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On the trace of idempotent matrices over group algebras

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Abstract Let G be a group and E an idempotent matrix with entries in the group algebra $\mathbf{C}G$. In this paper, we consider the embedding of $\mathbf{C}G$ into the von Neumann algebra $\mathcal{N}G$ and use the center-valued trace on the latter, in order to obtain some information about the coefficients of the Hattori-Stallings rank of E. Our results generalize the inequalities obtained previously by Kaplansky [11], Passi, Passmann, Luthar and Alexander [1,10,12], while providing at the same time a unified and coherent presentation of these, via the notion of moments that are associated with E.

Contents

0 Introduction

Let G be a group and $\mathbf{C}G$ the associated group algebra. A very useful tool in the study of the K-theory group $K_0(\mathbf{C}G)$ is the universal trace defined by Hattori [7] and Stallings [15]

$$r_{HS} : K_0(\mathbf{C}G) \longrightarrow \mathbf{C}G/[\mathbf{C}G, \mathbf{C}G].$$

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Since the vector space $\mathbf{C}G/[\mathbf{C}G, \mathbf{C}G]$ has a basis consisting of the set $\mathcal{C}(G)$ of G-conjugacy classes, the Hattori-Stallings rank $r_{HS}(E)$ of an idempotent $N \times N$ matrix E with entries in $\mathbf{C}G$ can be expressed as a linear combination of the form $\sum_{[g]\in\mathcal{C}(G)} r_g(E)[g]$, where $r_g(E) \in \mathbf{C}$ for all $g \in G$. The use of analytic techniques in the study of E was pioneered by Kaplansky [11], who proved that $r_1(E)$ is a totally real number, which satisfies the inequalities

$$0 \leq r_1(E) \leq N \tag{1}$$

and vanishes only if $E = 0$. Kaplansky's proof used the embedding of the group algebra $\mathbf{C}G$ into the reduced group C^*-algebra C_r^*G of G and employed the canonical trace τ on the latter. In fact, all known proofs of the implication $[r_1(E) = 0] \Longrightarrow [E = 0]$ involve analytic arguments (cf. [13]). Kaplansky's result was complemented by Zaleskii [17], who used reduction to positive characteristic in order to prove that $r_1(E) \in \mathbf{Q}$. Zaleskii's technique was subsequently used by Bass, who proved in [2] that the coefficients $r_g(E)$ are algebraic numbers for all $g \in G$. On the other hand, a modification of Kaplansky's trace argument was used by Weiss [16], in order to prove that for any idempotent $e \in \mathbf{C}G$ and any element $g \in G$ we have

$$|r_g(e)|^2 \leq [G : C_g] \cdot r_1(e),$$

where C_g denotes the centralizer of g in G. Of course, the inequality above has a substance only if the index $[G : C_g]$ is finite, i.e. only if g has finitely many conjugates in G. Weiss' inequality was generalized by Passi and Passman [12], who proved that

$$\sum_{[g]\in\mathcal{C}_f(G)} \frac{|r_g(e)|^2}{[G : C_g]} \leq r_1(e)$$

for any idempotent $e \in \mathbf{C}G$; here, $\mathcal{C}_f(G)$ denotes the subset of $\mathcal{C}(G)$ consisting of the finite conjugacy classes. This latter inequality was extended to idempotent matrices by Alexander in [1]; if E is an idempotent $N \times N$ matrix with entries in $\mathbf{C}G$, then

$$\sum_{[g]\in\mathcal{C}_f(G)} \frac{|r_g(E)|^2}{[G : C_g]} \leq N \cdot r_1(E). \tag{2}$$

Moreover, Alexander showed that the inequality above is an equality if and only if the matrix E is central in $\mathbf{M}_N(\mathbf{C}G)$ (i.e. if and only if $E = cI_N$ for some central idempotent $c \in \mathbf{C}G$). In the special case where G is a finite group, Alexander's inequality was proved by Luthar and Passi in [10].

Our goal in this paper is to present a general scheme that enables one to obtain various inequalities involving the coefficients $r_g(E)$ of the Hattori-Stallings rank of an idempotent matrix E with entries in the group algebra of a group G, for elements $g \in G$ that have only finitely many conjugates. In particular, we generalize and obtain a unified presentation of the inequalities (1) and (2). To that end, we follow Kaplansky and consider the embedding of the group algebra $\mathbf{C}G$ into the von Neumann algebra $\mathcal{N}G$. In this context, the embedding of $\mathbf{C}G$ into $\mathcal{N}G$ has been also considered by Eckmann [4,5] and Schafer [14]. On the other hand, the same idea of embedding into the von Neumann algebra has been extensively used in the recent literature, in order to study conjectures by Atiyah, Singer, Baum and Connes (cf. [9]). By means of this embedding, an idempotent $N \times N$ matrix E with entries in $\mathbf{C}G$ may be viewed as an idempotent in the von Neumann algebra $\mathbf{M}_N(\mathcal{N}G)$. This latter von Neumann algebra is finite, whereas its center is identified with the center $\mathcal{Z}G$ of $\mathcal{N}G$; therefore, we may consider the center-valued trace $a(E) \in \mathcal{Z}G$ of E (cf. [8, Chapter 8]). The abelian von Neumann algebra $\mathcal{Z}G$ is isomorphic with the algebra of essentially bounded measurable functions on some probability space (Ω, P), in such a way that the restriction of the canonical trace τ on $\mathcal{Z}G$ is identified with the expectation (integration) operator. Hence, any operator $c \in \mathcal{Z}G$ may be viewed as an integrable random variable $\widehat{c}$ on (Ω, P), whereas $\tau(c)$ is the expectation $\mathbf{E}(\widehat{c}) = \int \widehat{c}\, dP$. With that in mind, if $a(E) \in \mathcal{Z}G$ is the center-valued trace of an idempotent matrix E as above, we define the sequence of moments $(\mu_n(E))_n$ of E, by letting $\mu_n(E) = \tau(a(E)^n)$ for all n. We prove that the sequence $(\mu_n(E))_n$ is a decreasing sequence of non-negative real numbers; in particular, $0 \leq \mu_n(E) \leq 1$ for all n. These moments can be expressed in terms of the coefficients of the Hattori-Stallings rank $r_{HS}(E) = \sum_{[g]\in\mathcal{C}(G)} r_g(E)[g]$ of E; it turns out that

$$\mu_1(E) = \frac{1}{N} r_1(E) \quad \text{and} \quad \mu_2(E) = \frac{1}{N^2} \sum_{[g]\in\mathcal{C}_f(G)} \frac{|r_g(E)|^2}{[G : C_g]}.$$

In this way, Kaplansky's inequalities (1) are equivalent to the assertion that $\mu_1(E) \in [0, 1]$, whereas Alexander's inequality (2) is the assertion that $\mu_2(E) \leq \mu_1(E)$. Besides the inequalities that involve the higher moments of E, we may use the analytic properties of the center-valued trace $a(E)$ in order to obtain bounds for the absolute value of the coefficients of the Hattori-Stallings rank $r_{HS}(E)$. Moreover, the study of the asymptotic behavior of the moments of E gives information about its global behavior: If $e \in CG$ is an idempotent then the limit $\lim_n \mu_n(e)$ is a measure of the relative size of the two-sided ideal I generated by e inside the group algebra $\mathbf{C}G$ (cf. Remark 6.5).

The contents of the paper are as follows: In the first Section, we use a result of Bass, in order to conclude that the coefficients $r_g(E) \in \mathbf{C}$ vanish for any group element $g \in G$ of infinite order that has finitely many conjugates. In Section 2, we consider the von Neumann algebra $\mathbf{M}_N(\mathcal{N}G)$ and obtain a formula for the center-valued trace $a(E)$ of an idempotent matrix $E \in \mathbf{M}_N(\mathbf{C}G) \subseteq \mathbf{M}_N(\mathcal{N}G)$, in terms of the Hattori-Stallings rank $r_{HS}(E)$. In the following Section, we use that formula in order to bound the absolute value of the coefficients of $r_{HS}(E)$. The resulting inequalities are *local*, in the sense that they involve either a single group element or the elements in a finite cyclic subgroup of G. In Section 4, we introduce the moments of an idempotent matrix E as above and express them in terms of the coefficients of the Hattori-Stallings rank $r_{HS}(E)$. Using elementary (deterministic) inequalities among the powers of the random variable which is associated with the operator $a(E)$, we obtain a sequence of inequalities among the $r_g(E)$'s that generalize those of Kaplansky and Alexander. In the following Section, we study the differences between consecutive moments and provide lower bounds for them. Finally, in Section 6, we study the geometric significance of the limit $\lim_n \mu_n(E)$ and obtain a lower bound for it, in terms of E. We conclude the paper with an Appendix, where we extend Alexander's version of the inequality (2) for semi-simple elements in matrix rings over group algebras, by considering the higher moments of the idempotent matrices that are involved.

Notations and terminology. For any element g of a group G we denote by $[g]$ its conjugacy class and let $C_g = \{x \in G : xg = gx\}$ be its centralizer. The set of all conjugacy classes in G is denoted by $\mathcal{C}(G)$, whereas $\mathcal{C}_f(G) \subseteq \mathcal{C}(G)$ is the subset consisting of the finite ones. The subset (normal subgroup) of G consisting of those elements that have only finitely many conjugates is denoted by G_f; in other words, $G_f = \{g \in G : [g] \in \mathcal{C}_f(G)\} = \{g \in G : [G : C_g] < \infty\}$.

1 A consequence of Bass' vanishing criterion

For any ring R we consider the additive subgroup $[R, R] \subseteq R$ generated by the commutators $rr' - r'r$, $r, r' \in R$. The Hattori-Stallings rank $r_{HS}(E)$ of an idempotent matrix E with entries in R is the residue class of the trace $tr(E) \in R$ in the quotient group $T(R) = R/[R, R]$ (cf. [7,15]). Then, $r_{HS}(E)$ depends only upon the class of E in the K-theory group $K_0(R)$.

We are interested in the special case where $R = \mathbf{C}G$ is the group algebra of a group G. Then, $T(\mathbf{C}G) = \mathbf{C}G/[\mathbf{C}G, \mathbf{C}G]$ is a vector space with basis the set $\mathcal{C}(G)$ of G-conjugacy classes and hence the rank $r_{HS}(E)$ of an idempotent matrix E with entries in $\mathbf{C}G$ can be viewed as a complex-valued function on G, which is constant on conjugacy classes and vanishes in all but finitely many of them. If we denote by $r_g(E)$ the value of that function on the conjugacy class $[g]$ of any element $g \in G$, then we can write $r_{HS}(E) = \sum_{[g]\in\mathcal{C}(G)} r_g(E)[g]$. Working by reduction to positive characteristic, Zaleskii proved in [17] that the coefficient $r_1(E)$ is always a rational number. Bass used the same technique and proved in [2] that the coefficients $r_g(E)$ are algebraic numbers for all $g \in G$, obtaining at the same time a criterion for them to vanish.

