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The Development of Algebra: Confronting Historical and Psychological Perspectives

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COMBINING HISTORY AND PSYCHOGENESIS

Why did the development of algebra lag behind geometry for so many centuries? Why do today's pupils have difficulties with even the simplest word problems? What prevented generations of mathematicians from accepting the idea of the irrational and the negative numbers? What are the roots of the difficulties experienced by students confronted with the concept of complex number for the first time?

It is neither by chance, nor by mere carelessness, that my list of questions is a mixture of psychological and historical puzzles. As different as they seem at first glance, these two sets of problems may in fact have much in common. Indeed, there are good reasons to expect that, when scrutinized, the phylogeny and ontogeny of mathematics will reveal more than marginal similarities. At least, this is what follows from the constructivist view according to which learning consists in the reconstruction of knowledge.

Piaget—one of the first and most outspoken protagonists of constructivism, and thus of the thesis that "the historical-critical and psychogenetic studies [should] converge" (Garcia & Piaget, 1989, p. 108)—grounds his position in the claim that

> the advances made in the course of history of scientific thought from one period to the next, do not, except in rare instances, follow each other in random fashion, but can be seriated, as in psychogenesis, in the form of sequential 'stages.' (p. 28)

It is probably because of the inherent properties of knowledge itself, because of the nature of the relationship between its different levels, that similar recurrent phenomena can be traced throughout its historical development and its individual reconstruction. For the same reason, difficulties experienced by an individual learner at different stages of knowledge formation may be quite close to those

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that once challenged generations of mathematicians. The parallel terms, *epistemic subject* and *collective epistemic subject* used by some researchers to distinguish between the individual learner and the collective of the creators of knowledge, bespeak the widespread belief in these similarities.

In this article I make a very quick journey through the history of algebra, trying to detect certain recurrent phenomena in the development of abstract ideas. The point of departure for this pattern finding will be a theoretical model according to which the formation of mathematical knowledge is more or less a cyclic process, a process in which the transitions from one level to another follow some constant course. The particular scheme that will be used here pictures mathematics as a hierarchy in which what is conceived operationally (i.e., as a computational process) on one level is reified into an abstract object and conceived structurally on a higher level. The idea of an operational–structural duality of mathematical concepts with its numerous implications was presented in detail in Sfard (1991, 1992). For the convenience of the reader, a summary of the relevant elements of this framework is given in the Appendix. While traveling through the centuries, I confront, whenever possible, historical developments with examples taken from empirical studies on the ways in which today's students learn the subject.

During the hasty flight over history, our telescope will be directed at what may be considered turning points in the development of algebraic thinking. Here I try to fathom not only the mechanisms that put such developments in motion, but also the nature and the source of the cognitive difficulties which invariably pop up whenever a crucial step forward is to be made. This topic deserves special attention because the difficulties seem so ubiquitous both in history and in the classroom that they ought to be regarded as a regular part of the process of knowledge construction rather than as a madness with no method in it. Indeed, when history is considered, what seems to be the most striking common characteristic of the many ways in which new ideas entered the scene and then evolved, is the great deal of distrust and reluctance with which the candidates for citizenship in the kingdom of mathematics were invariably greeted. (The turbulent evolution of such concepts as function or number may serve as good examples; see, e.g., Hefendehl-Hebeker, 1991; Kleiner, 1988, 1989.)

According to a widespread belief often expressed by historians (e.g., Boyer, 1985), it was the lack of logical foundations that obstructed the acceptance of the new types of numbers. However, the scheme of concept development, as well as mathematicians' own utterances, suggest an additional explanation: In some cases the resistance to a new abstract object might have been of ontological rather than of purely logical origins. It could stem from the inability to reify a process. Reification is an act of turning computational operations into permanent object-like entities. For example, a complex number is born only when a person is able to view the process of extracting the square root of a negative number as a real entity, as a permanent thing in its own right. To some people, all this may seem to be conjuring up a new thing out of nothing.

Reification is a major change in the way of looking at things and as such is inherently difficult to achieve. There are several types of serious obstacles that lie in wait for those who dare to speak about new abstract objects (see the Appendix). A revolutionary change in basic beliefs on the nature of mathematics must sometimes occur before the new idea is fully accepted. A natural resistance to upheavals in tacit epistemological and ontological assumptions, which so often obstructed the historical growth of mathematics, can hardly be prevented from appearing in the classroom.

In the next section I concentrate on algebra. Its evolution will be presented as a constant (but not necessarily conscious) attempt at turning computational procedures into mathematical objects, accompanied by a strenuous struggle for reification. I hope that from this bird's-eye view of history a lesson of some practical importance will be learned regarding the nature and sources of the traps lying in wait for today's students throughout the curriculum. To those who shrug at my bold (some would say presumptuous) attempt to deal in one short presentation with the whole of the development of algebra, and from a dual perspective at that, let me say that history will be used here only to the extent which is necessary to substantiate the claims about historical and psychological parallels. No more than a very general view of algebra will be presented.

STAGE 1: FROM ANTIQUITY TO RENAISSANCE—TOWARD THE SCIENCE OF GENERALIZED NUMERICAL COMPUTATIONS

What Is Algebra?

When and where did algebra begin? The literature provides more than one opinion on this matter. "There are many historians of mathematics who trace the origins of algebra to various nations of antiquity: the Assyrians, Babylonians, Egyptians. Others, with more critical judgment, locate these origins at the school of Alexandria" say Garcia and Piaget (1989, p. 143), immediately expressing their disagreement and saying that for them algebra is a much more recent invention. Were algebra really known to the Greeks, they argue, pre-Euclidean and Euclidean geometry, fairly well developed anyway, would have opened up to become even more impressive achievements: "It is clear that the difficulties the Greeks encountered in resolving their numerous geometrical problems can be explained only by the absence of a science of algebra" (p. 143). These words indicate that the authors' disagreement with the others stems not so much from different historical information as from the fact that they obviously have their own interpretation of the term *algebra*. An answer to the question "What is algebra?" must therefore precede any historical account.

The majority of authors seem to be quite unanimous as to the early origins of algebra because they spot algebraic thinking wherever an attempt is made to treat computational processes in a somehow general way. Generality is one of these salient characteristics that make algebra different from arithmetic. Boyer (1985) explains his decision to call some problems solved in ancient Egypt *algebraic* by saying that they "do not concern specific concrete objects, such as bread and beer, nor do they call for operations on known numbers" (p. 16). Novy (1973), in the context of somewhat later developments, repeatedly states that "the search for a *general* [italics added] solution of . . . equations" was one of the two main themes of algebra. "The concept and the definition of realms of numbers" was the other (p. 25). Like Boyer, Kline (1972) agrees that algebraic methods were used as early as in ancient Mesopotamia and Egypt, and like Novy he grounds this claim in the fact that what was done there may be interpreted in the modern language as solving equations in a general way: "Though only concrete examples were given, many were intended to illustrate a general procedure for quadratics" (p. 9).

