ΚΕΦ. 5

ΠΕΡΙΕΧΟΜΕΝΑ

Ⓐ Κανόνας της Αλυσίδος / Αλυσιδωτή Παραγώγιση / Διαφ. σύνθεσης 2 συναρτήσεων
Θ. Μέσης Τιμής Δ.Λ.
(βλ. Θ. ενδιαμέσων τιμών)

Ⓑ Θ. Αντιστρόφου

Ⓖ Θ. Πεπλεγμένης Συνάρτησης

Ⓓ Εφ. Ευθεία, εφ. Επίπεδο

ΕΦΑΡΜΟΓΕΣ ΤΟΥ ΔΙΑΦΟΡΙΚΟΥ (κεφ. 5+6)

8ο Μάθημα
26/10/20

Ⓐ $f : A(\subseteq \mathbb{R}^n) \to \mathbb{R}$, f είναι $\underline{\underline{C^1}}$

$\iff \dfrac{\partial f}{\partial x_i}$, $i=1,2,\dots,n$ είναι συνεχείς στο A

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "\vec{g}"] \arrow[dr, "\vec{f}\circ\vec{g}"'] & \mathbb{R}^m \arrow[d, "\vec{f}"] \\
 & \mathbb{R}^l
\end{tikzcd}$$

Τι ζητάμε;

? Αν η $\vec{g}$ διαφορίσιμη, $\vec{f}$ διαφορίσιμη είναι η $\vec{f}\circ\vec{g}$? διαφορίσιμη; Ποιο είναι το διαφορικό της $\vec{f}\circ\vec{g}$; ?

Τι συμβαίνει για $n=m=l=1$;

Εάν $g:\mathbb{R}\to\mathbb{R}$, $\exists\ g'(x_0)$ $(x_0\in\mathbb{R})$
$f:\mathbb{R}\to\mathbb{R}$, $\exists\ f'(g(x_0))$. Τότε

$$\begin{tikzcd}
{} \arrow[r, "g"] \arrow[dr, "f\circ g"'] & {} \arrow[d, "f"] \\
 & {}
\end{tikzcd}$$

i) Υπάρχει η $(f\circ g)'(x_0)$ (*)

ii) $(f\circ g)'(x_0) = f'(g(x_0))\cdot g'(x_0)$

- $d(f\circ g)(x_0) : \mathbb{R}\to\mathbb{R}$, Γραμμική με "πίνακα" $((f\circ g)'(x_0))$ ①
- $dg(x_0) : \mathbb{R}\to\mathbb{R}$, Γραμμική με "πίνακα" $(g'(x_0))$ ②
- $df(g(x_0)) : \mathbb{R}\to\mathbb{R}$, Γραμμική με "πίνακα" $(f'(g(x_0)))$ ③

Από την (*) έχουμε

$$\boxed{d(f\circ g)(x_0) = df(g(x_0))\circ dg(x_0)}$$

$\updownarrow$ ① $(f\circ g)'(x_0)$ $\quad$ $\updownarrow$ ② $f'(g(x_0))$ $\quad$ $\updownarrow$ ③ $g'(x_0)$

Θεώρημα (Αλυσίδας - Διαφόρισης - σύνθεσης Συναρτ.)

Έστω $\vec{g}: A \to \mathbb{R}^m$, $\vec{x_0} \in A$ και $\exists\ d\vec{g}(\vec{x_0})$
όπου $\vec{x_0} \in A\ (\subseteq \mathbb{R}^n)$.
Επιπλέον θεωρούμε $\vec{f}: B(\subseteq \vec{g}(A)) \to \mathbb{R}^l$ και
$\exists\ d\vec{f}(\vec{g}(\vec{x_0}))$. Τότε

i) Υπάρχει το $d(\vec{f}\circ\vec{g})(\vec{x_0})$

$$d(\vec{f}\circ\vec{g})(\vec{x_0}) = d\vec{f}(g(\vec{x_0}))\circ d\vec{g}(\vec{x_0})$$

πίνακας Jacobi

ii) $J_{\vec{f}\circ\vec{g}}(\vec{x_0}) = J_{\vec{f}}(\vec{g}(\vec{x_0}))\cdot J_{\vec{g}}(\vec{x_0})$

Απόδ. → Παραλείπεται (είναι διαδικαστική)

