28º Μάθημα Πιθανοτήτων

14/5/2010

"ΑΣΚΗΣΕΙΣ ΦΥΛΛΑΔΙΟΥ ⑤"

Άσκηση: 2

(X,Y) με $f(x,y) = \begin{cases} c(y^2-x^2)e^{-y}, & y>0, \ -y<x<y \\ 0 & \text{αλλιώς} \end{cases}$

i) Πρέπει: $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1 \Leftrightarrow \int_0^{+\infty}\int_{-y}^{y} c(y^2-x^2)e^{-y}\,dx\,dy = 1 \Leftrightarrow$

$\Leftrightarrow \int_0^{+\infty} c\cdot e^{-y}\left(\int_{-y}^{y}(y^2-x^2)dx\right)dy = 1 \Leftrightarrow \int_0^{+\infty} c\cdot e^{-y}\left[y^2\cdot x - \frac{x^3}{3}\right]_{-y}^{y} = 1 \Leftrightarrow$

$\Leftrightarrow \int_0^{+\infty} c\cdot e^{-y}\frac{4}{3}y^3\,dy = 1 \Leftrightarrow \dots \Leftrightarrow c = \frac{1}{8}$

ii) $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{|x|}^{+\infty} c(y^2-x^2)e^{-y}\,dy = c\int_{|x|}^{+\infty} y^2 e^{-y}dy - cx^2\int_{|x|}^{+\infty} e^{-y}dy =$

$= \frac{1}{4}(1+|x|)e^{-|x|}.$

$f_Y(y) = \int_{-y}^{y} c(y^2-x^2)e^{-y}dx = cy^2\cdot e^{-y}\int_{-y}^{y} 1\,dx - ce^{-y}\int_{-y}^{y} x^2\,dx = \dots =$

$= \frac{1}{6}y^3 e^{-y}$

iii) $E[X] = \int_{-\infty}^{+\infty} x f_X(x)\,dx = \int_{-\infty}^{+\infty} \frac{1}{4}x(1+|x|)e^{-|x|}\,dx = \int_{-\infty}^{0}\dots + \int_0^{+\infty}\dots = \dots$

iv) $f_{X|Y}(x/y) = \frac{f(x,y)}{f_Y(y)} = \dots$ και $f_{Y/X}(y/x) = \frac{f(x,y)}{f_X(x)} = \dots$

Άσκηση: 3

i) σ.π.π. $\rightarrow \int_0^1\int_0^2 = 1$

ii) $f_X(x) = \int_0^2 f(x,y)dy = \frac{6}{7}(2x^2+x)$ και

$f_Y(y) = \int_0^1 f(x,y)dx = \frac{6}{7}\left(y^4+\frac{1}{3}\right)$

v) $E[X] = \int_0^1 x f_X(x)\,dx = 1$

vi) $E[Y] = \int_0^2 y f_Y(y)\,dx = \frac{11}{28}$

vii) $f_{Y/X}(y/x) = \frac{f(x,y)}{f_X(x)}$

$f_{X/Y}(x/y) = \frac{f(x,y)}{f_Y(y)}$

iii) $P[X > Y] = \int_0^2 \int_y^1 f(x,y)\,dx\,dy = \ldots = \frac{15}{56}$

iv) $P[Y > \frac{1}{2}, X < \frac{1}{2}] = \int_{1/2}^{2} \int_0^{1/2} f(x,y)\,dx\,dy$

$$P[Y > \tfrac{1}{2} / X < \tfrac{1}{2}] = \frac{P[Y > \frac{1}{2}, X < \frac{1}{2}]}{P[X < \frac{1}{2}]} = \frac{\int_{1/2}^{2} \int_0^{1/2} f(x,y)\,dx\,dy}{\int_0^{1/2} f_X(x)\,dx}$$

Άσκηση: 5

$$f(x,y) = \begin{cases} x+y & , \ 0 < x < 1, \ 0 < y < 1 \\ 0 & , \ \text{αλλιώς} \end{cases}$$

i) Δεν είναι ανεξάρτητες αφού η $f(x,y)$ δεν μπορεί να γραφεί ως γινόμενο δυο συναρτήσεων (μια του x και μια του y)
Για παράδειγμα, αν είχα $f(x,y) = \underbrace{3x^2}_{g(x)} \underbrace{y}_{h(y)}$ θα γραφόταν ως γινόμενο !

ii) $f_X(x) = \int_0^1 (x+y)\,dy = \left[xy + \frac{y^2}{2}\right]_0^1 = x + \frac{1}{2}$

iii) $P[X+Y<1] = \int_0^1 \int_0^{1-y} f(x,y)\,dx\,dy = \ldots = \frac{1}{3}$

$x+y<1$

$x<$

iv) $f_{Y/X}(y/x) = \frac{f(x,y)}{f_X(x)}$

Άσκηση: 6

$$f(x,y) = \begin{cases} 12xy(1-x) & , \ 0<x<1, \ 0<y<1 \\ 0 & , \ \text{αλλού} \end{cases}$$

i) $\underbrace{12x(1-x)}_{g(x)} \underbrace{y}_{h(y)}$ είναι ανεξάρτητες

ii) $E[X] = \int_0^1 x f_X(x)\,dx = \ldots = \frac{1}{2}$ iii) $E[Y] = \int_0^1 y f_Y(y)\,dy = \ldots = \frac{2}{3}$

iv) $Var[X] = E[X^2] - (E[X])^2 = \int_0^1 x^2 f_X(x)\,dx - (E[X])^2 = \ldots = 1/20$

v) $Var[Y] = \ldots = 1/18$.