On one point, therefore, there seems to be perfect agreement among all the authors, including Garcia and Piaget: Algebra is a science of generalized computations. Thus, the differences may only have their roots in the opinions about the means necessary to implement algebraic methods. For Garcia and Piaget, the symbolic notation, never heard of in Babylonia or Egypt, is clearly part and parcel of the branch of mathematics called algebra. Similarly, Unguru (1975, p. 77) names "operational symbolism" one of the "main features of algebraic way of thinking." According to other authors, the modern algebraic symbols are not the only possible vehicle of generality.

In this article, I join the latter school of thought. I use the term algebra with respect to any kind of mathematical endeavor concerned with generalized computational processes, whatever the tools used to convey this generality. This definition brings into full relief the operational origins of algebraic thinking. In the following sections the history of algebra will be presented as a sequence of steps toward ever greater generality and, at the same time, toward structurality. The three stages that will be listed—rhetorical and syncopated algebra, Vietan symbolic algebra, and abstract algebra—correspond roughly to what is taught today at, respectively, the primary, secondary, and tertiary levels.

Rhetorical and Syncopated Algebras

For many people, even if well versed in mathematics, it often comes as a surprise to learn that algebraic notation, which in our minds seems inseparable from algebra itself, is quite a recent invention. Until the 16th century, computational processes, whether generalized or not, were presented either verbally or in a mixture of words and symbols. Two examples of these early ways of expressing algebraic thought appear in Figure 1. The first, taken from Diophantus (c. 250 AD), includes some symbols and thus represents the so called syncopated algebra.

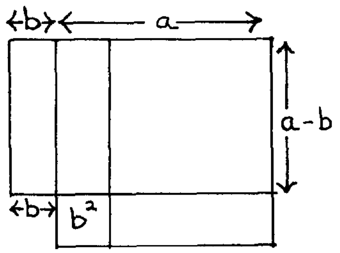
1. Syncopated algebra: from Diophantus, "Arithmetica," c. 250 AD
To find two numbers such that their sum and product are given numbers:
"Given sum 20, given product 96. $2x$ the difference of the required. Therefore the numbers are $10 + x$, $10 - x$. Hence $100 - x^2$ is 96. Therefore, x is 2 and the required numbers are 12, 8."

Remark: In fact, Diophantus used Greek letters as symbols. We translated them (after Fauvel and Grey, 1987, p. 218) into the modern signs for the sake of clarity.

2. Rhetorical algebra: from "Aryabhatia" by Aryabhata, AD 499
To find the number of elements in the arithmetic progression the sum of which is given: "Multiply the sum of the progression by eight times the common difference, add the square of the difference between twice the first term, and the common difference, take the square root of this, subtract twice the first term, divide by the common difference, add one, divide by two."

3. Geometric Algebra

a. Greek proof of identity equivalent to
$(a - b)(a + b) = a^2 - b^2$



b. Greek solution to the problem equivalent to the equation
$x^2 = ab$

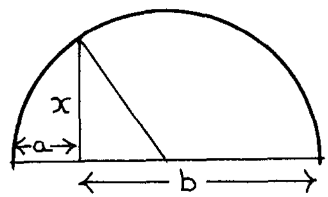


Figure 1. Rhetorical and syncopated algebra.

The second, coming from the Hindu *Aryabhatia* (AD 499), is purely verbal and as such belongs to the kind of algebra known as rhetorical.

The presymbolic algebra that began, as was already mentioned, almost 4,000 years ago in ancient Egypt and Mesopotamia developed—even if only moderately—in the Greece of Pythagoras and Euclid, made a considerable step forward thanks to the work of Diophantus, and for several centuries flourished in the medieval Hindu, Arab, and Italian writings. The salient trait of this algebra was its predominantly operational character. Garcia and Piaget would add the prefix *intra* to the *operational* to emphasize the fact that the computational processes were observed "from inside" rather than from a higher level perspective. Its sole concern was finding general prescriptions for computing unknown values out of concrete numerical data. Thus, algorithms were sought that could be used for entire families of problems.

For all their pursuit of generality, however, the ancient and even the medieval mathematicians usually explained their computational methods through concrete numerical examples rather than by universal prescriptions. Diophantus' solution in Figure 1 is a representative example of this. Although the problem was stated in general terms, concrete numbers were chosen to explain the solution; other questions of the same kind could from then on be worked out by analogy—by substitution of new numbers instead of those chosen in the example. Sometimes, a special name was given to the unknown quantity: *Length* and *width* evidently served this purpose in ancient Mesopotamia, a letter played this role in Diophantus' *Arithmetica, root* appeared in Al-Khowarizmi's (9th century) Al-jabr, and *tanto* in the writings of such Italian algebraists as Bombelli (c. 1526–1573). Only rarely, however, and relatively late, could an author be found who, like Aryabhata in Example 2, addressed both the given and the sought-for numbers by general names, thus paving the way for the later idea of a variable. In these rare cases, the language used by the writer might be quite imaginative as exemplified by the following instruction for finding the fourth item in a proportion: "Multiply the fruit by the desire and divide by the measure. The result will be the fruit of the desire" (*Aryabhatia,* from Boyer, 1985, p. 233).

Thus, until the 16th century the development of algebra was marked not by changes in either the general character of the endeavor or in the methods employed—these remained basically the same for more than 2 millennia—but by the gradual increase in the complexity of the investigated computational processes. What began with an "equivalent of solution of linear equations of the form $x + ax = b$ or $x + ax + ba = c$" (Boyer, 1985, p. 16) in the Ahmes Papyrus (c. 1650 BC), continued with quadratics everywhere from Mesopotamia to Greece to the medieval East and Europe, and ended with truly complicated prescriptions for solving equations of the third and fourth degree in Cardan's *Ars Magna* (1545; see Cardan, 1968).

It is quite obvious that in comparison to the modern notation, where formulas such as $12.3 + 2\sqrt{6}$ or $a^2 + 23b$ concisely represent both a computational process and its product and thus facilitate a transition to structural thinking, the rhetorical and syncopated expressions, with their prolixity and tediously sequential character, impose an operational outlook. The operational mode of thinking, however, puts a substantial burden on the working memory and thus is more strenuous and less effective than the structural approach induced by the modern notation. This may be clearly seen when the Aryabhata's prescription is confronted with its symbolic counterpart. It would be quite reasonable, therefore, to expect that the moment today's student gains access to the algebraic symbolism he or she becomes willing to use it in every possible context. We can only guess that this is exactly what Cardan would have done had algebraic symbols been available to him at the time he toiled to explain the solutions of the cubic and quartic equations.