ΚΑΝΟΝΑΣ ΑΛΥΣΙΔΑΣ

$\vec{g} = (g_1, g_2, \dots, g_n)$, $\vec{x_0} \in A\ (\subseteq \mathbb{R}^n)$
$\vec{f} = (f_1, f_2, \dots, f_l)$

$\vec{h} = \vec{f}\circ\vec{g} : A\ (\subseteq \mathbb{R}^n) \to \mathbb{R}^l$
$\vec{h} = (h_1, h_2, \dots, h_l)$

$$\begin{tikzcd}
(x_1, x_2, \dots, x_n) \arrow[r, "\vec{g}"] \arrow[dr, "\vec{h} = \vec{f}\circ\vec{g}"'] & (t_1, t_2, \dots, t_m) \arrow[d, "\vec{f}"] \\
& \mathbb{R}^l
\end{tikzcd}$$

Γράφουμε την (ii) (του θεωρήματος)

$$\begin{pmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \dots & \frac{\partial h_1}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial h_l}{\partial x_1} & \frac{\partial h_l}{\partial x_2} & \dots & \frac{\partial h_l}{\partial x_n} \end{pmatrix}(\vec{x_0}) = \begin{pmatrix} \frac{\partial f_1}{\partial t_1} & \frac{\partial f_1}{\partial t_2}, & \dots, & \frac{\partial f_1}{\partial t_m} \\ \vdots & \vdots & & \\ \frac{\partial f_l}{\partial t_1} & \frac{\partial f_l}{\partial t_2} & \dots & \frac{\partial f_l}{\partial t_m} \end{pmatrix}(g(\vec{x_0})) \cdot \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \dots & \frac{\partial g_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial g_m}{\partial x_1} & \dots & \frac{\partial g_m}{\partial x_n} \end{pmatrix}(\vec{x_0})$$

$$\frac{\partial h_1}{\partial x_1}(\vec{x_0}) = \frac{\partial f_1}{\partial t_1}\frac{\partial g_1}{\partial x_1} + \frac{\partial f_1}{\partial t_2}\cdot\frac{\partial g_2}{\partial x_1} + \dots + \frac{\partial f_1}{\partial t_m}\cdot\frac{\partial g_m}{\partial x_1} =$$

$$= \sum_{k=1}^{m} \frac{\partial f_1}{\partial t_k}(\vec{g}(\vec{x_0}))\cdot\frac{\partial g_k(\vec{x_0})}{\partial x_1}$$

Γενικά $\boxed{\frac{\partial h_i}{\partial x_j}(\vec{x_0}) = \sum_{k=1}^{m} \frac{\partial f_i(\vec{g}(\vec{x_0}))}{\partial t_k} \cdot \frac{\partial g_k(\vec{x_0})}{\partial x_j}, \quad \begin{matrix} i=1,2,\dots,\ell \\ j=1,2,\dots n \end{matrix}}$

Ειδική Περίπτωση !!

$n=1, \quad m \geq 2, \quad \ell=1$

$$\begin{tikzcd} \mathbb{R}\,(t) \arrow[r, "\vec{r}"] \arrow[dr, "f\circ\vec{r}"'] & \vec{r}(t) \in \underset{(x_1,x_2,\dots,x_m)}{\mathbb{R}^m} \arrow[d, "f"] \\ & \mathbb{R} \end{tikzcd}$$

$\vec{r} : I \to \mathbb{R}^m$, $I=$ ανοικτό διάστημα του $\mathbb{R}$

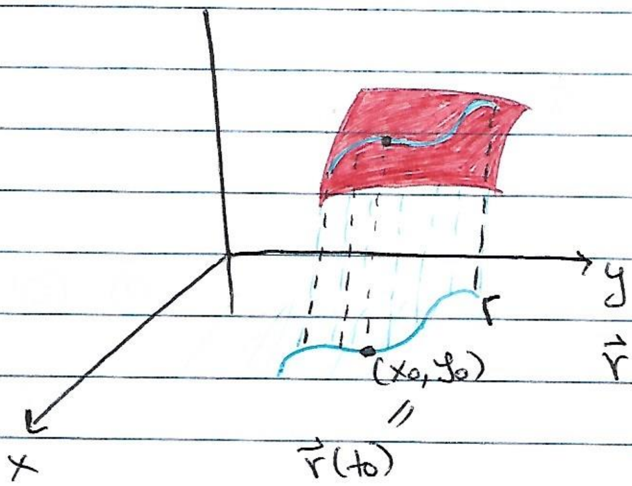
$\vec{r}(t) = (r_1(t), \dots, r_m(t))$

$\vec{r}\,'(t_0) = (r_1'(t_0), \dots, r_m'(t_0))$

$$\nabla f(\vec{r}(t_0)) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_m}\right)(\vec{r}(t_0))$$