Theorem 1.1 (cf. [2, Theorem 8.1(c)]) *Let G be a group, E an idempotent matrix with entries in the group algebra $\mathbf{C}G$ and $r_{HS}(E) = \sum_{[g]\in\mathcal{C}(G)} r_g(E)[g]$ its Hattori-Stallings rank. Then, there is a positive integer u such that for any element $g \in G$ with $r_g(E) \neq 0$, we have $[g] = \left[g^{p^u}\right] \in \mathcal{C}(G)$ for all but finitely many prime numbers p.* $\square$

Corollary 1.2 *Let G be a group, E an idempotent matrix with entries in the group algebra $\mathbf{C}G$ and $r_{HS}(E) = \sum_{[g]\in\mathcal{C}(G)} r_g(E)[g]$ its Hattori-Stallings rank. If $g \in G_f$ is an element of infinite order, then $r_g(E) = 0$.*

Proof We argue by contradiction, assuming that there exists an element $g \in G_f$ of infinite order such that $r_g(E) \neq 0$. In that case, Theorem 1.1 implies that there is a prime number p and a positive integer u such that $[g] = \left[g^{p^u}\right] \in \mathcal{C}(G)$; then, $[g] = \left[g^{p^{nu}}\right] \in \mathcal{C}(G)$ and hence $g^{p^{nu}} \in [g]$ for all $n \geq 1$. Since g is an element of infinite order, it follows that its conjugacy class is an infinite set. But this is absurd, since $g \in G_f$. $\square$

2 The center-valued trace on $\mathbf{M}_n(\mathcal{N}G)$

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ the algebra of bounded linear operators on it. The weak operator topology (WOT) on $\mathcal{B}(\mathcal{H})$ is the locally convex topology defined by the family of semi-norms $(P_{\xi,\eta})_{\xi,\eta\in\mathcal{H}}$, where $P_{\xi,\eta}(a) = \mid \langle a(\xi), \eta\rangle \mid$ for any two vectors $\xi, \eta \in \mathcal{H}$ and any operator $a \in \mathcal{B}(\mathcal{H})$. We consider a von Neumann algebra $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$; in other words, $\mathcal{N}$ is a WOT-closed $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ (cf. [8, Chapter 6]). A vector $\xi \in \mathcal{H}$ is called separating for $\mathcal{N}$ if the evaluation map $ev_\xi : \mathcal{N} \longrightarrow \mathcal{H}$ is injective. Two projections $e, f \in \mathcal{N}$ are called equivalent in $\mathcal{N}$ if there is a partial isometry $u \in \mathcal{N}$ such that $u^*u = e$ and $uu^* = f$. The von Neumann algebra $\mathcal{N}$ is called finite if there is no projection $e \in \mathcal{N}$ with $e \neq 1$, which is equivalent to 1 in $\mathcal{N}$. A finite von Neumann algebra $\mathcal{N}$ has a center-valued trace, as stated in the following result.

Theorem 2.1 (cf. [8, Chapter 8]) *Let $\mathcal{N}$ be a finite von Neumann algebra with center $\mathcal{Z}$. Then, there is a linear map $t : \mathcal{N} \longrightarrow \mathcal{Z}$ having the following properties:*

(i) t is a trace, i.e. $t(ab) = t(ba)$ for all $a, b \in \mathcal{N}$.

(ii) $t(a) = a$ for all $a \in \mathcal{Z}$.

(iii) The element $t(a^*a) \in \mathcal{Z}$ is self-adjoint and $t(a^*a) > 0$ for all $a \in \mathcal{N} \setminus \{0\}$.

(iv) t is a $\mathcal{Z}$-module map, i.e. $t(ab) = at(b)$ for all $a \in \mathcal{Z}$ and $b \in \mathcal{N}$.

(v) If $\tau : \mathcal{N} \longrightarrow \mathbf{C}$ is a norm-continuous trace functional then $\tau = \tau' \circ t$, where τ' is the restriction of τ to $\mathcal{Z}$.

The linear map t is uniquely characterized by properties (i), (ii) and (iii). □

Remarks 2.2 (i) Let $\mathcal{N}$ be a finite von Neumann algebra with center $\mathcal{Z}$ and consider its center-valued trace t. If $e \in \mathcal{N}$ is an idempotent then the element $t(e) \in \mathcal{Z}$ is self-adjoint and satisfies the inequalities $0 \leq t(e) \leq 1$. Moreover, $t(e) = 0$ (resp. $t(e) = 1$) if and only if $e = 0$ (resp. $e = 1$). In order to verify these assertions, we fix a projection $p \in \mathcal{N}$ such that $e = pe$ and $p = ep$.[1] Since t is a trace, we have $t(e) = t(p)$; on the other hand, $p = p^*p$ and hence the element $t(p) \in \mathcal{Z}$ is self-adjoint with $t(p) \geq 0$ (cf. Theorem 2.1 (iii)). If $t(e) = 0$ then $t(p) = 0$ and hence $p = 0$ (loc.cit.); therefore, it follows that $e = pe = 0$. Considering the idempotent $1 - e$, we conclude that $t(e) \leq 1$ with strict inequality if $e \neq 1$.

(ii) Let $\mathcal{N}$ be a finite von Neumann algebra with center $\mathcal{Z}$ and consider its center-valued trace t. If $e \in \mathcal{N}$ is an idempotent such that $t(e) \in \mathcal{Z}$ is also an idempotent, then $e \in \mathcal{Z}$ (and hence $t(e) = e$).[2] Indeed, if $t(e)^2 = t(e)$ then the element $e' = e - et(e) = e(1 - t(e))$ is idempotent and

$$t(e') = t(e) - t(et(e)) = t(e) - t(e)t(e) = t(e) - t(e)^2 = 0,$$

where the second equality follows since t is a $\mathcal{Z}$-module map (cf. Theorem 2.1 (iv)). In view of (i) above, we conclude that $e' = 0$ and hence $e = et(e)$; therefore,

[1] One can find such a projection p for any idempotent e in a C^*-algebra; this can be shown, for example, by using the argument employed in the proof of [3, Theorem 3.1].

[2] Conversely, if the idempotent $e \in \mathcal{N}$ is central then $t(e) = e$ and hence $t(e) \in \mathcal{Z}$ is an idempotent.

it follows that the element $e'' = t(e) - e$ is idempotent. Since t is the identity on $\mathcal{Z}$ (cf. Theorem 2.1 (ii)), we have

$$t(e'') = t(t(e)) - t(e) = t(e) - t(e) = 0.$$

Invoking (i) above once again, we conclude that $e'' = 0$ and hence $e = t(e) \in \mathcal{Z}$.

(iii) Let $\mathcal{N}$ be a von Neumann algebra with center $\mathcal{Z}$ and assume that there is a linear map $t : \mathcal{N} \longrightarrow \mathcal{Z}$, having properties (i) and (iii) in the statement of Theorem 2.1. Then, the algebra $\mathcal{N}$ is finite. Indeed, let $e \in \mathcal{N}$ be a projection which is equivalent to 1 in $\mathcal{N}$. This means that there is a partial isometry $u \in \mathcal{N}$ such that $e = u^*u$ and $1 = uu^*$. Then, the element $1 - e \in \mathcal{N}$ is a projection and

$$t(1-e) = t(1) - t(e) = t(uu^*) - t(u^*u) = 0,$$

where the last equality follows from the trace property of t. In view of the faithfulness of t, we have $1 - e = 0$ and hence $e = 1$.

Let $\mathcal{N}$ be a von Neumann algebra of operators acting on the Hilbert space $\mathcal{H}$ with center $\mathcal{Z}$. Then, for any positive integer n the algebra $\mathbf{M}_n(\mathcal{N})$ of $n \times n$ matrices with entries in $\mathcal{N}$ is a von Neumann algebra of operators acting on the n-fold direct sum $\mathcal{H} \oplus \cdots \oplus \mathcal{H}$ with center $\mathcal{Z}_n = \{aI_n : a \in \mathcal{Z}\}$. We note that the map $aI_n \mapsto a$, $aI_n \in \mathcal{Z}_n$, is an isometric isomorphism $\mathcal{Z}_n \xrightarrow{\sim} \mathcal{Z}$; in the sequel, we identify $\mathcal{Z}_n$ with $\mathcal{Z}$ by means of this isomorphism. If the algebra $\mathcal{N}$ is finite then the center-valued trace $t : \mathcal{N} \longrightarrow \mathcal{Z}$ induces a trace

$$t_n : \mathbf{M}_n(\mathcal{N}) \longrightarrow \mathcal{Z},$$

which maps any matrix $A = (a_{ij})_{i,j} \in \mathbf{M}_n(\mathcal{N})$ onto $\sum_{i=1}^n t(a_{ii}) \in \mathcal{Z}$ for all $n \geq 1$.

Proposition 2.3 *Let $\mathcal{N}$ be a finite von Neumann algebra with center $\mathcal{Z}$. We fix an integer $n \geq 1$ and consider the matrix algebra $\mathbf{M}_n(\mathcal{N})$ and the trace t_n defined above. Then, the von Neumann algebra $\mathbf{M}_n(\mathcal{N})$ is finite, whereas its own center-valued trace $t_{(n)}$ maps any matrix $A = (a_{ij})_{i,j} \in \mathbf{M}_n(\mathcal{N})$ onto $\frac{1}{n}t_n(A) \in \mathcal{Z}$.*

Proof It is easily seen that the map $A \mapsto \frac{1}{n}t_n(A)$, $A \in \mathbf{M}_n(\mathcal{N})$, has properties (i), (ii) and (iii) in the statement of Theorem 2.1; hence, $\mathbf{M}_n(\mathcal{N})$ is finite (cf. Remark 2.2 (iii)). The proof is now finished by invoking the uniqueness assertion of Theorem 2.1. □

Let G be a group and consider its action on the Hilbert space $\ell^2 G$ by left translations. The associated algebra homomorphism

$$L : \mathbf{C}G \longrightarrow \mathcal{B}(\ell^2 G)$$

identifies $\mathbf{C}G$ with the $*$-algebra $L(\mathbf{C}G) = \{L_a : a \in \mathbf{C}G\}$ of operators on $\ell^2 G$, in such a way that $L_g^* = L_{g^{-1}}$ for any element $g \in G$. Then, the group von Neumann algebra $\mathcal{N}G$ is defined as the WOT-closure of $L(\mathbf{C}G)$ in $\mathcal{B}(\ell^2 G)$. We consider the linear functional

$$\tau : \mathcal{N}G \longrightarrow \mathbf{C},$$

which is defined by letting $\tau(a) = \langle a(\delta_1), \delta_1\rangle$ for all $a \in \mathcal{N}G$; here, δ_1 denotes the vector of the canonical orthonormal basis $(\delta_g)_{g\in G}$ of $\ell^2 G$ that corresponds to the element $1 \in G$. We record for future reference the following well-known result:

Lemma 2.4 *Let G be a group, $\mathcal{N}G$ the associated von Neumann algebra and τ the linear functional defined above. Then:*

(i) τ is a WOT-continuous trace with $\tau(I) = 1$, where $I \in \mathcal{N}G$ is the identity operator.

(ii) The vector $\delta_1 \in \ell^2 G$ is separating for $\mathcal{N}G$.

(iii) τ is positive and faithful, i.e. $\tau(a^*a) > 0$ for all $a \in \mathcal{N}G \setminus \{0\}$.