More than one empirical study has shown, however, that in reality things look quite different. What was noticed for the first time in a series of experiments performed by Clement and his colleagues (Clement, Lochhead, & Soloway, 1979; Soloway, Lochhead, & Clement, 1982) found its further confirmation in the systematic studies by myself (Sfard, 1987) and by Harper (1987). All the data unanimously showed that even pupils with several years of (symbolic) algebra behind them may do better with verbal than with symbolic methods. In 1979, Clement and his colleagues discovered that a large proportion of college students could not translate such simple sentences as "There are six times as many students as professors" into equations and in 1982 they found that students were much more successful when required to write appropriate computer programs. These observations led me to the conjecture that it may be the operational character of the computer encoding that makes this seemingly awkward representation somehow easier for pupils than the structural algebraic symbolism. To test this supposition, an experiment was carried out in which two groups of high-school students, 14 to 17 years of age, were presented with questions like the one presented in Figure 2. The participants were asked to choose, from among three possibilities, a formula that matched the situation described in a problem. In another questionnaire, they were required to find a verbal prescription for solving a similar problem. Both groups succeeded in the latter kind of task significantly better than in the former. In light of these findings, it no longer comes as a surprise that, as was shown by Harper, students often choose the rhetorical method if not obliged to use algebraic symbols. In his experiment, Harper asked a group of pupils of different ages to solve one of Diophantus' problems (similar

PROBLEM	OPERATIONAL SOLUTION	STRUCTURAL SOLUTION
In a class the boys outnumber the girls by four	To find the number of girls we have to:	x = number of girls y = number of boys
	a. add 4 to the number of boys	
	b. subtract 4 from the number of boys	a. $x + 4 = y$ b. $x = y + 4$
	c. none of the above	c. $y > x + 4$

Figure 2. A Problem used in the experiment by Sfard (1987).

to the one presented in Figure 1). He noted that not only in the youngest students, but also among the older there was a distinct preference for verbal prescriptions.

In all these experiments the authors emphasize that the discovered phenomenon cannot be regarded as a mere outcome of classroom experience because the students were never trained in constructing verbal solutions to word problems. Thus, the rhetorical method was used spontaneously, independently of instruction. It seems, therefore, that the precedence of operational over structural thinking must be, at least in this case, one of those developmental invariants we are looking for in this article—it was observed in the historical development of mathematics as well as in the process of individual learning. All these findings speak with force for the thesis of the inherent difficulty of the transition from an operational to a structural approach. In the next chapter, I look more deeply at the impediments to progress at the junction between rhetorical/syncopated and symbolic algebra.

Geometric Algebra

Our account of presymbolic algebra would be seriously incomplete without a mention of the so called *geometric algebra,* a very special breed of mathematics which developed in ancient Greece. Its story enlightens another aspect of the difficulty with reification.

As its name suggests, geometric algebra was a result of a merger between the two central components of ancient mathematics. Basically, it consisted in interpreting quantities expressed with letters as lengths of line segments and the operations on these quantities as finding lengths, areas, or volumes of the figures built from these segments. Thus, solving equations could be translated into finding the geometrical construction that would produce a segment of the length equal to the sought-for quantity (see Example 3b in Figure 1). Algebraic identities could also be proved in this way (see Example 3a). To be sure, some authors oppose the view that geometrical algebra was "algebra dressed up" and claim that the problems dealt with were essentially geometric and were tackled only for this reason (Unguru, 1975, p. 69). This discussion, however, is irrelevant to the present subject because it was agreed that all those events that contributed to the science of generalized computations, even if only indirectly, qualify to be included in the historical account.

The desire to marry the science of computations with geometry is not much younger than the two disciplines themselves, and geometric algebra is one of its first results. The main reason usually brought by historians (see, e.g., Boyer, 1985; Kline, 1972, 1980) to explain this ancient urge for unification is the fact that, unlike algebra, geometry was considered in Greece to be a paragon of consistency and mathematical rigor. Those who tackled concrete quantities and unknowns through lengths and areas evidently hoped that in this way the advantages of geometry would be transmitted to the science of computations. Another explanation brings the problematic idea of irrational number as the trigger for separating the concept of number from that of continuous magnitude. Because this distinction first arose in the context of incommensurability, it was only natural to view the continuous quantities as tightly connected to the realm of geometry.

Our model of concept development suggests an additional explanation of the phenomenon of geometric algebra. Greek geometry, with its "thinking embodied in, fused with graphic, diagrammatic representation" (Unguru, 1975, p. 76) was clearly at its structural stage, whereas algebra, preoccupied with verbally represented computational processes, could be conceived in no other way than operationally. The structurality of the geometry facilitated thinking and enabled effective investigations. The operational rhetoric of algebra made it cumbersome and unyielding. No deeply penetrating, generalizing insight was possible. What algebra needed for further development was reification of its basic concepts. At this time, no better means were available to help in reification of the growingly complex computations than the palpable geometric objects. Geometric figures rendered some kind of tangible existence not only to the idea of irrational quantity, but also to the elusive concept of variable magnitude. However, although initially helpful, these tools soon proved restrictive. They created a system of prohibitions that greatly limited the range of problems qualifying for algebraic treatment. For instance, because the unknown was usually referred to as *length,* its square as *area,* and its cube as *volume,* adding different powers remained for some time entirely out of the question. Also, no power greater than 3 was admitted in calculations.

Once again an important lesson can be learned from history by teachers and psychologists. The current studies on visualization (e.g., Dreyfus, 1991) leave little doubt as to the effectiveness of graphical representations even in learning such abstract subjects as algebra. No wonder, then, that the Greeks found it useful to give numerical computations a geometrical interpretation. For the same reason, graphical means are offered today to those who teach algebra. However, while employing geometry to support the science of computation we should remember that, if used without precautions and treated too literally, the models may become restrictive rather than helpful. The following declaration by Bell (1951) is pertinent here: "Real mischief is done when the credulous pupils acquire an ineradicable belief that their purely metaphorical language describes an 'existent space' or an 'objective reality'" (p. 140). This statement may sound too emphatic but, stripped of its exaggerated rhetoric, what it really says is probably this: Algebra is an inherently abstract discipline and one cannot escape teaching it as such.

STAGE 2: FROM VIÈTE TO PEACOCK—ALGEBRA AS A SCIENCE OF UNIVERSAL COMPUTATIONS

Viète's Invention: Variable as a Given

To find the right tools for the reification of generalized computational procedures took many centuries, which shows once more how genuinely difficult the process was. How the structural stage in algebra was eventually attained is told in this section. It is the story of modern algebraic symbolism.