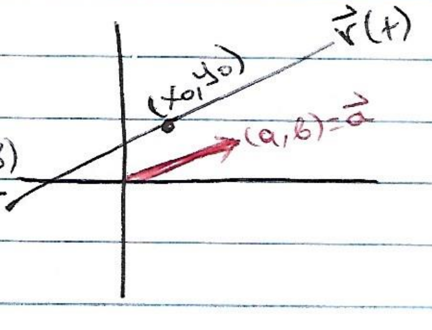
Τότε κανόνας Αλυσίδας δίνει:

$$\boxed{\frac{d\,(f\circ\vec{r})}{dt}(t_0) = \nabla f(\vec{r}(t_0)\cdot\vec{r}\,'(t_0)}$$



$\vec{r}(t_0) = (x_0, y_0)$

- Για $\vec{r}(t) = (x_0, y_0) + t\,(a, b)$, $\|(a,b)\| = 1$



$$\left.\frac{d(f\circ\vec{r})(t)}{dt}\right|_{t=0} = \lim_{t\to0}\frac{f(x_0+ta, y_0+tb) - f(a,b)}{t}$$

$$= D_{\vec{a}} f(x_0, y_0) = \nabla f(x_0, y_0)\cdot(a, b)$$

ΑΣΚΗΣΕΙΣ

1) $f(u,v) = (u+v, u\cdot v)$ $\quad \begin{pmatrix} f_1(u,v) = u+v \\ f_2(u,v) = u\cdot v \end{pmatrix}$

$g(x,y) = (x^2+y^2, x^3, x+y)$ $\quad \begin{pmatrix} g_1(x,y) = x^2+y^2 \\ g_2(x,y) = x^3 \\ g_3(x,y) = x+y \end{pmatrix}$

$\vec{h} = \vec{g}\circ\vec{f}$ $(=h_1, h_2, h_3)$

$$\begin{tikzcd}(u,v) \arrow[r, "\vec{f}"] \arrow[dr, "\vec{h}"'] & \mathbb{R}^2(x,y) \arrow[d, "\vec{g}"] \\ & \mathbb{R}^3\end{tikzcd}$$

Να υπολ. $\dfrac{\partial h_1}{\partial u}$, $\dfrac{\partial h_3}{\partial v}$

κατ' ευθείαν και με καν. Αλυσίδας

Λύση

$\vec{h}(u,v) = \vec{g}(u+v, u\cdot v) =$
$= ((u+v)^2 + u^2v^2, (u+v)^3, (u+v)+uv)$

$h_1(u,v) = (u+v)^2 + u^2+v^2$
$h_3(u,v) = u+v+uv$

$\dfrac{\partial h_1}{\partial u} = 2(u+v) + 2u\cdot v^2$ ①

$\dfrac{\partial h_3}{\partial v} = 1+u$ ①

Ισχύει ①, ②

κανόνας Αλυσίδας

$$\frac{\partial h_1}{\partial u} = \frac{\partial g_1}{\partial x}\cdot\frac{\partial f_1}{\partial u} + \frac{\partial g_1}{\partial y}\cdot\frac{\partial f_2}{\partial u}$$

$$= 2x\Big|_{(u+v,uv)} \cdot 1 + 2y\Big|_{(u+v,uv)} \cdot \cancel{v} = 2(u+v) + 2uv^2$$ ②

$$\bullet\ \frac{\partial h_3}{\partial v}=\frac{\partial g_3}{\partial x}\,\frac{\partial f_1}{\partial u}+\frac{\partial g_3}{\partial y}\,\frac{\partial f_2}{\partial v}=$$

$$=1\cdot 1+1\cdot u=1+u \quad (2')$$

Ισχύει η ισότης των (1'), (2')

|| *Ας γίνει εξάσκηση σε Ασκήσεις με Επαλήθευση || του κανόνα της Αλυσίδας.