The trace τ is called the canonical trace on $\mathcal{N}G$. □

An immediate consequence of Lemma 2.4 is that the von Neumann algebra $\mathcal{N}G$ associated with a group G is finite. This follows by repeating the argument used in Remark 2.2 (iii), with the canonical trace τ in the place of t therein. In order to identify the corresponding center-valued trace, we note that the center $Z(L(\mathbf{C}G))$ of the algebra $L(\mathbf{C}G)$ is contained in the center $\mathcal{Z}G$ of $\mathcal{N}G$.[3] Indeed, if $a \in Z(L(\mathbf{C}G))$ then we have $ab = ba$ for all $b \in L(\mathbf{C}G)$. Since multiplication in $\mathcal{B}(\ell^2 G)$ is separately WOT-continuous, it follows that $ab = ba$ for all $b \in \mathcal{N}G$ and hence $a \in \mathcal{Z}G$. On the other hand, for any $g \in G_f$ the element $c(g) = \sum_{x\in[g]} x$ is easily seen to be central in $\mathbf{C}G$; in fact, the set $\{c(g) : [g] \in \mathcal{C}_f(G)\}$ is a basis of the center $Z(\mathbf{C}G)$ of $\mathbf{C}G$. It follows that the operator $L_{c(g)} = \sum_{x\in[g]} L_x$ is central in $L(\mathbf{C}G)$ and hence $L_{c(g)} \in Z(L(\mathbf{C}G)) \subseteq \mathcal{Z}G$ for any $g \in G_f$. We can obtain an explicit formula for the center-valued trace on elements of the subalgebra $L(\mathbf{C}G) \subseteq \mathcal{N}G$ in terms of these operators; the reader may find a proof of the next result in [6, Proposition 2.7].

Proposition 2.5 *Let G be a group and consider an element $g \in G$ and the center-valued trace t on the von Neumann algebra $\mathcal{N}G$.*

(i) If $g \in G_f$ then $t(L_g)$ is the operator $\frac{1}{[G:C_g]} L_{c(g)} = \frac{1}{[G:C_g]} \sum_{x\in[g]} L_x$.

(ii) If $g \notin G_f$ then $t(L_g) = 0$. □

Let G be a group. For any positive integer n we consider the composition

$$\mathbf{M}_n(\mathbf{C}G) \xrightarrow{L_n} \mathbf{M}_n(\mathcal{N}G) \xrightarrow{t_{(n)}} \mathcal{Z}G, \tag{3}$$

where L_n is the homomorphism induced by L, $\mathcal{Z}G$ the center of $\mathcal{N}G$ and $t_{(n)} = \frac{1}{n} t_n$ the center-valued trace of the von Neumann algebra $\mathbf{M}_n(\mathcal{N}G)$ (cf. Proposition 2.3). We also consider the Hattori-Stallings trace maps $r_{HS} : \mathbf{M}_n(\mathbf{C}G) \longrightarrow T(\mathbf{C}G)$ and $r_{HS} : \mathbf{M}_n(\mathcal{N}G) \longrightarrow T(\mathcal{N}G)$. Then, there is a commutative diagram

$$\begin{array}{ccccc} \mathbf{M}_n(\mathbf{C}G) & \xrightarrow{L_n} & \mathbf{M}_n(\mathcal{N}G) & \xrightarrow{t_{(n)}} & \mathcal{Z}G \\ r_{HS} \downarrow & & r_{HS} \downarrow & & \| \\ T(\mathbf{C}G) & \xrightarrow{T(L)} & T(\mathcal{N}G) & \xrightarrow{\frac{1}{n}\bar{t}} & \mathcal{Z}G \end{array} \tag{4}$$

where $T(L)$ is the map induced from L by passage to the quotients and $\bar{t}$ is that induced by the center-valued trace t on $\mathcal{N}G$.

[3] One can show that $\mathcal{Z}G$ coincides with the WOT-closure of $Z(L(\mathbf{C}G))$ in $\mathcal{B}(\ell^2 G)$, but we shall not make any use of this fact below.

Proposition 2.6 (cf. [9, §9.5.2]) *Let G be a group and $\mathbf{C}G$ the associated group algebra. We consider the von Neumann algebra $\mathcal{N}G$, its center $\mathcal{Z}G$ and fix a positive integer n. Then, the image of an idempotent matrix $E \in \mathbf{M}_n(\mathbf{C}G)$ with Hattori-Stallings rank $r_{HS}(E) = \sum_{[g]\in\mathcal{C}(G)} r_g(E)[g]$ under the composition (3) is the operator*

$$a = \frac{1}{n}\sum_{g\in G_f} \frac{r_g(E)}{[G:C_g]} L_g = \frac{1}{n}\sum_{g\in G_{f,tor}} \frac{r_g(E)}{[G:C_g]} L_g \in \mathcal{Z}G,$$

where $G_{f,tor} \subseteq G_f$ is the subset consisting of the torsion elements $g \in G_f$.

Proof We compute

$$\begin{aligned}
\big(t_{(n)} \circ L_n\big)(E) &= \tfrac{1}{n}\left(\bar{t} \circ T(L) \circ r_{HS}\right)(E) \\
&= \tfrac{1}{n}\left(\bar{t} \circ T(L)\right)\left(\sum_{[g]\in\mathcal{C}(G)} r_g(E)[g]\right) \\
&= \tfrac{1}{n}\sum_{[g]\in\mathcal{C}(G)} r_g(E)\left(\bar{t} \circ T(L)\right)[g] \\
&= \tfrac{1}{n}\sum_{[g]\in\mathcal{C}(G)} r_g(E)\, t(L_g) \\
&= \tfrac{1}{n}\sum_{[g]\in\mathcal{C}_f(G)} \tfrac{r_g(E)}{[G:C_g]} L_{c(g)} \\
&= \tfrac{1}{n}\sum_{g\in G_f} \tfrac{r_g(E)}{[G:C_g]} L_g.
\end{aligned}$$

In the above chain of equalities, the first one follows from the commutativity of diagram (4) and the fifth one from Proposition 2.5. This completes the proof, in view of Corollary 1.2. □

Theorem 2.7 *Let G be a group, $\mathbf{C}G$ the associated group algebra, n a positive integer and $E \in \mathbf{M}_n(\mathbf{C}G)$ an idempotent matrix with Hattori-Stallings rank $r_{HS}(E) = \sum_{[g]\in\mathcal{C}(G)} r_g(E)[g]$. We consider the von Neumann algebra $\mathcal{N}G$ and let*

$$a = a(E) = \frac{1}{n}\sum_{g\in G_f} \frac{r_g(E)}{[G:C_g]} L_g = \frac{1}{n}\sum_{g\in G_{f,tor}} \frac{r_g(E)}{[G:C_g]} L_g \in \mathcal{N}G,$$

where $G_{f,tor} \subseteq G_f$ is the subset consisting of the torsion elements $g \in G_f$. Then:

(i) The operator $a \in \mathcal{N}G$ is central, self-adjoint and satisfies the inequalities $0 \le a \le I$.

(ii) $a = 0$ (resp. $a = I$) if and only if $E = 0$ (resp. $E = I_n$).

(iii) $a = a^2$ if and only if $E = cI_n$ for some central idempotent $c \in \mathbf{C}G$.

Proof Let $E' = L_n(E) \in \mathbf{M}_n(\mathcal{N}G)$ be the idempotent matrix obtained from E by applying the homomorphism L to its entries. Then, Proposition 2.6 implies that $a = t_{(n)}(E')$, where $t_{(n)}$ is the center-valued trace on $\mathbf{M}_n(\mathcal{N}G)$. Therefore, assertions (i) and (ii) are immediate consequences of Remark 2.2 (i). Moreover, it follows from Remark 2.2 (ii) that the operator a is idempotent if and only if the matrix E' is central in $\mathbf{M}_n(\mathcal{N}G)$. Since the inverse image $L^{-1}(\mathcal{Z}G) \subseteq \mathbf{C}G$ of the center $\mathcal{Z}G$ of $\mathcal{N}G$ is the center of the group algebra, this latter condition is equivalent to the existence of a central element $c \in \mathbf{C}G$, such that $E = cI_n$. □

As a prelude to the results that will be obtained in the following Sections, we conclude the present one by stating an immediate consequence of Theorem 2.7.

Corollary 2.8 (cf. [16, proof of Theorem 2]) *Let G be a group, E an idempotent matrix with entries in the group algebra $\mathbf{C}G$ and $r_{HS}(E) = \sum_{[g]\in\mathcal{C}(G)} r_g(E)[g]$ its Hattori-Stallings rank. Then, $r_{g^{-1}}(E) = \overline{r_g(E)}$ for all $g \in G_f$.*

Proof It follows from Theorem 2.7 that the operator $\sum_{g\in G_f} \frac{r_g(E)}{[G:C_g]} L_g \in \mathcal{B}(\ell^2 G)$ is self-adjoint. Since $C_g = C_{g^{-1}}$, we conclude that $r_{g^{-1}}(E) = \overline{r_g(E)}$ for any element $g \in G_f$. □

Remarks 2.9 Let G be a group, E an idempotent matrix with entries in $\mathbf{C}G$ and $r_{HS}(E) = \sum_{[g]\in\mathcal{C}(G)} r_g(E)[g]$ its Hattori-Stallings rank.

(i) As we have already noted, Bass has proved in [2] that the complex number $r_g(E)$ is algebraic for all $g \in G$. In the special case where the element g has finitely many conjugates, this result is an immediate consequence of Corollary 2.8. Moreover, in that case, the algebraic number $r_g(E)$ has totally real modulus. Indeed, let us consider an element $g \in G_f$ and the complex number $x = r_g(E)$. Then, for any automorphism σ of the field $\mathbf{C}$ we have

$$\sigma(\overline{x}) = \sigma\left(\overline{r_g(E)}\right) = \sigma\left(r_{g^{-1}}(E)\right) = r_{g^{-1}}(\sigma(E)) = \overline{r_g(\sigma(E))} = \overline{\sigma\left(r_g(E)\right)} = \overline{\sigma(x)},$$

where $\sigma(E)$ is the idempotent matrix obtained from E by applying σ to the complex numbers that are involved in its entries. Since there are both real and non-real transcendental numbers, whereas the group $Aut(\mathbf{C})$ acts transitively on them, it follows that x is algebraic. Moreover, if $y = \sqrt{x\overline{x}}$ is the modulus of x, then for any $\sigma \in Aut(\mathbf{C})$

$$\sigma(y)^2 = \sigma(y^2) = \sigma(x\overline{x}) = \sigma(x)\,\sigma(\overline{x}) = \sigma(x)\,\overline{\sigma(x)}$$

is a non-negative real number and hence $\sigma(y) \in \mathbf{R}$. Therefore, y is totally real.[4]

(ii) If $g \in G_f$ is an element which is conjugate to its inverse, then Corollary 2.8 implies that the complex number $r_g(E)$ is real (and hence totally real, in view of (i) above). In particular, $r_g(E)$ is totally real if $g \in G_f$ is an element of order 2.

3 Local inequalities

Let G be a group and consider an idempotent matrix E with entries in the group algebra $\mathbf{C}G$ and an element $g \in G_f$. In this Section, we prove certain inequalities involving the absolute value of the coefficient $r_g(E) \in \mathbf{C}$ of the Hattori-Stallings rank $r_{HS}(E)$ of E. In view of Corollary 1.2, the interesting case is that where g is an element of finite order.

[4] This argument was communicated to me by T. Schick.