Today historians seem united in the opinion that although letters were often used by mathematicians before the 16th century, it was the way François Viète (1540–1603) employed them which made the real difference. The French mathematician was the first to replace numerical givens with symbols. To put it in modern language, Viète was the inventor of parametric equations, equations with literal coefficients. Until this point, letters were used in algebra to symbolize the sought-for unknown quantities. Viète decided to denote them by vowels. Given numbers, namely those which were assumed to be known and provided as data, had to be represented by consonants. Thanks to this convention, entire families of problems (equations) could now be dealt with by means of concisely stated algorithms. Thus, the introduction of parameter was a great step toward the generality so intensely pursued in mathematics in general and in algebra in particular. To use Boyer's (1985) imaginative expression, Viète's "givens" helped convert algebra from Diophantus' "bag of tricks" into a genuine science of general computations (p. 334). Viète himself was aware of the fact that he had added a floor to the hierarchical edifice of mathematical generalization and abstraction. According to his own description, whereas arithmetic is the science of concrete numbers (*logica numerosa*), his type of algebra is a science of species (*logica speciosa*) or of types of things rather than of the things themselves. Thus, this is probably where the concept of variable was born.

To fully appreciate Viète's achievement one has to consider its impact on mathematics in general. Employing letters as givens, together with the subsequent symbolism for operations and relations, condensed and reified the whole of existing algebraic knowledge in a way that made it possible to handle it almost effortlessly, and thus to use it as a convenient basis for entirely new layers of mathematics. In algebra itself, symbolically represented equations soon turned into objects of investigation in their own right and the purely operational method of solving problems (by reverse calculations) was replaced by formal manipulations on propositional formulas. These manipulations are addressed in the sequel as secondary operations, as opposed to the underlying arithmetic processes referred to as primary.

The advent of symbolic algebra was soon followed by the emergence of an entirely new kind of natural science. For the first time in history, mathematics had the means for dealing with changing magnitudes and not just constant quantities. It was only natural that scientists would seize the new invention to represent all kinds of natural processes. Along with the intense investigation of the mathematical structure of physical movement, the concept of function, innately and indissolubly linked with the idea of variable, began to arise. It is this development that made it possible for physics to be translated into the precise language of mathematics.

The impact of symbolic algebra was also felt in geometry. As emphasized in the previous section, this ancient branch of mathematics, with its easily visualizable basic objects, was up to this point predominantly structural. According to our model of abstract knowledge formation, the next step in the development of geometry should be a transition to an operational mode of thinking at a higher level. To attain more generality, it had to detach itself from concrete triangles and pyramids and concentrate on the constructions and transformations by which these primary objects are governed. The means for this could only come from an independently developing algebra: "Only algebra . . . would have enabled [the Greeks] to formulate [certain unsolved geometrical] problems in terms of operations" (Garcia & Piaget, 1989, p. 143). And indeed, the transition from operational to structural thinking in algebra was soon echoed in a substantial step toward higher level operational thinking in geometry. Descartes (1596–1650) and Fermat (1601–1665) were those who employed symbolic algebra in geometry for the first time. Geometrical figures and their transformations were represented through the appropriate computational processes. This invention was later named analytic geometry because the method of investigation based on manipulating algebraic symbols (which later included differential and integral calculus) became known as analysis. One may say that in this way algebra, which once turned to geometry for help in reification and verification, and which thus came to be viewed as a "minor appendage to geometry" (Kline, 1980, p. 123), could eventually pay its debt. It reciprocated with the means for capturing generality and conveying operational thought. According to Kline, if Greek algebra expressed through geometry was called geometric, then Descartes' invention should be known as algebraic geometry. (Unfortunately, this name was given to another branch of mathematics which developed much later.) Such terminology would aptly reflect the symmetry of mutual services rendered by algebra and geometry to each other.

To our modern eyes, the idea of a variable as any number seems so obvious and simple, we can hardly understand why it did not appear many centuries earlier. After all, letters were used in mathematics already in antiquity (e.g., in Euclid's *Elements*, c. 300 BC.). This fact, however, becomes much less surprising when one realizes that a variable as a given imposes functional thinking—it requires an ability to think simultaneously about entire families of numbers rather than about any specific quantity. Thus, the introduction of a parameter demands a very sharp change of perspective for which structural understanding of computational processes is indispensable.

In this context, it is worth mentioning that Viète's invention was not always fully appreciated by historians of mathematics. (Garcia & Piaget credit Jacob Klein and his book *Die griechische Logistic und die Entstehung der Algebra*, 1934, with the first attempt at reassessment.) The same mechanism that concealed the import of Viète's achievement from the eyes of historians often hides from today's teachers the height of the step to be climbed by students when parametric equations are presented to them for the first time. Such ignorance sometimes has grave consequences. I can clearly remember a traumatic experience of my own. It happened many years ago when I was teaching simultaneous linear equations to two groups of 10th graders. At that time I was quite insensitive to the huge conceptual difference between equations with numerical coefficients and equations with parameters. With my rich and versatile experience of variables and functions, I had come to treat these two kinds of problems as almost indistinguishable. And, obviously, so did the authors of the textbook I was using. They scattered problems with parameters all over the chapter, concealing them among other questions. They did it without warning. Not knowing what I was doing, I gave my pupils some of the parametric equations for homework. The price of this ill-calculated deed was high: For a fortnight, I was stuck doing things I had never planned. My students would not let me talk about anything other than problems with parameters. They could not cope with this kind of task themselves and one or two examples with adequate general explanations were obviously not enough. After five or six meetings and two tests devoted solely to this topic, my pupils still seemed somewhat shaky in their understanding. The fact that the difficulties I witnessed were not something particular to my students (or, for that matter, to me as a teacher) was quite obvious for several reasons. First, exactly the same happened quite independently in both my groups. Second, my colleagues reported similar experiences. After 2 weeks of grappling with the difficulties, I could clearly see their deep roots. Eventually, I became aware of the vast conceptual change that occurs during the transition from the concrete to parametrically given problems. I learned it the hard way.

This story is an anecdote rather than a piece of scientifically designed research. More systematic evidence for the existence of the problem may be found in studies by Lee and Wheeler (1989), Booth (1988), and the Assessment of Performance Unit (1980). The aforementioned investigation by Harper (1987) shows, along with the pupils' tendency to use rhetorical algebra, their inability to use the Vietan kind of variable. Nearly half of the oldest participants in Harper's experiment, when asked to show that two numbers may always be found if their sum and difference are given, preferred the Diophantan type of argument to the Vietan: They chose arbitrary concrete numbers rather than letters for their givens. This research is one of the few studies that makes explicit use of history to predict students' behavior.