2) $\hat{f}:\mathbb{R}^2\to\mathbb{R}$ διαφ. $\vec{g}(r,\varphi)=(r\cdot \text{συν}\varphi,\ r\cdot \text{ημ}\varphi)$

$f=\hat{f}\circ\vec{g}$, $f(r,\varphi)=\hat{f}(r\cdot\text{συν}\varphi,\ r\cdot\text{ημ}\varphi)$

ΝΔΟ $\|\nabla\hat{f}\|^2=\left(\frac{\partial f}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial f}{\partial \varphi}\right)^2$

Λύση

$$\begin{tikzcd}
(r,\varphi) \arrow[r, "\vec{g}"] \arrow[dr, "f"'] & (x,y)=(r\,\text{συν}\varphi,\ r\,\text{ημ}\varphi) \arrow[d, "\hat{f}"] \\
 & \mathbb{R}
\end{tikzcd}$$

$$\bullet\ \frac{\partial f}{\partial r}=\frac{\partial \hat{f}}{\partial x}\,\frac{\partial x}{\partial r}+\frac{\partial \hat{f}}{\partial y}\,\frac{\partial y}{\partial r}$$

$$\frac{\partial f}{\partial r}=\frac{\partial \hat{f}}{\partial x}\cdot(\text{συν}\varphi)+\frac{\partial \hat{f}}{\partial y}\cdot(\text{ημ}\varphi) \quad ①$$

$$\bullet\ \frac{\partial f}{\partial \varphi}=\frac{\partial \hat{f}}{\partial x}\cdot\frac{\partial x}{\partial \varphi}+\frac{\partial \hat{f}}{\partial y}\cdot\frac{\partial y}{\partial \varphi} \ (=)$$

$$\frac{\partial f}{\partial \varphi}=\frac{\partial \hat{f}}{\partial x}\cdot(-r\cdot\text{ημ}\varphi)+\frac{\partial \hat{f}}{\partial y}\,(r\,\text{συν}\varphi) \quad ②$$

$$\left(\frac{\partial f}{\partial r}\right)^2+\frac{1}{r^2}\cdot\left(\frac{\partial f}{\partial \varphi}\right)^2=\left(\frac{\partial \hat{f}}{\partial x}\right)\overset{\overset{1}{\shortparallel}}{(\text{συν}^2\varphi+\text{ημ}^2\varphi)}+\left(\frac{\partial \hat{f}}{\partial y}\right)^2\overset{\overset{1}{\shortparallel}}{(\text{ημ}^2\varphi+\text{συν}^2\varphi)}=$$

$$=\|\nabla\hat{f}\|^2$$

(Αλλαγή Συμβολισμού: $\hat{f}(x,y)\equiv f(r\,\text{συν}\varphi,\ r\,\text{ημ}\varphi)$)

3) $a\cdot b \neq 0$ και $f:\mathbb{R}^2\to\mathbb{R}$

με $a\cdot f_x + b f_y = 0$ ⊛

Ν.Δ.Ο., $f(x,y) = g(bx-ay)$

(Υποδ. $u = bx-ay$, $v = bx+ay$)

Λύση

$$\begin{tikzcd}(x,y) \arrow[r, "\vec{g}"] \arrow[dr, "f"'] & (u=bx-ay,\ y=bx+ay) \arrow[d, "\hat{f}"] \\ & \mathbb{R}\end{tikzcd}$$

$f = \vec{g}\circ\hat{f}$

$\left\{ f(x,y) = \right.$
$= \hat{f}(\underset{u}{\underset{\parallel}{bx-ay}}, \underset{v}{\underset{\parallel}{bx+ay}})$

- $\dfrac{\partial f}{\partial x} = \dfrac{\partial \hat{f}}{\partial u}\cdot\dfrac{\partial(bx-ay)}{\partial x} + \dots$

$\dots + \dfrac{\partial \hat{f}}{\partial v}\cdot\dfrac{\partial(bx+ay)}{\partial x} =$

$= f_x = \dfrac{\partial \hat{f}}{\partial y}\cdot b + \dfrac{\partial \hat{f}}{\partial v}\, b$

⊛

- $f_y = \dfrac{\partial f}{\partial u}\cdot(-a) + \dfrac{\partial f}{\partial v}\cdot a$

$0 \overset{⊛}{=} 2ab\,\dfrac{\partial \hat{f}}{\partial v}$ ⊛

άρα $\dfrac{\partial \hat{f}(u,v)}{\partial v} = 0, \ \forall u\in\mathbb{R}$

$\hat{f}(u,v) = g(u)$

$f(x,y) = g(bx-ay)$

(Για $a=3$, $b=2$, οχτ: 20, 2,5)