Lemma 3.1 *Let G be a group and consider the canonical trace τ on the von Neumann algebra $\mathcal{N}G$. Let $e \in \mathcal{N}G$ be an idempotent and a a self-adjoint operator in the center $\mathcal{Z}G$ of $\mathcal{N}G$, such that $0 \leq a \leq I$, where $I \in \mathcal{N}G$ is the identity operator. Then, the complex number $\tau(ea)$ is real and*

$$\max\{0, \tau(a) + \tau(e) - 1\} \leq \tau(ea) \leq \min\{\tau(e), \tau(a)\}.$$

Proof Since the element $e \in \mathcal{N}G$ is idempotent, its center-valued trace $t(e) \in \mathcal{Z}G$ is self-adjoint and ≥ 0 (cf. Remark 2.2 (i)). It follows that $0 \leq t(e)a \leq t(e)$ and hence we may invoke the positivity of τ in order to conclude that $0 \leq \tau(t(e)a) \leq \tau(t(e))$. On the other hand, we have $\tau(t(e)a) = \tau(t(ea)) = \tau(ea)$ and $\tau(t(e)) = \tau(e)$ (cf. Theorem 2.1 (iv), (v)). Hence, we have proved that

$$0 \leq \tau(ea) \leq \tau(e). \tag{5}$$

Replacing e by the complementary idempotent $I - e$, we obtain the inequalities

$$0 \leq \tau((I-e)a) \leq \tau(I-e). \tag{6}$$

It is easily seen that the combination of (5) and (6) above gives precisely the inequalities in the statement. □

Proposition 3.2 *Let G be a group and a a self-adjoint operator in the center of the von Neumann algebra $\mathcal{N}G$, such that $0 \leq a \leq I$, where $I \in \mathcal{N}G$ is the identity operator. For any element $g \in G$ we consider the complex number $a_g = \langle a(\delta_1), \delta_g\rangle$; in particular, $a_1 = \tau(a)$, where τ is the canonical trace on $\mathcal{N}G$. If $g \in G$ is an element of finite order n and $\zeta \in \mathbf{C}$ an n-th root of unity, then the complex number $\sum_{i=1}^{n-1} \zeta^i a_{g^i}$ is real and*

$$-\min\{a_1, (n-1)(1-a_1)\} \leq \sum_{i=1}^{n-1} \zeta^i a_{g^i} \leq \min\{1-a_1, (n-1)a_1\}.$$

Proof We consider the element $e = \frac{1}{n}\sum_{i=0}^{n-1} \zeta^{-i} L_{g^i} \in \mathcal{N}G$; it is easily seen that e is an idempotent with $\tau(e) = \frac{1}{n}$. On the other hand, we have

$$\begin{aligned}
\tau(ea) &= \tfrac{1}{n}\sum_{i=0}^{n-1} \zeta^{-i}\tau(L_{g^i}a)\\
&= \tfrac{1}{n}\sum_{i=0}^{n-1} \zeta^{-i}\langle L_{g^i}a(\delta_1), \delta_1\rangle\\
&= \tfrac{1}{n}\sum_{i=0}^{n-1} \zeta^{-i}\langle a(\delta_1), L_{g^i}^*(\delta_1)\rangle\\
&= \tfrac{1}{n}\sum_{i=0}^{n-1} \zeta^{-i}\langle a(\delta_1), \delta_{g^{-i}}\rangle\\
&= \tfrac{1}{n}\sum_{i=0}^{n-1} \zeta^{-i} a_{g^{-i}}\\
&= \tfrac{1}{n}\sum_{i=0}^{n-1} \zeta^{i} a_{g^{i}}.
\end{aligned}$$

Therefore, Lemma 3.1 implies that

$$\max\left\{0, a_1 + \frac{1}{n} - 1\right\} \le \frac{1}{n}\sum_{i=0}^{n-1}\zeta^i a_{g^i} \le \min\left\{\frac{1}{n}, a_1\right\}$$

and hence

$$\max\{0, na_1 + 1 - n\} \le \sum_{i=0}^{n-1}\zeta^i a_{g^i} \le \min\{1, na_1\}.$$

It is easily seen that the inequalities in the statement follow from the ones above by subtracting a_1 from all sides. □

Corollary 3.3 *Let G be a group and $\mathbf{C}G$ the associated group algebra. We fix a positive integer N and consider an idempotent matrix $E \in \mathbf{M}_N(\mathbf{C}G)$ with Hattori-Stallings rank $r_{HS}(E) = \sum_{[g]\in\mathcal{C}(G)} r_g(E)[g]$. If $g \in G_f$ is an element of finite order n and $\zeta \in \mathbf{C}$ an n-th root of unity, then the complex number $\sum_{i=1}^{n-1}\zeta^i \frac{r_{g^i}(E)}{[G:C_{g^i}]}$ is real and*

$$-\min\{r_1(E), (n-1)(N - r_1(E))\} \le \sum_{i=1}^{n-1}\zeta^i \frac{r_{g^i}(E)}{[G:C_{g^i}]} \le \min\{N - r_1(E), (n-1)r_1(E)\}.$$

Proof Let $a = \frac{1}{N}\sum_{g\in G_f}\frac{r_g(E)}{[G:C_g]}L_g \in \mathcal{N}G$ be the operator which is associated with E as in Theorem 2.7. Then, the proof is finished by applying Proposition 3.2 to a and multiplying through the resulting inequalities by N. □

Remarks 3.4 (i) Let n be a positive integer and fix a primitive n-th root of unity $\zeta \in \mathbf{C}$ and complex numbers $s_0, s_1 \dots, s_{n-1}$. If we define the complex numbers $\sigma_0, \sigma_1, \dots, \sigma_{n-1}$ by letting $\sigma_j = \sum_{i=0}^{n-1}\zeta^{ij}s_i$ for all j, then it is easily seen that $s_i = \frac{1}{n}\sum_{j=0}^{n-1}\zeta^{-ij}\sigma_j$ for all i.

(ii) Let G be a group, $\mathbf{C}G$ the associated group algebra and $E \in \mathbf{M}_N(\mathbf{C}G)$ an idempotent matrix with Hattori-Stallings rank $r_{HS}(E) = \sum_{[g]\in\mathcal{C}(G)} r_g(E)[g]$. We consider an element $g \in G_f$ of order $n > 1$ and a primitive n-th root of unity $\zeta \in \mathbf{C}$. If we define $\rho_j = \sum_{i=0}^{n-1}\zeta^{ij}\frac{r_{g^i}(E)}{[G:C_{g^i}]}$ for all $j = 0, 1, \dots, n-1$, then we have $\frac{r_g(E)}{[G:C_g]} = \frac{1}{n}\sum_{j=0}^{n-1}\zeta^{-j}\rho_j$ (cf. (i) above). Therefore, the inequalities of Corollary 3.3 imply that $\frac{|r_g(E)|}{[G:C_g]} \le \min\{N, nr_1(E)\}$ and hence, considering the matrix $I_N - E$, we conclude that $\frac{|r_g(E)|}{[G:C_g]} \le \min\{N, nr_1(E), n(N - r_1(E))\}$.

Lemma 3.5 *Let G be a group and τ the canonical trace on the von Neumann algebra $\mathcal{N}G$. We consider two operators $b, c \in \mathcal{N}G$ and assume that c is self-adjoint and ≥ 0. Then, we have $|\tau(bc)| \le \|b\|\,\tau(c)$.*

Proof Using the Cauchy-Schwartz inequality, we have

$$\begin{aligned}|\,\tau(d^*bd)\,| &= |\ \langle d^*bd(\delta_1), \delta_1\rangle\ | \\ &= |\ \langle bd(\delta_1), d(\delta_1)\rangle\ | \\ &\leq \|\, bd(\delta_1)\,\| \cdot \|\, d(\delta_1)\,\| \\ &\leq \|\, b\,\| \cdot \|\, d(\delta_1)\,\|^2 \\ &= \|\, b\,\|\ \langle d(\delta_1), d(\delta_1)\rangle \\ &= \|\, b\,\|\ \langle d^*d(\delta_1), \delta_1\rangle \\ &= \|\, b\,\|\ \tau(d^*d)\end{aligned}$$

For all $b, d \in \mathcal{N}G$. Hence, τ being a trace, we have $|\,\tau(bdd^*)\,| \leq \|\, b\,\|\ \tau(dd^*)$ for all $b, d \in \mathcal{N}G$. This finishes the proof, since any self-adjoint operator c in $\mathcal{N}G$ with $c \geq 0$ is equal to dd^* for a suitable $d \in \mathcal{N}G$. □

Corollary 3.6 *Let G be a group and $\mathbf{C}G$ the associated group algebra. We consider a positive integer N and let $E \in \mathbf{M}_N(\mathbf{C}G)$ be an idempotent matrix with Hattori-Stallings rank $r_{HS}(E) = \sum_{[g]\in\mathcal{C}(G)} r_g(E)[g]$. Then, $\frac{|r_g(E)|}{[G:C_g]} \leq \min\{r_1(E), N - r_1(E)\}$ for all $g \in G_f \setminus \{1\}$.*

Proof Let $a = \frac{1}{N}\sum_{g\in G_f} \frac{r_g(E)}{[G:C_g]} L_g \in \mathcal{N}G$ be the operator which is associated with E as in Theorem 2.7 and fix an element $g \in G_f \setminus \{1\}$. Since a is self-adjoint and ≥ 0, Lemma 3.5 implies that

$$\frac{1}{N}\frac{|\,r_g(E)\,|}{[G:C_g]} = |\,\langle a(\delta_1), \delta_g\rangle\,| = |\,\langle a(\delta_1), L_g(\delta_1)\rangle\,| = |\,\tau(L_g^* a)\,| \leq \tau(a) = \frac{1}{N} r_1(E),$$

i.e. that $\frac{|r_g(E)|}{[G:C_g]} \leq r_1(E)$. Considering the complementary idempotent $I_N - E$, it follows that $\frac{|r_g(E)|}{[G:C_g]} \leq N - r_1(E)$. □

Remark 3.7 Let n, N be positive integers and r a real number with $r \in [0, N]$. Then, it is easily seen that

$$\min\{r, N - r\} \leq \min\{N, nr, n(N - r)\}.$$

Hence, it follows that the inequalities of Corollary 3.6 are sharper than those resulting from Corollary 3.3 (cf. Remark 3.4 (ii)).

4 Moment inequalities

The Gelfand representation theorem asserts that any commutative C^*-algebra is isomorphic with the algebra of continuous complex-valued functions on some compact space. The following result is a representation theorem of the same type, identifying any abelian von Neumann algebra with the algebra of equivalence classes of essentially bounded measurable functions on some measure space.

Theorem 4.1 (cf. [8]) *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{Z} \subseteq \mathcal{B}(\mathcal{H})$ an abelian von Neumann algebra with separating vector ξ of norm 1. Then, there is a probability space (Ω, P) and an isometric $*$-isomorphism $\phi : L^\infty(\Omega) \longrightarrow \mathcal{Z}$, such that $\int f\, dP = \langle\phi(f)(\xi), \xi\rangle$ for any $f \in L^\infty(\Omega)$.* □

We are interested in the special case where $\mathcal{Z}$ is the center $\mathcal{Z}G$ of the von Neumann algebra $\mathcal{N}G$ of a group G. We recall that the unit vector $\delta_1 \in \ell^2 G$ is a separating vector for $\mathcal{N}G$ (cf. Lemma 2.4 (ii)) and hence for $\mathcal{Z}G$ as well.