There is another aspect of the passage to symbolic algebra that has attracted the attention of researchers. More often than not, solving equations in a rhetorical way was based on reversing computational processes, or undoing what was done to the unknown quantity (see Example 1 in Figure 1). Much evidence has been collected for the particular difficulty of the transition from such a working backward technique to the method involving the so-called permissible operations on both sides of an equation. A survey of the relevant research was given by Kieran (1992): "A major turnaround must occur [in algebra] when students are asked to think in terms of the forward operations that represent the structure of the problem rather than in terms of the solving operations [which reverse the process of computation]" (p. 403). This turnaround corresponds to the point in history where rhetorical algebra gave way to the symbolic. The transition is problematic because it requires this difficult change of perspective that has already been mentioned several times in this article: Operational thinking must be replaced by structural.

Peacock and the British School: Dearithmetization of Algebra

Although soon after Viète symbolic algebra began to flourish, some of the most prominent thinkers voiced their qualms about using it. Newton, for example, claimed that "algebra is the analysis of bunglers in mathematics" (Kline, 1980, p. 124). Once again, historians tend to ascribe these doubts to the fact that the new discipline lacked a logical basis. As in all the previously discussed cases, however, this impediment did not seem serious enough to prevent widespread use of the effective analytic method. Indeed, Kline admits that "by 1750 the reluctance to use algebra had been overcome," although "by that time algebra was a full-grown tree with many branches but no roots" (p. 125).

Like the other cases we dealt with, a scrutinizing look into history will reveal that, along with the concern about internal consistency, doubts of an ontological nature popped up here and there in various writings. It was the unreifiable notion of variable that was the core of the problem. This notion, the exact meaning of which cannot be easily explained through a rigorous definition, may well be one of the most problematic in the whole of mathematics—so problematic, in fact, that doubts permeate the professional literature even today. According to Bell (1951):

> to state fully what a variable is would take a book. And the outcome might be a feeling of discouragement, for our attempts to understand variables would lead us into a morass of doubt concerning the meanings of the fundamental concepts of mathematics. (p. 101)

And then he immediately added his own imprecise description: "Variable is something which changes." This operational component of change, which resisted all attempts at reification, is what made the concept of variable unacceptable even in the eyes of some 20th-century mathematicians. This is certainly what bothered Frege (1970, p. 107) who required "elimination of time" because of its being "alien to Analysis." The impossibility of doing it is what eventually forced him to reject the whole idea by saying "The word 'variable' . . . has no justification in pure Analysis." This is a quite radical opinion with which not many mathematicians would agree. Usually doubts about ontological origins gradually dissolve with persistent use of a notion. And, indeed, by the end of the 18th century mathematicians were obviously familiar enough with algebra and its techniques to use them without further ado.

But not all of them would be able to calm their conscience by pushing the ontological questions aside. In the 1830s and 1840s, a debate on the meaning of algebra and its symbolism was led by prominent British mathematicians such as Augustus de Morgan (1806–1871) and Sir William Rowan Hamilton (1805–1865). De Morgan's associate George Peacock (1791–1858) is generally regarded as the leading figure in the innovative school of thought which developed as a result of this dispute. (Individual contributions of the different mathematicians are presented in detail by Novy, 1973.)

Until then, algebra had been regarded as "universal arithmetic"—a discipline which specialized in expressing in a general way the rules governing numerical operations. This interpretation of symbols and symbolic manipulations necessarily restricted the scope and force of algebraic laws. The British mathematicians felt an urge to provide algebra with a sound logical basis that would set it free from such limitations.

The tendency to broaden the scope of concepts by gradually loosening different restraints on their meaning is one of the typical processes that can be traced throughout the history of mathematics. The concept of negative number resulted from the removal of the embargo on subtracting a number from a smaller one (this, of course, was done not without reluctance and hesitation, but that is another story). Similarly, complex number was born when extracting square roots from negative quantities ceased to be seen as a taboo. It is probably the 19th century's gradual reconciliation with the seemingly unacceptable idea of complex number that gave British mathematicians the courage to claim the emancipation of algebra from the yoke of its original meaning. They had an almost mystical feeling that the laws of algebra must be treated as completely universal and that this principle of universality has to be accepted as superior to any consideration other than the consistency of the theory. Thus, the fact that certain numerical operations did not comply with some rules was not regarded as a sufficient reason for restricting the rules. This stance found its most emphatic expression in Peacock's "principle of permanence": "Whatever form is algebraically equivalent to another form expressed in general symbols, must continue to be equivalent, whatever the symbols denote" (cited in Novy, 1973, p. 191). The term *form* probably means here algebraic expression and *algebraically equivalent* means equivalent through symbolic manipulations.

Peacock's immediate conclusion from this principle may be formulated as follows: If a number does not obey a law, the number rather than the law would be the one to go. A variable should no longer be seen as a generalized number but must be treated as a thing in itself, devoid of any external sense. Variables are thus mere symbols that denote nothing. They, of course, may be interpreted in different ways in different contexts but they have no meaning of their own. From here Peacock promptly arrived at a complete dearithmetization of algebra. The meaning of symbols should no longer be expected to come from their nonexistent designata, but rather, must be sought in the way the formulas are transformed and combined with each other. These transformations are subordinate to rules given by axioms. The axioms themselves are arbitrary. Thus, Peacock and his colleagues may be regarded as precursors of the formalist school of thought that developed fully only several decades later when David Hilbert (1862–1943) generalized the idea of semantically void symbols beyond algebra and applied it to the whole of mathematics.

The dearithmetization of algebra is a typical example of severing a mathematical idea from its operational origins in order to attain full reification. Things like that happened in mathematics many times before. The concept of number, for example, originated in the operations of counting and measuring but these interpretations had to be given up if the idea of complex number was to be accepted. Several decades later, the mathematicians who converted the function into a simple set of ordered pairs sacrificed the algorithmic underpinnings of the concept. One may say, using Garcia and Piaget's terminology, that with the dearithmetization of algebra, the interoperational stage was finally attained where the primary operations were reified, and the relationships between them, rather than their internal structure, became the central object of attention.