4) i) $z(x,y) = f\left(\frac{x}{y}\right)$ ΝΔΟ $\overset{x}{\cancel{x}}\cdot\dfrac{\partial z}{\partial x} + y\cdot\dfrac{\partial z}{\partial y} = 0$

ii) $z\cdot(x,y) = f\cdot\left(\dfrac{x+y}{x-y}\right)$ ΝΔΟ $\overset{x_2}{\cancel{x}}\cdot\dfrac{\partial z}{\partial x} + y\cdot\dfrac{\partial z}{\partial y} = 0$

Λύση

i)

$$\begin{tikzcd}(x,y) \arrow[r, "g"] \arrow[dr, "f\circ g(x,y)"'] & \frac{x}{y} = t \in\mathbb{R} \arrow[d, "f"] \\ & f(t)\in\mathbb{R}\end{tikzcd}$$

έχουμε ότι

$Z = f\circ g(x,y)\in\mathbb{R}$

↷

$$\frac{\partial z}{\partial y} = f'\left(\frac{x}{y}\right)\cdot\left(\frac{1}{y}\right)$$

$$\frac{\partial z}{\partial y} = f'\left(\frac{x}{y}\right)\cdot\left(-\frac{1}{y^2}\right)$$

$$\underline{x\cdot z_x} + y\cdot z_y = \frac{x}{y}\, f'\left(\frac{x}{y}\right) + x\cdot\left(-\frac{1}{y^2}\right)\cdot f'\left(\frac{x}{y}\right) = 0$$

ii) Ανάλογα

5) (**) $f: \mathbb{R}^n \setminus \{\vec{0}\} \longrightarrow \mathbb{R}$, $f(t\vec{x}) = t^{\alpha} f(\vec{x})$

$t>0,\ \vec{x}\neq\vec{0}$ και $f=$ διαφορίσιμη

Η f καλείται α-τάξης ομογενής

Ν.Δ.Ο. $\vec{x}\cdot\nabla f(\vec{x}) = \alpha\cdot f(\vec{x})$ → Εξίσωση Euler

(π.χ. $n=2$, $\left(x\cdot\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = \alpha f\right)$

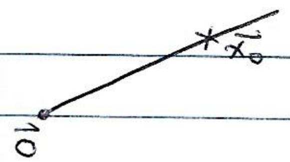
Λύση

$\vec{x}_0 \neq \vec{0}$

$\vec{r}(t) = t\,\vec{x}_0$ → $\vec{r}(0)=\vec{0}$

→ $\vec{r}(1)=\vec{x}_0$

→ $\vec{r}(t)=\vec{x}_0$



$g = \underline{f\circ\vec{r}}$

① $\frac{d}{dt}(f\circ\vec{r})(t) = \nabla f(\vec{r}(t)\cdot\vec{r}'(t))$

$g'(1) = \nabla f(\vec{x}_0)\cdot\vec{x}_0$

② $\frac{d}{dt}\, g(t) = \frac{d}{dt}\left(f(t\vec{x}_0)\right) = \frac{d}{dt}\left(t^{a} f(\vec{x}_0)\right) =$

$= a\cdot t^{a-1}\cdot f(\vec{x}_0)$

$g'(1) = a\cdot f(\vec{x}_0)$ Άρα έχουμε ότι,

$\nabla f(\vec{x})\cdot\vec{x} = \alpha\cdot f(\vec{x})$

1) ΟΡΙΣΜΟΙ πολύ καλά (σκεφτόμαστε)
2) ΑΣΚΗΣΕΙΣ

Σχόλια

i) Ισχύει και το αντίστροφο. Αν $\nabla f(\vec{x})\circ\vec{x} = a f(\vec{x})$
$\Rightarrow$ η $f =$ ομογενής a- τάξης

ii) Οι συναρτήσεις της ασκ. 4 είναι ομογενείς 0- τάξης

$$\left(f(tx,ty) = f\left(\frac{tx}{\underset{\swarrow\, t}{\cancel{t}}\,y}\right) = f\left(\frac{x}{y}\right) = f(x,y) \right.$$

$$\left. f(tx,ty) = t^{0}\cdot f(x,y) \right)$$