Corollary 4.2 *Let G be a group, $\mathcal{Z}G$ the center of the von Neumann algebra $\mathcal{N}G$ and τ the canonical trace on the latter. Then, there is a probability space (Ω, P) and an isometric $*$-isomorphism $\phi : L^\infty(\Omega) \longrightarrow \mathcal{Z}G$, such that $\int f\, dP = \tau(\phi(f))$ for any $f \in L^\infty(\Omega)$.* □

In view of Corollary 4.2, any operator $a \in \mathcal{Z}G$ may be regarded as an integrable random variable f on some probability space (Ω, P), in such a way that the sequence $(\tau(a^n))_n$ coincides with the sequence of moments $(\mathbf{E}(f^n))_n = \left(\int f^n\, dP\right)_n$ of f. By an obvious abuse of language, we shall refer to the sequence $(\tau(a^n))_n$ as the sequence of moments of a. Our goal in this Section is to study the sequence of moments of the operator which is induced from an idempotent matrix with entries in the group algebra $\mathbf{C}G$, as in Theorem 2.7.

Proposition 4.3 *Let G be a group and τ the canonical trace on the von Neumann algebra $\mathcal{N}G$. We consider a self-adjoint operator $a \in \mathcal{N}G$, such that $0 \le a \le I$, where $I \in \mathcal{N}G$ is the identity operator. Then:*

(i) The sequence $(\tau(a^n))_n$ is a decreasing sequence of non-negative real numbers; in particular, $0 \le \tau(a^n) \le 1$ for all n.

(ii) Let n be a positive integer. Then, $\tau(a^n) = 0$ (resp. $\tau(a^n) = 1$) if and only if $a = 0$ (resp. $a = I$). Moreover, $\tau(a^n) = \tau(a^{n+1})$ if and only if $a = a^2$.

(iii) For all $n \ge 1$ we have $0 \le \tau(a^n) - \tau(a^{n+1}) \le \frac{n^n}{(n+1)^{n+1}}$.

Proof We note that, using the Gelfand representation theorem, the self-adjoint operator a may be viewed as a function with values in $[0, 1]$.

(i) In view of our assumption, the operator a^n is self-adjoint and satisfies the inequality $0 \le a^n \le I$ for all n, whereas the sequence $(a^n)_n$ is operator-decreasing. Taking into account the positivity of τ (cf. Lemma 2.4 (iii)), we conclude that $\tau(a^n)$ is a real number contained in the interval $[0, 1]$ for all n, while the sequence $(\tau(a^n))_n$ is decreasing.

(ii) Since $a^n = 0$ (resp. $a^n = 1$) if and only if $a = 0$ (resp. $a = 1$), the first part is an immediate consequence of the faithfulness of τ (loc.cit.). Now assume that $\tau(a^n) = \tau(a^{n+1})$; then, $a^n - a^{n+1} \ge 0$ and $\tau(a^n - a^{n+1}) = 0$. Invoking the faithfulness of τ again, we conclude that $a^n - a^{n+1} = 0$ and hence $a^n = a^{n+1}$. This latter equality is clearly equivalent to the equality $a = a^2$.

(iii) Using elementary calculus, it is easily seen that $0 \le x^n - x^{n+1} \le \frac{n^n}{(n+1)^{n+1}}$ for any number $x \in [0, 1]$ (the maximum is obtained when $x = \frac{n}{n+1}$). It follows that $0 \le a^n - a^{n+1} \le \frac{n^n}{(n+1)^{n+1}} I$. The proof is therefore finished, invoking the positivity of τ. □

In order to apply the inequalities of Proposition 4.3 to the operator which is induced from an idempotent matrix with entries in the group algebra of a group G as in Theorem 2.7, we need a formulary for the moments of an operator in $L(\mathbf{C}G) \subseteq \mathcal{N}G$. To that end, we consider an element $\rho = \sum_g \rho_g g \in \mathbf{C}G$, where $\rho_g \in \mathbf{C}$ for all

$g \in G$, and the associated operator $\varrho = L_\rho = \sum_g \rho_g L_g \in \mathcal{N}G$. Then, for any element $g \in G$ we compute

$$\begin{aligned} \tau\left(L_g\varrho^n\right) &= \langle L_g\varrho^n(\delta_1), \delta_1\rangle \\ &= \langle \varrho^n(\delta_1), L_g^*(\delta_1)\rangle \\ &= \langle \varrho^n(\delta_1), \delta_{g^{-1}}\rangle \\ &= \sum\{\rho_{g_1}\rho_{g_2}\cdots\rho_{g_n} : (g_1, g_2, \ldots, g_n) \in G^n \text{ and } g_1g_2\cdots g_n = g^{-1}\} \end{aligned} \tag{7}$$

for all $n \geq 1$. In particular,

$$\tau\left(\varrho^n\right) = \sum\{\rho_{g_1}\rho_{g_2}\cdots\rho_{g_n} : (g_1, g_2, \ldots, g_n) \in G^n \text{ and } g_1g_2\cdots g_n = 1\}.$$

Theorem 4.4 *Let G be a group, N a positive integer and $E \in \mathbf{M}_N(\mathbf{C}G)$ an idempotent matrix with Hattori-Stallings rank $r_{HS}(E) = \sum_{[g]\in\mathcal{C}(G)} r_g(E)[g]$. We consider the sequence $(\mu_n)_n$, which is defined by letting $\mu_0 = 1$ and*

$$\mu_n = \frac{1}{N^n}\sum\left\{\frac{r_{g_1}(E)}{[G:C_{g_1}]}\frac{r_{g_2}(E)}{[G:C_{g_2}]}\cdots\frac{r_{g_n}(E)}{[G:C_{g_n}]} : (g_1, g_2, \ldots, g_n) \in G_f^n \right.$$
$$\left. \text{and } g_1g_2\cdots g_n = 1\right\}$$

for all $n \geq 1$. Then:

(i) The sequence $(\mu_n)_n$ is a decreasing sequence of non-negative real numbers; in particular, $0 \leq \mu_n \leq 1$ for all n.

(ii) Let n be a positive integer. Then, $\mu_n = 0$ (resp. $\mu_n = 1$) if and only if $E = 0$ (resp. $E = I_N$). Moreover, $\mu_n = \mu_{n+1}$ if and only if $E = cI_N$ for some central idempotent $c \in \mathbf{C}G$.

(iii) For all $n \geq 1$ we have $\mu_{n+1} \leq \mu_n \leq \mu_{n+1} + \frac{n^n}{(n+1)^{n+1}}$.

The sequence $(\mu_n)_n = (\mu_n(E))_n$ will be referred to as the sequence of moments of the idempotent matrix E.

Proof Let $\mathcal{N}G$ be the von Neumann algebra of G and consider the operator

$$a = \frac{1}{N}\sum_{g\in G_f}\frac{r_g(E)}{[G:C_g]}L_g \in \mathcal{N}G.$$

Then, the sequence $(\mu_n)_n$ is precisely the sequence of moments $(\tau(a^n))_n$ of a and hence the result follows from Proposition 4.3, in view of Theorem 2.7. □

We shall now make explicit the inequalities obtained above for small values of the parameter n. Part (i) of the result below is due to Kaplansky (cf. [11]), whereas two thirds of part (ii) are due to Alexander [1] (see also [10] and [12]).

Corollary 4.5 *Let G be a group, $\mathbf{C}G$ the associated group algebra, N a positive integer and $E \in \mathbf{M}_N(\mathbf{C}G)$ an idempotent matrix with Hattori-Stallings rank $r_{HS}(E) = \sum_{[g]\in\mathcal{C}(G)} r_g(E)[g]$. Then:*

(i) The complex number $r_1(E)$ is real and $0 \leq r_1(E) \leq N$; moreover, $r_1(E) = 0$ (resp. $r_1(E) = N$) if and only if $E = 0$ (resp. $E = I_N$).

(ii) We have

$$\sum_{[g]\in\mathcal{C}_f(G)} \frac{|r_g(E)|^2}{[G:C_g]} \overset{(*)}{\leq} Nr_1(E) \leq \sum_{[g]\in\mathcal{C}_f(G)} \frac{|r_g(E)|^2}{[G:C_g]} + \frac{N^2}{4};$$

moreover, the inequality $(*)$ *is an equality if and only if* $E = cI_N$ *for some central idempotent* $c \in \mathbf{C}G$.

Proof Let $(\mu_n)_n$ be the sequence of moments of E, as defined in Theorem 4.4. We note that $\mu_1 = \frac{1}{N} r_1(E)$, whereas

$$\begin{aligned}
\mu_2 &= \frac{1}{N^2} \sum_{g\in G_f} \frac{r_g(E)}{[G:C_g]} \frac{r_{g^{-1}}(E)}{[G:C_{g^{-1}}]} \\
&= \frac{1}{N^2} \sum_{g\in G_f} \frac{|r_g(E)|^2}{[G:C_g]^2} \\
&= \frac{1}{N^2} \sum_{[g]\in\mathcal{C}_f(G)} \frac{|r_g(E)|^2}{[G:C_g]}
\end{aligned}$$

(cf. Corollary 2.8). Therefore, (i) follows since $\mu_1 \in [0, 1]$, whereas $\mu_1 = 0$ (resp. $\mu_1 = 1$) if and only if $E = 0$ (resp. $E = I_N$). In the same way, (ii) follows since $\mu_2 \leq \mu_1 \leq \mu_2 + \frac{1}{4}$, whereas $\mu_2 = \mu_1$ if and only if E is central. □

Remarks 4.6 Let G be a group, $\mathbf{C}G$ the associated group algebra and $\mathcal{N}G$ the von Neumann algebra of G. We consider a positive integer N and an idempotent matrix $E \in \mathbf{M}_N(\mathbf{C}G)$ with Hattori-Stallings rank $r_{HS}(E) = \sum_{[g]\in\mathcal{C}(G)} r_g(E)[g]$. We also consider the induced idempotent matrix $E' \in \mathbf{M}_N(\mathcal{N}G)$, the center-valued trace

$$a = t_{(N)}(E') = \frac{1}{N} \sum_{g\in G_f} \frac{r_g(E)}{[G:C_g]} L_g \in \mathcal{N}G$$

(cf. Theorem 2.7) and the sequence of moments $(\mu_n)_n$ of E, defined in Theorem 4.4.

(i) Since a is a central operator in $\mathcal{N}G$, we may apply Corollary 4.2 in order to identify a with an integrable random variable $f = \phi^{-1}(a)$ on some probability space (Ω, P). The variance $var(f)$ of f is equal to $\mu_2 - \mu_1^2$ and hence

$$var(f) = \frac{1}{N^2} \sum_{[g]\in\mathcal{C}_f(G)\setminus\{[1]\}} \frac{|r_g(E)|^2}{[G:C_g]}.$$

In particular, it follows that $var(f) = 0$ if and only if $r_g(E) = 0$ for any element $g \in G_f \setminus \{1\}$. Invoking Zaleskii's theorem on the rationality of $r_1(E)$ (cf. [17]) and the criterion of [6, Proposition 2.3 (ii)], we conclude that $var(f) = 0$ if and only if there is a positive integer n such that the block diagonal matrix $E' \oplus E' \oplus \cdots \oplus E' \in \mathbf{M}_{nN}(\mathcal{N}G)$ is unitarily equivalent with a diagonal matrix with zeroes and ones along the diagonal. Equivalently, $var(f) = 0$ if and only if the class of E' has finite order in the reduced K-theory group $\widetilde{K_0}(\mathcal{N}G)$.