The 19th-century discussion between British mathematicians was one of the incentives for a recent study carried out by one of my colleagues and myself (Linchevski & Sfard, 1991). Fascinated by the fact that mathematicians themselves could hardly make up their minds as to the best possible interpretation of algebra, we decided to turn to today's students in an attempt to find out their implicit beliefs about the meaning of symbolic formulas and manipulations. In our study many questions, sometimes quite nonroutine and surprising, were asked regarding such concepts as equivalence of equations and inequalities, permissible operations, and solution of an equation or inequality (see examples in Figure 3). Before the experiment, we conjectured that the students would interpret propositional formulas in one of two ways: either as generalized arithmetic

1. Are the following equations (inequalities) equivalent or not? Explain your answers.

a. $4x^2 > 9$	b. $4x - 11 = 2x - 7$	c. $x^2 + 1 = 0$
$2x > 3$	$(x - 2)^2 = 0$	$x^2 + 5 = 0$

2. Complete the following sentences:
   a. To solve an equation means..............
   b. When we solve an equation we are allowed to execute the following operations: ....
   c. These operations are permitted because.................
   d. When we finish solving the equation, what we get in the end is..................

Figure 3. Sample questions from the research by Linchevski and Sfard (1991).

expressions, which can only be understood through the underlying mathematical operations (primary processes) and through the sets of numbers to be substituted instead of the letters, or as mere strings of symbols that draw their meaning from the manipulations (secondary processes) people use to perform on them. The result of our investigation was quite unequivocal: Whatever question or problem was presented to the students, the answers invariably showed a clear bias toward the latter type of interpretation. The majority of pupils viewed algebraic expressions as meaningless symbols governed by arbitrary established transformations.

This second approach to algebraic symbolism deceptively reminds us of Peacock's ideas. Thus, on the face of it, our students revealed surprising maturity by treating algebraic symbols in the way recommended by mathematicians themselves. In fact, however, our pupils' conceptions were nothing like the ideas promoted by Peacock. What the British mathematician proposed may seem quite simple but a closer look discloses its truly sophisticated nature. Peacock's ideas were generated by a well-appreciated necessity to free algebra from the burden of the initially helpful but now restraining semantic load. They resulted from a familiarity with the alternative interpretations of algebra and with their disadvantages. To put it another way, Peacock's semantically emptied concepts originated in the conscious decision of a person who knew exactly what he or she was going to give up and who was perfectly able to go back to the renounced meanings whenever appropriate. No such return is possible for the student whose eyes have never been opened to the alternative options. Indeed, our students' ostensibly structural thinking did not seem battered by any operational underpinnings and there was no reason to believe that what could be seen was but an upper layer of a sophisticated conceptual structure. Their conceptions appeared one-dimensional and shallow. Such comprehension is not very likely to lead to the flexibility in interpreting variable that is the basis of successful problem solving. This is probably why a tone of exasperation is usually assumed when results of studies are reported showing that students' "algebraic language is empty, having only syntax" (Burton, 1988, p. 2).

As was argued by Linchevski and Sfard (1991), it is probably the modern structural way of teaching that may be partly responsible for this situation. In the discussion of how to teach algebra, I would, therefore, vote for courses which take the historical facts into consideration and compromise the modern definitions for the sake of a less advanced but to the learner more accessible operational approach. (One such course was recently developed in England; see the National Mathematics Project, 1987, and a brief description of this course in Kieran, 1992.)

Paraphrasing Picasso, who reportedly claimed that one must be able to draw realistically before becoming an abstract painter, we may say that in mathematics a pupil should be a Platonic realist before turning into a formalist and being able to deal with pure abstraction brought into being by stipulation.

STAGE 3: FROM GALOIS TO BOURBAKI—ALGEBRA AS A SCIENCE OF ABSTRACT STRUCTURES

First Step: The Emergence of Group Theory

Peacock took the liberty of introducing the component of arbitrariness into algebra. By doing this and by suggesting that the axiomatic method exempts mathematicians from ontological confinements, he laid a cornerstone for a new kind of mathematics, the spirit of which is mocked in the rhyme of dubious didactic value: "Minus times minus equals plus, the reasons for this we need not discuss." From now on mathematicians could invent new mathematical objects fearing nothing and nobody but the laws of logic. The internal consistency of a new idea should be their sole concern and no philosophical questions about the nature of the formally defined object or about its relationship with the real world ought to seem relevant any more. After this ontological breakthrough, introducing new mathematical objects through axiomatic systems gradually became common practice. (For a deeper insight into the process of change which, in fact, was more complex than may be understood from this concise description, see, e.g., Kleiner, 1986.) Algebra's bonds with numbers and numerical computations were loosened even further and it gradually turned into a science of abstract structures.

Hamilton's invention of quaternions in the 1850s may be regarded as the first act of such free creation. His earlier work on complex numbers, which he presented simply as pairs of real numbers ruled by formally defined operations, brought him close to the position held by today's mathematicians: He began to realize that nothing more than a consistent axiomatic system is needed to legitimize the existence of an abstract object. (Hamilton was quite ahead of his times as may be seen in the following statement by his friend John Graves: "I have not yet any clear view as to the extent to which we are at liberty to create imaginaries, and to endow them with supernatural properties," Kleiner, 1987, p. 233.) He immediately decided to take advantage of this new approach by pushing the idea of number-like n-tuples a little further. The quaternions are 4-tuples of real numbers, subordinated to a regular type of addition and to a noncommutative operation of multiplication. Although Hamilton introduced them to mathematics on the sole basis of formal definition, he still felt an urge to justify his creation by pointing out its possible physical applications. Those who came after him soon freed themselves even from this kind of consideration. Nevertheless, his step toward the new kind of mathematics was so decisive that in the eyes of some historians "all of modern algebra owes its origins to Hamilton's creation of quaternions" (Kline, 1980, p. 295).

Another milestone in the history of abstract algebra was the emergence of the concept of group. Its origins go back to pre-Hamiltonian times, to the works of Joseph Louis Lagrange (1736–1813) and Paolo Ruffini (1765–1822). Both these mathematicians were preoccupied with one of the central problems of 18th- and

19th-century algebra—the question of the possibility of solving equations of degree 5 or more by radicals. Both of them noticed that important information about the equation may be collected through a study of certain functions of its roots and through counting the number of different values such functions obtain when the roots are permuted in all possible ways. The notion of permutation gradually overshadowed the other auxiliary concepts until it became the center of attention. In no time, what was basically a process, an operation of rearranging a sequence of entities, came to be treated as an abstract object. The first step toward reification of the concept was made by Augustin Louis Cauchy (1789–1857) who explicitly talked about manipulating and combining the permutations in certain well-defined ways, thus viewing them as inputs to higher level procedures. The operations on permutations, in their turn, were soon to become the central object of inquiry. Evariste Galois (1811–1832) was the one who eventually defined the notion of group, namely, he explicitly declared his interest in the structure imposed on a set of permutations by the operations which can be performed on them (the so-called substitutions). The name *group,* although not consistently used, was introduced by him to denote this new mathematical object of unprecedented abstractness and richness. The English mathematician Arthur Cayley (1821–1895) took the construction of the concept a step further by freeing it from any commitments to the nature of the basic elements of a group. They could be anything: permutations, quaternions, matrices, and so forth. Thus, Cayley ultimately shifted the emphasis from the manipulated entities to the manipulations themselves.