$f_X(x) = 6x(1-x)$, $f_Y(y) = 2y$.

Άσκηση: 8

$$f(x,y) = \begin{cases} x \cdot e^{-x(y+1)}, & x>0, y>0 \\ 0, & \text{αλλιώς} \end{cases}$$

i) $f_{X/Y}(x/y) = \dfrac{f(x,y)}{f_Y(y)}$ $\qquad f_{Y/X}(y/x) = \dfrac{f(x,y)}{f_X(x)}$

$$f_Y(y) = \int_0^{+\infty} x \cdot e^{-x(y+1)}\, dx = \int_0^{+\infty} x \left(\frac{e^{-x(y+1)}}{-(y+1)}\right)' dx = \left[-\frac{x \cdot e^{-x(y+1)}}{y+1}\right]_0^{+\infty} \circledast +$$

$$+ \int_0^{+\infty} \frac{e^{-x(y+1)}}{y+1}\, dx = 0 + 0 + \left[\frac{e^{-x(y+1)}}{-(y+1)^2}\right]_0^{\infty}$$

$\circledast$ $x \cdot e^{-x(y+1)} \overset{\infty \cdot 0}{=} \dfrac{e^{-x(y+1)}}{\frac{1}{x}} \underset{=}{\overset{DLH}{=}} \dfrac{-e^{-x(y+1)}(y+1)}{-\frac{1}{x^2}}$

$\downarrow$

$\dfrac{x}{e^{-x(y+1)}} \underset{DLH}{\overset{\left(\frac{\infty}{\infty}\right)}{=}} 0$

$\rightarrow f_X(y) = \dfrac{1}{(y+1)^2}$, $f_X(x) = \displaystyle\int_0^{+\infty} x\, e^{-x(y+1)}\, dy = x \cdot e^{-x} \int_0^{+\infty} e^{-yx}\, dy = e^{-x}$

ii) $Z = X \cdot Y$, $F_Z(z) = P(Z \le z) = P(XY \le z) = \displaystyle\int_0^{+\infty}\int_0^{z/y} x \cdot e^{-x(y+1)}\, dx\, dy$
$= 1 - e^{-z} \Rightarrow f_Z(z) = e^{-z}$

Άσκηση: 10

X, Y ανεξαρτ. $\sim Bin(n,p)$

$P[X=i] = P[Y=i] = \binom{n}{i} p^i (1-p)^{n-i}$

i) $P[X=i / X+Y=m] = \dfrac{P[X=i, X+Y=m]}{P[X+Y=m]} = \dfrac{P[X=i, Y=m-i]}{P[X+Y=m]}$ $\underline{\text{ανεξ.}}$

$$= \frac{P[X=i] \cdot P[Y=m-i]}{P[X+Y=m]} = \frac{\binom{n}{i} p^i (1-p)^{n-i} \binom{n}{m-i} p^{m-i} (1-p)^{n-m+i}}{\binom{2n}{m} p^m (1-p)^{2n-m}} =$$

$$= \frac{\binom{n}{i}\binom{n}{m-i}}{\binom{2n}{m}}$$

ii) $2n$ φορές $\quad X = \#$ κ στις πρώτες n δοκιμές $\sim Bin$

$\qquad\qquad\qquad Y = \#$ κ στις επόμενες n δοκιμές $\sim Bin$

Άσκηση: 9

Υπόδειξη: $P[\underbrace{X+Y}_{\sim NegBin}=n]$

Άσκηση: 7

2 ζάρια... $X = max$ $\quad P_{Y|X}(y|x) = ;$

$Y = min$

$P[Y=j \mid X=i] = P[min=j \mid max=i]$

Ισχύει: $j \leq i$.

- $j=i \Rightarrow P[Y=j \mid X=i] = \dfrac{P[Y=j, X=i]}{P[X=i]} = \dfrac{1/36}{P[X=i]}$

- $j<i \Rightarrow$ $\quad$ >> $\quad = \quad$ >> $\quad = \dfrac{2/36}{P[X=i]}$

Όμως: $\sum_{j=1}^{i} P(Y=j \mid X=i) = 1 \Leftrightarrow \dfrac{1/36}{P[X=i]} + \sum_{j=1}^{i-1} \dfrac{2/36}{P[X=i]} = 1$

$$\frac{1/36}{P[X=i]} + \frac{2/36 \cdot (i-1)}{P[X=i]} = 1 \Leftrightarrow \frac{2i-1}{36P[X=i]} = 1 \Leftrightarrow P[X=i] = \frac{2i-1}{36}$$

Άρα: $P[Y=j \mid X=i] = \begin{cases} \dfrac{1}{2i-1}, & j=1 \\ \dfrac{2}{2i-1}, & j<1 \end{cases}$