(ii) Let n be a positive integer. It follows from the proof of Proposition 4.3 (iii) and the faithfulness of the canonical trace τ that the inequality $\mu_n \leq \mu_{n+1} + \frac{n^n}{(n+1)^{n+1}}$ of Theorem 4.4 (iii) is an equality if and only if $a = \frac{n}{n+1} I$, where $I \in \mathcal{N}G$ is the identity operator. Therefore, using again [6, Proposition 2.3 (ii)], we conclude that $\mu_n = \mu_{n+1} + \frac{n^n}{(n+1)^{n+1}}$ if and only if the block diagonal matrix $E' \oplus E' \oplus \cdots \oplus E' \in \mathbf{M}_{(n+1)N}(\mathcal{N}G)$ is unitarily equivalent (in $\mathbf{M}_{(n+1)N}(\mathcal{N}G)$) with a diagonal matrix with N zeroes and nN ones along the diagonal. This latter condition is equivalent to the equality $(n+1)[E'] = n[I_N] \in K_0(\mathcal{N}G)$. In particular, it follows that $\mu_1 = \mu_2 + \frac{1}{4}$ if and only if the matrices $\begin{pmatrix} E' & 0 \\ 0 & E' \end{pmatrix}, \begin{pmatrix} I_N & 0 \\ 0 & 0 \end{pmatrix} \in \mathbf{M}_{2N}(\mathcal{N}G)$ are unitarily equivalent, i.e if and only if $2[E'] = [I_N] \in K_0(\mathcal{N}G)$.

5 The gaps between consecutive moments

Let G be a group and E an idempotent $N \times N$ matrix with entries in the group algebra $\mathbf{C}G$. The inequalities of Theorem 4.4 (iii) provide us with an upper bound for the difference $\mu_n - \mu_{n+1}$ between two consecutive moments of E. These upper bounds are absolute, in the sense that they don't depend on E, and optimal, in the sense that they are attained for certain E's (cf. Remark 4.6 (ii)). Since any two consecutive moments are equal if the matrix E is central in $\mathbf{M}_N(\mathbf{C}G)$ (cf. Theorem 4.4 (ii)), the only non-negative absolute lower bound for these differences is 0.

Our next goal is to obtain lower bounds for the difference between consecutive moments that depend on E. To that end, we consider the Hattori-Stallings rank $r_{HS}(E) = \sum_{[g]\in\mathcal{C}(G)} r_g(E)[g]$ and the operator $a = \sum_{g\in G_f} a_g L_g \in \mathcal{N}G$, where $a_g = \frac{1}{N} \frac{r_g(E)}{[G:C_g]}$ for all $g \in G_f$ (cf. Theorem 2.7). Let

$$X = \{g \in G_f : a_g \neq 0\} = \{g \in G_f : r_g(E) \neq 0\};$$

then, Corollary 2.8 implies that for any element $g \in G_f$ we have $g \in X$ if and only if $g^{-1} \in X$. We consider the cardinality

$$\lambda = \lambda(E) = \text{card}\, X = \sum \{[G : C_g] : [g] \in \mathcal{C}_f(G) \text{ and } r_g(E) \neq 0\} \tag{8}$$

and the complex number

$$\sigma = \sigma(E) = \sum_{g\in G_f} a_g = \frac{1}{N} \sum_{g\in G_f} \frac{r_g(E)}{[G : C_g]} = \frac{1}{N} \sum_{[g]\in\mathcal{C}_f(G)} r_g(E). \tag{9}$$

Remarks 5.1 Let G be a group, $E \in \mathbf{M}_N(\mathbf{C}G)$ an idempotent matrix and $X \subseteq G_f$ the set defined above.

(i) In view of Kaplansky's theorem (Corollary 4.5 (i)), the non-negative integer λ defined in Eq. (8) vanishes (i.e. $X = \emptyset$) if and only if $E = 0$.

(ii) Since G_f is a normal subgroup of G, we may consider the quotient group $\overline{G} = G/G_f$. The natural map $G \longrightarrow \overline{G}$ induces a homomorphism between the associated group algebras and hence a homomorphism $\mathbf{M}_N(\mathbf{C}G) \longrightarrow \mathbf{M}_N\left(\mathbf{C}\overline{G}\right)$; let $\overline{E}$ be the image of E under this latter homomorphism. Since $r_{\overline{1}}\left(\overline{E}\right)$ is equal to the sum $\sum_{[g]\in\mathcal{C}_f(G)} r_g(E)$, we may apply the theorems of Zaleskii [17] and Kaplansky (Corollary 4.5 (i)) to $\overline{E}$, in order to conclude that the complex number σ defined in Eq. (9) above is rational and contained in the interval $[0, 1]$.

Using the set X introduced above, we shall now define the sequence $(X_n)_{n\geq 0}$ of subsets of G, by letting $X_0 = \{1\}$ and $X_n = \{g_1 g_2 \cdots g_n : g_1, g_2, \ldots, g_n \in X\}$ for all $n \geq 1$; in particular, $X_1 = X$.

Lemma 5.2 *Let G be a group, $\mathbf{C}G$ the associated group algebra and τ the canonical trace on the von Neumann algebra $\mathcal{N}G$. We consider an idempotent matrix E with entries in $\mathbf{C}G$, the operator $a \in \mathcal{N}G$ defined in Theorem 2.7 and the sequence $(X_n)_n$ of subsets of G defined above. Then:*

(i) If $E \neq 0$ then the sequence $(X_n)_n$ is increasing, i.e. $X_n \subseteq X_{n+1}$ for all n.

(ii) Let n be a non-negative integer and $g \in G_f$ a group element such that $\tau(L_g a^n) \neq 0$. Then, $g \in X_n$.

(iii) Let n be a non-negative integer, $Y \subseteq G_f$ a subset containing X_n and σ the complex number defined in Eq. (9). Then, $\sum_{g\in Y} \tau(L_g a^n) = \sigma^n$.

Proof (i) Since $E \neq 0$, Kaplansky's theorem (Corollary 4.5 (i)) implies that $r_1(E) \neq 0$ and hence $1 \in X$. Then, the inclusions $X_n \subseteq X_{n+1}$, $n \geq 0$, follow immediately from the definitions.

(ii) In view of Eq.(7), we have

$$\tau\left(L_g a^n\right) = \sum\{a_{g_1} a_{g_2} \cdots a_{g_n} : (g_1, g_2, \ldots, g_n) \in G_f^n \text{ and } g_1 g_2 \cdots g_n = g^{-1}\}.$$

Since $\tau\left(L_g a^n\right) \neq 0$, we conclude that there are $g_1, g_2, \ldots, g_n \in X$ such that $g^{-1} = g_1 g_2 \cdots g_n$. The set X is closed under inversion and hence $g_1^{-1}, g_2^{-1}, \ldots, g_n^{-1} \in X$ as well; it follows that $g = g_n^{-1} \cdots g_2^{-1} g_1^{-1} \in X_n$.

(iii) We compute

$$\begin{aligned}
\sum_{g\in Y} \tau(L_g a^n) &= \sum_{g\in G_f} \tau(L_g a^n) \\
&= \sum_{g\in G_f} \sum\{a_{g_1} a_{g_2} \cdots a_{g_n} : (g_1, g_2, \ldots, g_n) \in G_f^n \\
&\qquad\qquad \textit{and } g_1 g_2 \cdots g_n = g^{-1}\} \\
&= \sum\{a_{g_1} a_{g_2} \cdots a_{g_n} : (g_1, g_2, \ldots, g_n) \in G_f^n\} \\
&= \left(\sum_{g\in G_f} a_g\right)^n \\
&= \sigma^n.
\end{aligned}$$

In the above chain of equalities, the first one is an immediate consequence of (ii) above, in view of our assumption that $Y \supseteq X_n$, whereas the second one follows from Eq. (7). □

Proposition 5.3 *Let G be a group, E a non-zero idempotent matrix with entries in the group algebra $\mathbf{C}G$ and $(\mu_n)_n$ the associated sequence of moments, defined in Theorem 4.4. Then,*

$$0 \leq \frac{\sigma^n - \sigma^{n+1}}{\lambda^{n+1}} \leq \mu_n - \mu_{n+1}$$

for all $n \geq 0$, where the numbers λ, σ are defined in Eqs. (8) and (9) respectively.

Proof Let $\mathcal{N}G$ be the von Neumann algebra of G and $a \in \mathcal{N}G$ the operator associated with E as in Theorem 2.7. Then, $\mu_n = \tau(a^n)$ for all n, where τ is the canonical trace on $\mathcal{N}G$. We consider the sequence $(X_n)_n$ of subsets of G, which is defined in the paragraph before Lemma 5.2, and fix a non-negative integer n. If $b_{n+1} = \sum_{g \in X_{n+1}} L_g \in \mathcal{N}G$ then

$$\| b_{n+1} \| \leq \sum_{g \in X_{n+1}} \| L_g \| = \operatorname{card} X_{n+1} \leq \lambda^{n+1}.$$

Since the operator $a^n - a^{n+1} \in \mathcal{N}G$ is self-adjoint and ≥ 0, Lemma 3.5 implies that

$$\begin{aligned} | \tau(b_{n+1}(a^n - a^{n+1})) | \ &\leq \ \| b_{n+1} \| \ \tau(a^n - a^{n+1}) \\ &= \ \| b_{n+1} \| \ (\mu_n - \mu_{n+1}) \\ &\leq \ \lambda^{n+1}(\mu_n - \mu_{n+1}). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \tau(b_{n+1}(a^n - a^{n+1})) &= \tau(b_{n+1}a^n) - \tau(b_{n+1}a^{n+1}) \\ &= \sum_{g \in X_{n+1}} \tau(L_g a^n) - \sum_{g \in X_{n+1}} \tau(L_g a^{n+1}) \\ &= \sigma^n - \sigma^{n+1}, \end{aligned}$$

where the last equality follows from Lemma 5.2 (i), (iii). This finishes the proof, since the complex number $\sigma^n - \sigma^{n+1}$ is real and non-negative (cf. Remark 5.1 (ii)). $\square$

It is clear that the inequalities of Proposition 5.3 complement those of Theorem 4.4 (iii). In particular, the following result complements the inequality $(*)$ of Corollary 4.5 (ii).

Corollary 5.4 *Let G be a group, $\mathbf{C}G$ the associated group algebra, N a positive integer and $E \in \mathbf{M}_N(\mathbf{C}G)$ an idempotent matrix with Hattori-Stallings rank $r_{HS}(E) = \sum_{[g] \in \mathcal{C}(G)} r_g(E)[g]$.*

(i) If $\varsigma = \sum_{[g] \in \mathcal{C}_f(G)} r_g(E)$ then ς is a rational number with $0 \leq \varsigma \leq N$.

(ii) We have

$$\frac{\varsigma(N - \varsigma)}{\lambda^2} + \sum_{[g] \in \mathcal{C}_f(G)} \frac{|r_g(E)|^2}{[G : C_g]} \leq N r_1(E),$$

where the integer λ is defined in Eq. (8).

