυλινδρικός μετασχηματισμός και τα r, θ, z κυλινδρικές συντεταγμένες.

Ο περιορισμός του $\vec{T}: (0, \infty) \times [0,2\pi) \times \boldsymbol{R} \to \boldsymbol{R}^3 \backslash \{(0,0,z), z \in \boldsymbol{R}\}$ είναι 1-1 και επί.



Σχέση Καρτεσιανών Κυλινδρικών συντεταγμένων.

$$\begin{cases} x = r\ \sigma\upsilon\nu\theta \\ y = r\ \eta\mu\theta \\ z = z \end{cases} \qquad (r, \theta, z) \in [0, \infty) \times \boldsymbol{R} \times \boldsymbol{R}$$

$$\begin{cases} r = \sqrt{x^2 + y^2} \\ \varepsilon\varphi\theta = \frac{y}{x} \ \ \alpha\nu\ x \neq 0. A\nu\ x = 0: \theta = \frac{\pi}{2} \ \ \gamma\iota\alpha\ y > 0, \theta = \frac{3\pi}{2} \ \ \gamma\iota\alpha\ y < 0 \\ z = z \end{cases} \quad (x, y, z) \in \boldsymbol{R}^3 \backslash \{(0,0,z)\}$$

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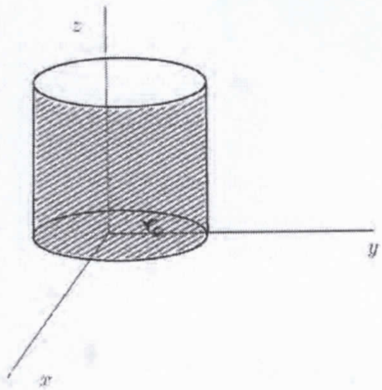
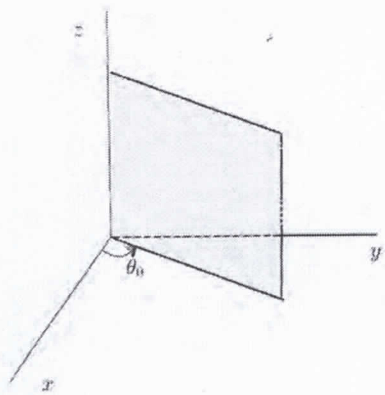
<u>Κυλινδρικές επιφάνειες $r = r_0(> 0), \theta = \theta_0, z = z_0$ (στο καρτεσιανό σύστημα)</u>



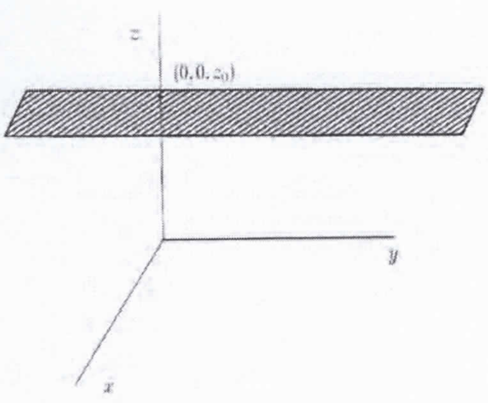
$S_1: r = r_0$, κύλινδρος $\{(x, y, z) \in \boldsymbol{R}^3 : x^2 + y^2 = r_0^2\}$

$\{\vec{r}_1(\theta, z) = (r_0 συν\theta, r_0 ημ\theta, z), \theta \in [0,2\pi], z \in \boldsymbol{R}\}$



$S_2: \theta = \theta_0$, ημιεπίπεδο

$\{\vec{r}_2(r, z) = (r συν\theta_0, r ημ\theta_0, z), r \geq 0, z \in \boldsymbol{R}\}$



$S_3: z = z_0$, επίπεδο $\{(x, y, z) \in \boldsymbol{R}^3 : z = z_0, (x, y) \in \boldsymbol{R}^2\}$

$\{\vec{r}_3(r, \theta) = (r συν\theta, r ημ\theta, z_0), r > 0, \theta \in [0,2\pi]\}$

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Το $(x_0, y_0, z_0) = (r_0 \eta\mu\theta_0, r_0 \sigma\upsilon\nu\theta_0, z_0)$ είναι το σημείο τομής των επιφανειών

$$S_1 : r = r_0, S_2 : \theta = \theta_0, S_3 : z = z_0$$