It is noteworthy that although Lagrange, Ruffini, and Cauchy were only one small step distant from the idea of group they never actually arrived at it. What prevented them from going further was probably the fact that, as far as their approach to mathematics was concerned, they still belonged to the preformalist school of thought. Indeed, the next move involved the kind of change in the basic philosophical assumptions that was achieved only slightly later by such writers as Hamilton and Peacock. (Although the work of Galois may be regarded as prior to the emergence of formalism, it was not recognized until much later.) The decisive step could not be taken by people who felt that certain external factors, which go beyond mathematics itself, restrict their freedom to create new abstract beings.

Further Development: The Proliferation of Abstract Beings

After the invention of the concept of group, nothing could stop algebra from turning into a science of abstract structures. What happened in mathematics after the first successful attempts at free creation may be described as a true baby boom. No longer fettered by ontological considerations mathematicians felt free to conjure up new algebraic beings without ever asking about their relationship to the physical world or even about their prospective applications in the natural sciences. The richness of the new structures and their links with other regions of mathematics seemed more important. Mathematics stopped being the servant of natural science and from then on was developed for its own sake.

Three salient traits of this pure mathematics in general, and of the new algebra in particular, were its great abstractness, its concern for logical foundations, and its tendency to split into loosely tied subdisciplines. On the one hand, the abstract structures provided the "loftier points of view from which many fields of mathematics, both ancient and modern, [could] be seen as wholes and not as rococo patchworks of dislocated special problems" (Bell, 1951, p. 15). In the language of Garcia and Piaget (1989) the trans-operational stage in algebra was attained where the computational processes investigated at the previous stage could now be viewed from a much higher vantage point. On the other hand, the "passion of abstraction, sometimes quite furious" (Bell, 1951, p. 158) led to the emergence of ever new structures and ever new branches of algebra. The concept of group was accompanied by the notion of invariant and by the theory of matrices, and followed by the ideas of field, ring, and linear space. Eventually algebra "mushroomed into a welter of smaller developments that have little relation to each other or to the original concrete fields" (Kline, 1972, p. 1157).

No wonder, therefore, that at a certain stage the necessity of reunification of algebra and maybe even of the whole of mathematics could be felt among mathematicians. Algebra itself provided the means for this endeavor. In his *Erlanger Program* (1872), Felix Klein united the different geometries, both Euclidean and non-Euclidean, into one theory by characterizing each of them with the help of a certain group of transformations and by saying that from now on geometry should be treated as the study of their invariants. Much later, in the middle of the 20th century, the Bourbaki group set itself an even more ambitious goal: The whole of mathematics was to be reduced to three mother structures. Algebraic structure, with its laws of composition, was proposed as one of them; the other two were the order structure and the topological structure. Eventually the theory of categories was developed that purported to unify all the branches of mathematics.

Here our journey through the history of mathematics ends. As in the previous sections, I would like to compare the past developments to the experiences of those who learn the subject. Not much systematic research has been done that can provide the relevant data. I can point to only one study that seems closely related to the present subject. In this investigation, carried out in Israel by Harel (1985), a teaching unit on linear spaces was developed and taught to secondary-school students. Classroom observations led the researcher to the conclusion that "the objects populating vector spaces are not tangible, thus they are not considered by the students as objects at all; it is only natural, therefore, that the space itself is not conceived as a mathematical object" (p. 64). These findings are hardly surprising. For the last 100 years mathematicians themselves have not seemed to have much difficulty with accepting even the strangest mathematical object on the sole basis of the inner coherence of the resulting system of concepts. It was the dismissal of external criteria for legitimization of mathematical ideas that brought logical considerations to the fore. The intellectual maturity, however, which gave mathematicians the strength to resist the ontological questions, cannot be expected from beginners. An axiomatic system is certainly not enough to convince the mathematically unsophisticated learner about the existence of an object that he or she has no way to see, touch, or just imagine. The doubts as to the nature of such objects, as well as to the legitimacy of the very act of free creation, may impede students' understanding in exactly the same way in which similar considerations obstructed the historical development of abstract structures until the 19th century. Indeed, there is no reason to assume that our student is more mathematically mature than Lagrange or Cauchy.

When no underlying computational process may be offered to make the introduction of a new mathematical object more smooth and natural, the computer may provide some help. The machine has almost unlimited power of reification. The figments of a mathematician's imagination materialize on the screen so that it becomes quite natural to treat them as if they were independent beings, external to the human mind. The conjecture about the possible influence of the computer on learning abstract algebra is now being tested in a study carried out in Israel and in the United States by Leron and Dubinsky (1995). Its results will certainly provide much new information on students' ability to learn advanced mathematics. If the computer proves itself a tool for reification, it may even lead the researchers to the conclusion that the history of algebra would have taken a different course had a powerful number cruncher been available several centuries earlier.

CONCLUDING REMARKS

The history of algebra was presented here as a long sequence of acts of creation in which generations of mathematical objects of increasing abstractness were brought into existence. Students who learn algebra have to recreate these objects for themselves. Some empirical data have been brought forward to enlighten several aspects of this process. It is not surprising that what was far from easy for mathematicians invariably proved to be quite difficult for the learner.

Many examples have been provided here to reinforce the thesis that didactic problems are likely to appear at all those junctions at which mathematicians themselves faltered and asked questions. For those who teach, therefore, familiarity with the history of mathematics is not just optional; rather, it seems indispensable to make them alert to the deeply hidden difficulties concerned with new concepts. The ontological obstacles are ubiquitous and at the same time they are elusive and difficult to detect. Pupils' fundamental problems with such ideas as complex number or variable may be overlooked by the teacher because the latter's own implicit beliefs make him or her oblivious to the very possibility of somebody having a different ontological stance. What helps in concealing ontological stumbling blocks is the fact that a student may become quite skillful in manipulating such mathematical objects as number, function, or algebraic expressions even without reifying them.