Proof (i) Both claims follow from Remark 5.1 (ii), since $\varsigma = N\sigma$, where σ is the number defined in Eq. (9). (In fact, $\varsigma = r_{\mathbb{T}}\big(E\big)$ in the notation of loc.cit.)

(ii) This follows from Proposition 5.3 by letting $n = 1$ therein, in view of the formulae for μ_1 and μ_2 given in the proof of Corollary 4.5. □

6 The mass of the central carrier

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$ a von Neumann algebra with center $\mathcal{Z}$. For any projection $p \in \mathcal{N}$ the central carrier $c(p)$ of p is the smallest projection of $\mathcal{Z}$ which dominates p (cf. [8, Chapter 6]); in other words,

$$c(p) = \inf\{q \in \mathcal{Z} : q \text{ is a projection and } p \leq q\}.$$

Let $V = [\mathcal{N}p(\mathcal{H})]^-$ be the closed linear span of the set $\mathcal{N}p(\mathcal{H}) = \{ap(\xi) : a \in \mathcal{N}, \xi \in \mathcal{H}\}$. It is easily seen that V has the following properties:

(i) V is invariant under the action of both $\mathcal{N}$ and its commutant $\mathcal{N}'$,
(ii) V contains the subspace $im\ p = p(\mathcal{H})$ and
(iii) V is the smallest closed subspace of $\mathcal{H}$ having properties (i) and (ii).

It follows that the central carrier $c(p)$ of p is the orthogonal projection onto V.

Remarks 6.1 Let $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra and consider an idempotent $e \in \mathcal{N}$.

(i) As we have already noted in Remark 2.2 (i), there exists a projection $p \in \mathcal{N}$ with $e = pe$ and $p = ep$. Then, $p(\mathcal{H}) = e(\mathcal{H})$ and hence $c(p)$ is the orthogonal projection onto the closed subspace $V \subseteq \mathcal{H}$, which is generated by the set $\mathcal{N}e(\mathcal{H}) = \{ae(\xi) : a \in \mathcal{N}, \xi \in \mathcal{H}\}$. We call $c(p)$ the central carrier of the idempotent e and denote it by $c(e)$.

(ii) Assume that there is a vector $\xi \in \mathcal{H}$, which is both separating and cyclic for the action of $\mathcal{N}$; this means that the evaluation map

$$ev_\xi : \mathcal{N} \longrightarrow \mathcal{H}$$

is injective and has a dense image. Then, $V = [\mathcal{N}e(\mathcal{H})]^-$ is easily seen to coincide with the closed linear span of the set $\{aeb(\xi) : a, b \in \mathcal{N}\}$. In other words, letting $J = \mathcal{N}e\mathcal{N}$ be the two-sided ideal of $\mathcal{N}$ generated by e, there is a commutative diagram

$$\begin{array}{ccc} J & \xrightarrow{ev_\xi} & V \\ \downarrow & & \downarrow \\ \mathcal{N} & \xrightarrow{ev_\xi} & \mathcal{H} \end{array}$$

whose vertical arrows are the inclusion maps and whose horizontal ones are injective with dense image. Since the central carrier $c = c(e)$ is the orthogonal projection onto V (cf. (i) above), we may regard it as a measure of the size of V inside $\mathcal{H}$ and hence of the size of the two-sided ideal $J = \mathcal{N}e\mathcal{N}$ inside $\mathcal{N}$.

We shall now consider the case of a finite von Neumann algebra and obtain a formula for the central carrier of an idempotent, in terms of the center-valued trace. In order to facilitate the notation in the next Lemma, we denote the WOT-limit of a (WOT-convergent) sequence $(a_n)_n$ of operators in a Hilbert space simply by $\lim_n a_n$.

Lemma 6.2 *Let $\mathcal{H}$ be a Hilbert space and consider a finite von Neumann algebra $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$ with center $\mathcal{Z}$ and an idempotent $e \in \mathcal{N}$ with center-valued trace $a = t(e) \in \mathcal{Z}$. Then:*

(i) The sequence $(a^n)_n$ converges in the weak operator topology to an operator $c \in \mathcal{Z}$.

(ii) The operator $c = \lim_n a^n$ is the greatest central projection of $\mathcal{N}$ which is $\leq a$.

(iii) If $c(e) \in \mathcal{Z}$ is the central carrier of e then $c(e) = 1 - \lim_n (1 - a)^n$.

Proof (i) We know that the operator $a \in \mathcal{Z}$ is self-adjoint and $0 \leq a \leq 1$ (cf. Remark 2.2 (i)). Therefore, the sequence $(a^n)_n$ is a decreasing sequence of positive operators in the von Neumann algebra $\mathcal{Z}$. As such, it admits a WOT-limit $c = \lim_n a^n \in \mathcal{Z}$.

(ii) It is clear that the operator $c \in \mathcal{Z}$ is self-adjoint and $0 \leq c \leq a$. Since multiplication in $\mathcal{B}(\mathcal{H})$ is separately WOT-continuous, we have $a^n \cdot c = a^n \cdot \lim_m a^m = \lim_m a^{n+m} = \lim_m a^m = c$ for all $n \geq 1$ and hence $c^2 = c \cdot c = (\lim_n a^n) \cdot c = \lim_n (a^n \cdot c) = \lim_n c = c$. It follows that the operator $c \in \mathcal{Z}$ is a projection. On the other hand, if $c' \in \mathcal{Z}$ is a projection with $c' \leq a$ then $c' = c'^n \leq a^n$ for all n and hence $c' \leq \lim_n a^n = c$.

(iii) Let $p \in \mathcal{N}$ be a projection with $e = pe$ and $p = ep$. Then, $t(p) = t(e) = a$, in view of the trace property of t, and $c(e) = c(p)$ (cf. Remark 6.1 (i)). Therefore, we may replace e by p and assume that $e \in \mathcal{N}$ is a projection. We claim that a central projection $c' \in \mathcal{N}$ dominates e if and only if $a \leq c'$. Indeed, if $e \leq c'$ then $a = t(e) \leq t(c') = c'$, in view of the positivity of t. Conversely, if $a \leq c'$ then $t(e) = a = ac' = t(e)c'$ and hence the projection $e - ec' = e(1 - c') \in \mathcal{N}$ is such that $t(e - ec') = t(e) - t(ec') = t(e) - t(e)c' = 0$. Invoking the faithfulness of t, it follows that $e - ec' = 0$ and hence $e = ec' \leq c'$. Having established the claim, we conclude that the central carrier $c(e)$ is the smallest central projection which is $\geq a$. In other words, $1 - c(e)$ is the greatest central projection which is $\leq 1 - a$. Using (ii) above for the idempotent $1 - e$, whose center-valued trace is $t(1 - e) = 1 - a$, it follows that $1 - c(e) = \lim_n (1 - a)^n$. □

We now specialize the above discussion to the case where $\mathcal{N}$ is an algebra of matrices with entries in the von Neumann algebra of a group.

Proposition 6.3 *Let G be a group, N a positive integer and E an idempotent $N \times N$ matrix with entries in the group algebra $\mathbf{C}G$. We consider the von Neumann algebra $\mathcal{N}G$, the induced idempotent matrix $E' \in \mathbf{M}_N(\mathcal{N}G)$ and let $a = t_{(N)}(E')$ be its center-valued trace and $c = c(E')$ its central carrier. Then, $\tau(c) = 1 - \lim_n \mu_n(I_N - E)$, where τ is the canonical trace on $\mathcal{N}G$ and $(\mu_n(I_N - E))_n$ the sequence of moments which is associated with the complementary idempotent matrix $I_N - E$, as in Theorem 4.4.*

Proof Since $\mu_n(I_N - E) = \tau\big[(I - a)^n\big]$ for all n, where $I \in \mathcal{N}G$ is the identity operator, the result follows from Lemma 6.2 (iii), in view of the WOT-continuity of τ. □

Proposition 6.4 *Let G be a group, $\mathbf{C}G$ the associated group algebra and N a positive integer. We consider an idempotent $N \times N$ matrix E with entries in $\mathbf{C}G$, its Hattori-Stallings rank $r_{HS}(E) = \sum_{[g] \in \mathcal{C}(G)} r_g(E)[g]$ and let $E' \in \mathbf{M}_N(\mathcal{N}G)$ be* *the induced matrix with entries in the von Neumann algebra $\mathcal{N}G$. If $c = c(E')$ is the central carrier of E' then*

$$\tau(c) \geq \frac{\sigma}{\lambda + \sigma - 1},$$

where τ is the canonical trace on $\mathcal{N}G$ and the numbers $\lambda = \lambda(E)$ and $\sigma = \sigma(E)$ are those defined in Eqs. (8) and (9) respectively.

Proof The inequality is obviously valid if $E = 0$ or $E = I_N$ and hence we may assume that $E \neq 0, I_N$. Let $(\mu_n(I_N - E))_n$ be the sequence of moments of the idempotent matrix $I_N - E$. It is clear that $\sigma(I_N - E) = 1 - \sigma(E) = 1 - \sigma$, whereas our assumption on E implies, in view of Kaplansky's theorem (cf. Corollary 4.5 (i)), that $\lambda(I_N - E) = \lambda(E) = \lambda$. Therefore, applying the inequality of Proposition 5.3 to the idempotent matrix $I_N - E$, we conclude that

$$\frac{(1-\sigma)^n - (1-\sigma)^{n+1}}{\lambda^{n+1}} \leq \mu_n(I_N - E) - \mu_{n+1}(I_N - E)$$

for all $n \geq 0$ and hence

$$\sum_{n=0}^{\infty} \frac{(1-\sigma)^n - (1-\sigma)^{n+1}}{\lambda^{n+1}} \leq \sum_{n=0}^{\infty} \left[\mu_n(I_N - E) - \mu_{n+1}(I_N - E)\right].$$

Summing up the series, it follows that

$$\frac{\sigma}{\lambda + \sigma - 1} \leq 1 - \lim_n \mu_n(I_N - E).$$

This finishes the proof, in view of Proposition 6.3. □

Remark 6.5 Let G be a group, $\mathbf{C}G$ the associated group algebra and $\mathcal{N}G$ the von Neumann algebra of G. Then, the composition

$$\mathbf{C}G \xrightarrow{L} \mathcal{N}G \xrightarrow{ev_{\delta_1}} \ell^2 G$$

is injective and has a dense image; in particular, the separating vector $\delta_1 \in \ell^2 G$ for the action of $\mathcal{N}G$ is also cyclic. We consider an idempotent $e \in \mathbf{C}G$, the induced idempotent $e' \in \mathcal{N}G$ and its central carrier $c = c(e')$. Then, $\tau(c) \in [0, 1]$ is a numerical measure of the size of $V = im\, c$ inside $\ell^2 G$. Let $I = \mathbf{C}G \cdot e \cdot \mathbf{C}G$ be the two-sided ideal of $\mathbf{C}G$ generated by e and $J = \mathcal{N}G \cdot e' \cdot \mathcal{N}G$ the two-sided ideal of $\mathcal{N}G$ generated by e'. Applying the considerations of Remark 6.1 (ii) in this special case, it follows that there is a commutative diagram

$$\begin{array}{ccccc} I & \xrightarrow{L} & J & \xrightarrow{ev_{\delta_1}} & V \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{C}G & \xrightarrow{L} & \mathcal{N}G & \xrightarrow{ev_{\delta_1}} & \ell^2 G \end{array}$$

whose vertical arrows are the inclusion maps. We note that both compositions in the horizontal direction in the diagram above are injective and have dense image. Therefore, $\tau(c)$ (a lower bound for which is provided by Proposition 6.4) may be regarded as a numerical measure of the size of I inside $\mathbf{C}G$.