One important lesson to be learned from history may be somewhat at variance with the pedagogical beliefs of the modern teacher. The stories just told seem to imply that the reification that is needed for a deep understanding of a concept (say complex number) cannot be expected before some familiarity with secondary processes (e.g., operations on complex numbers) has been attained. On the other hand, without the reification these processes cannot be truly meaningful. The surprising pedagogical conclusion follows from here: Sometimes the teacher and the students must put up with the necessity of practicing techniques even before they are fully understood. In light of this, it appears that in learning and teaching a crucial role is played by patience and persistence. Indeed, history has already shown that these may be the basic weapons against ontological difficulties. Cardan insisted on using complex numbers regardless of "the mental tortures involved" (Kleine, 1980, p. 116). History proved he was right—mathematicians eventually reconciled themselves to the concept. Similarly, today's student should be disciplined enough to work with algebraic techniques and manipulate abstract objects even if he or she has doubts as to their meaning. The teacher should tame his or her impatience when facing deficiencies in learners' understanding. If persistently used, the concepts will eventually become easier to reify—and to accept.

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APPENDIX

The Genesis of Mathematical Objects

While glancing every now and then at the long and turbulent history of number systems, I briefly summarize here one of the possible general scenarios of concept development. This theoretical model bears on the ideas initiated by Piaget and seems to be compatible with the theoretical frameworks proposed recently by some of his followers (e.g., Dubinsky, 1991; Harel & Kaput, 1991). It was first presented in much greater detail in Sfard (1991, 1992).

In all branches of mathematics and, in particular, in computational sciences, one can clearly distinguish two kinds of components: abstract objects and computational processes. Numbers are good examples of the former. The way people think about them and mentally operate upon them resembles the manner in which they perceive and manipulate physical objects. The abstract objects, in turn, serve as inputs and outputs to certain computational procedures—this second ingredient of the mathematical universe.

A closer look at these two separate and ostensibly dissimilar components will reveal an interesting relationship between them. As was explained in detail in Sfard (1991), abstract objects and computational processes, as different as they may seem, are but opposite sides of the same coin—two facets of the same thing. In a sense, the abstract objects are just an alternative way of referring to computational processes: Natural and rational numbers are metaphors for counting and measuring, respectively, and the concepts of negative and complex numbers refer, in fact, to nothing other than the operation of subtracting a number from a smaller one and to the process of extracting a square root of a negative number. Using the terminology introduced in Sfard (1991) I would thus say that any number (like any other mathematical concept, in fact) may be conceived in two ways: operationally, as a process, and structurally, as an object. (I emphasize the word *conceived* to make it clear that I am talking about the way a person perceives, thinks, and talks about abstract ideas and not about the nature of the mathematical entities themselves, whatever the words "the nature of an entity" may mean when considered independently of the epistemic subject.)

Thus, one may say that rational, irrational, negative, and complex numbers are just the more mature incarnations of certain computational processes. When their historical development is scrutinized, it invariably turns out that no sooner did the new numbers enter the scene than a certain nontraditional computational process began to gain recognition. The idea of irrational number stemmed from measuring procedures that could not be encoded as pairs (ratios) of integers. The notions of negative and complex numbers may be traced back to the work of Cardan (*Ars Magna,* 1545) in which the algorithms for solving cubic equations were shown to lead occasionally to such nonroutine procedures as subtractions of a number from a smaller one and extraction of the square root from the products of such subtractions. The interesting thing is that although the two concepts did not really catch on until nearly three centuries later, the persistent reluctance to accept them as legitimate objects did not prevent mathematicians from using them.

The development of the number concept has just been presented as a chain of transitions from operational to structural conceptions. As seen more than once, however, even before the processes which engendered the new kinds of numbers were reified, namely turned into full-blown objects, mathematicians were able to perform them and even to combine them with other operations to obtain more complex computations. I shall say, therefore, that the processes were interiorized and even condensed: They could be easily performed (so were interiorized) and they could be referred to as procedures executed inside a "black box"—something that no longer had to be described in full detail when considered as a part of a composite process (so they were condensed). This three-component pattern, interiorization/condensation/reification, seems to repeat itself at almost every turning point in the history of mathematical ideas—and in the process of learning.

From these theoretical reflections mathematics emerges as a hierarchy of abstract realms built in a sequence of almost identical steps: Time and again, processes performed on certain abstract objects turn into new objects in order to serve as inputs to higher level processes. With respect to a given concept, say, that of negative number, one can distinguish between primary and secondary processes, those which underlie the concept and those which are applied to it, respectively. In the case of negative numbers the subtraction $a - b$, from which the restriction $a \geq b$ has been removed, is a primary process, whereas the arithmetic operations extended to all its results are the secondary processes.

It should be mentioned here that the aforementioned scheme of concept construction is similar in some respects to the model proposed by Garcia and Piaget in their book *History of Science and Psychogenesis*. Their ideas, like those presented here, are based on the assumption of the cyclic nature of the process of knowledge formation common to historical developments and to individual learning. Like the idea of hierarchical construction where the same notions are being conceived and used differently at different levels, Garcia and Piaget's cycle of intraoperational, interoperational, and trans-operational stages reflects the change of perspective which takes place in the course of concept evolution. The intra- and interoperational stages roughly correspond to the phases of primary and secondary processes. The transoperational stage is attained only when the vantage point is pushed even higher and instead of concentrating on individual numbers one shifts attention to the overall structure imposed upon the given set of objects by the secondary operations. Soon the nature of the elements in the set loses its importance and the structure-imposing operations remain the only object of interest. Using our language and ideas we may say that such structures as groups or fields, the emergence of which indicates that the transoperational stage with respect to the concept of number has been attained, are, in a sense, nothing more than a combination of the most general computational process treated as autonomous wholes, thus already condensed, maybe even reified.

One final remark concerning the difficulty of reification: As an ontological shift it is an inherently complicated process. At least two serious reasons for its being very difficult to attain may be mentioned. One of them was called the vicious circle of reification. The use of this name refers to the fact that the reification of primary processes (those which underlie the given concept) seems to be the precondition for the ability to deal with secondary processes (those which are applied to the given concept), whereas the latter seem, in turn, to be a precondition for the former. For example, $3 - 5$ must be treated as a legitimate mathematical object before it can be manipulated and combined—through secondary processes—with other numbers. On the other hand, to speak about such operations as $3 - 5$ and $1 - 3.5$ as numbers, one must be able in advance to use them as inputs to the secondary processes. After all, it is the only way in which one may realize that $3 - 5$ and $1 - 3.5$ obey the same rules as 2, 5, 12 and the like, thus behaving like genuine numbers; such a realization is indispensable to justify and to motivate reification.

The second type of obstacle arises when some semantic concessions must be made before the new abstract object is fully accepted. For example, to talk about a square root of a negative quantity as a number, people must free themselves from their deeply rooted conviction that number is something which expresses quantity—a result of a measuring procedure. It is thus the very process that engendered the concept of number that must now be given up. In this article, I observed this phenomenon of alienation from the primary operational roots time and again while surveying the development of algebra.