A Semi-simple elements

Our goal in this Appendix is to complement Alexander's version of the inequality of Corollary 4.5 (ii) for semi-simple elements in matrix rings over group algebras (cf. [1]) in two directions:

(i) we replace the group algebra by the associated von Neumann algebra and
(ii) we also consider the higher moments of the idempotents that are involved.

Lemma A.1 *Let R be a complex algebra and consider orthogonal idempotents $e_1, \ldots, e_m \in R$ and distinct non-zero complex numbers $\lambda_1, \ldots, \lambda_m \in \mathbf{C}$. Then, the element $a = \lambda_1 e_1 + \cdots + \lambda_m e_m$ is central in R if and only if e_i is central in R for all $i = 1, \ldots, m$.*

Proof (cf. the argument used in [1, p. 2430]) It is clear that a is central if the e_i's are central. Conversely, let us assume that a is central. In view of our assumption on the e_i's, it follows that $a^n = \lambda_1^n e_1 + \cdots + \lambda_m^n e_m$ for all $n \geq 1$. Therefore, for any element $r \in R$ and any integer $n \geq 1$ we have $\lambda_1^n [e_1, r] + \cdots + \lambda_m^n [e_m, r] = \left[a^n, r\right] = 0$, where $[x, y] = xy - yx$ for all $x, y \in R$. We conclude that $f(\lambda_1)[e_1, r] + \cdots + f(\lambda_m)[e_m, r] = 0$ for any polynomial $f(X) \in \mathbf{C}[X]$ with $f(0) = 0$. In view of our assumption on the λ_i's, for any i there is an interpolation polynomial $f_i(X) \in \mathbf{C}[X]$ with $f_i(0) = 0$, $f_i(\lambda_j) = 0$ for all $j \neq i$ and $f_i(\lambda_i) = 1$. It follows that $[e_i, r] = 0$ for all $r \in R$ and hence e_i is central for all i. □

Proposition A.2 *Let G be a group and τ the canonical trace on the von Neumann algebra $\mathcal{N}G$. Assume that $a_1, \ldots, a_m \in \mathcal{N}G$ are commuting self-adjoint operators with $a_i \geq 0$ for all i and $a_1 + \cdots + a_m \leq I$, where $I \in \mathcal{N}G$ is the identity operator.*

(i) If $\lambda_1, \ldots, \lambda_m$ are complex numbers and $a = \lambda_1 a_1 + \cdots + \lambda_m a_m \in \mathcal{N}G$, then we have $\tau(a^* a) \leq |\lambda_1|^2 \tau(a_1) + \cdots + |\lambda_m|^2 \tau(a_m)$.

(ii) Assume that the complex numbers $\lambda_1, \ldots, \lambda_m$ are non-zero and distinct. Then, the inequality of (i) above is an equality if and only if $a_1, \ldots, a_m$ are orthogonal projections.

Proof (i) For a fixed index i, we have $\sum_{j \neq i} a_j \leq I - a_i$ and hence $\sum_{j \neq i} a_i a_j = a_i \sum_{j \neq i} a_j \leq a_i(I - a_i) = a_i - a_i^2$. Therefore, we conclude that

$$|\lambda_i|^2 \, \tau\left(\sum_{j \neq i} a_i a_j - a_i + a_i^2\right) \leq 0,$$

i.e. that

$$|\lambda_i|^2 \sum_{j \neq i} \tau(a_i a_j) - |\lambda_i|^2 \, \tau(a_i) + |\lambda_i|^2 \, \tau(a_i^2) \leq 0. \tag{10}$$

Moreover, we have $a_i a_j \geq 0$ and hence

$$0 \leq |\lambda_i - \lambda_j|^2 \, \tau(a_i a_j) \tag{11}$$

for all i, j. Adding up the inequalities (10) for all $i = 1, \ldots, m$ and the inequalities (11) for all $i, j = 1, \ldots, m$ with $i < j$, it follows that

$$\sum_i |\lambda_i|^2 \tau(a_i^2) + 2 \sum_{i<j} \mathrm{Re}\left(\lambda_i \overline{\lambda_j}\right) \tau(a_i a_j) \leq \sum_i |\lambda_i|^2 \tau(a_i). \tag{12}$$

This finishes the proof, since $a^*a = \sum_i |\lambda_i|^2 a_i^2 + 2 \sum_{i<j} \mathrm{Re}\left(\lambda_i \overline{\lambda_j}\right) a_i a_j$.

(ii) In view of our assumption on the λ_i's and the faithfulness of τ, the inequality (10) (resp. (11)) is an equality for some i (resp. for some pair (i, j) with $i \neq j$) if and only if $\sum_{j \neq i} a_i a_j - a_i + a_i^2 = 0$ (resp. if and only if $a_i a_j = 0$). It follows that the inequality (12) is an equality if and only if $\sum_{j \neq i} a_i a_j - a_i + a_i^2 = 0$ for all i and $a_i a_j = 0$ for all $i \neq j$, i.e. if and only if $a_i = a_i^2$ for all i and $a_i a_j = 0$ for all $i \neq j$. □

Corollary A.3 *Let G be a group and m, n two positive integers. We consider m orthogonal idempotent $n \times n$ matrices $E_1, \ldots, E_m$ with entries in the von Neumann algebra $\mathcal{N}G$ and let $a_1 = t_{(n)}(E_1), \ldots, a_m = t_{(n)}(E_m)$ be their center-valued traces. We also consider m complex numbers $\lambda_1, \ldots, \lambda_m$ and positive integers $k_1, \ldots, k_m$.*

(i) If $a = \lambda_1 a_1^{k_1} + \cdots + \lambda_m a_m^{k_m} \in \mathcal{N}G$ then

$$\tau(a^*a) \leq |\lambda_1|^2 \tau\left(a_1^{k_1}\right) + \cdots + |\lambda_m|^2 \tau\left(a_m^{k_m}\right),$$

where τ is the canonical trace on $\mathcal{N}G$.

(ii) Assume that the complex numbers $\lambda_1, \ldots, \lambda_m$ are non-zero and distinct. Then, the inequality of (i) above is an equality if and only if the matrix $A = \lambda_1 E_1 + \cdots + \lambda_m E_m$ is central in $\mathbf{M}_n(\mathcal{N}G)$.

Proof (i) In view of Remark 2.2 (i), the operator a_i is central, self-adjoint and satisfies the inequalities $0 \leq a_i \leq I$, where $I \in \mathcal{N}G$ is the identity operator; it follows that $0 \leq a_i^{k_i} \leq a_i$ for all $i = 1, \ldots, m$. On the other hand, our assumption on the orthogonality of the E_i's implies that the matrix $E_1 + \cdots + E_m$ is also idempotent and hence $0 \leq a_1 + \cdots + a_m \leq I$ (loc.cit.). It follows that $a_1^{k_1} + \cdots + a_m^{k_m} \leq I$ and hence the result is an immediate consequence of Proposition A.2 (i).

(ii) Since the λ_i's are non-zero and distinct, Lemma A.1 implies that the matrix A is central if and only if this is the case for the matrices $E_1, \ldots, E_m$. Since these idempotent matrices are orthogonal, Remark 2.2 (ii) implies that they are central if and only if their center-valued traces $a_1, \ldots, a_m$ are orthogonal projections. On the other hand, considering the C^*-algebra generated by the commuting self-adjoint and non-negative operators $a_1, \ldots, a_m$, we conclude that this latter condition is equivalent to the condition that the operators $a_1^{k_1}, \ldots, a_m^{k_m}$ be orthogonal projections. Hence, the proof is finished by invoking Proposition A.2 (ii). □

Let G be a group and n a positive integer. Then, the canonical trace τ on the von Neumann algebra $\mathcal{N}G$ induces a trace

$$\tau_n : \mathbf{M}_n(\mathcal{N}G) \longrightarrow \mathbf{C},$$

which maps any matrix $A = (a_{ij})_{i,j} \in \mathbf{M}_n(\mathcal{N}G)$ onto $\sum_{i=1}^n \tau(a_{ii}) \in \mathbf{C}$. If $\mathcal{Z}G$ is the center of $\mathcal{N}G$ and $a = t_{(n)}(A) \in \mathcal{Z}G$ the center-valued trace of a matrix A as above, then $\tau_n(A) = n\tau(a) \in \mathbf{C}$ (cf. Theorem 2.1 (v)).

Corollary A.4 *Let G be a group, n a positive integer and $E_1, \dots, E_m$ orthogonal idempotent $n \times n$ matrices with entries in the von Neumann algebra $\mathcal{N}G$. We fix an m-tuple of complex numbers $(\lambda_1, \dots, \lambda_m)$ and consider the matrix $A = \lambda_1 E_1 + \cdots + \lambda_m E_m \in \mathbf{M}_n(\mathcal{N}G)$ and its center-valued trace $a = t_{(n)}(A) \in \mathcal{Z}G$, where $\mathcal{Z}G$ is the center of $\mathcal{N}G$. Then,*

$$n\tau(a^*a) \leq |\lambda_1|^2\, \tau_n(E_1) + \cdots + |\lambda_m|^2\, \tau_n(E_m),$$

where τ is the canonical trace on $\mathcal{N}G$ and τ_n its extension to $\mathbf{M}_n(\mathcal{N}G)$. The inequality above is an equality if and only if the matrix A is central in $\mathbf{M}_n(\mathcal{N}G)$.

Proof Without any loss of generality, we may assume that the λ_i's are non-zero and distinct. Then, the result follows from Corollary A.3 by letting $k_1 = \cdots = k_m = 1$ therein. □

Let G be a group, n a positive integer and A an $n \times n$ matrix with entries in the group algebra $\mathbf{C}G$. Then, the residue class of the trace $tr(A) \in \mathbf{C}G$ in the quotient $T(\mathbf{C}G) = \mathbf{C}G/[\mathbf{C}G, \mathbf{C}G]$ can be written as a sum $\sum_{[g]\in\mathcal{C}(G)} r_g(A)[g]$, where $r_g(A) \in \mathbf{C}$ for all $g \in G$. If A' is the induced $n \times n$ matrix with entries in the von Neumann algebra $\mathcal{N}G$, then $r_1(A) = \tau_n(A')$, where τ_n is the extension of the canonical trace τ of $\mathcal{N}G$ to the matrix algebra $\mathbf{M}_n(\mathcal{N}G)$. Moreover, the arguments in the proof of Proposition 2.6 can be used in order to show that the center-valued trace $a = t_{(n)}(A')$ is the operator

$$a = \frac{1}{n} \sum_{g\in G_f} \frac{r_g(A)}{[G : C_g]} L_g \in \mathcal{N}G.$$

Corollary A.5 (*cf.* [1]) *Let G be a group, n a positive integer and $E_1, \dots, E_m$ orthogonal idempotent $n \times n$ matrices with entries in the group algebra $\mathbf{C}G$. We fix an m-tuple of complex numbers $(\lambda_1, \dots, \lambda_m)$ and consider the matrix $A = \lambda_1 E_1 + \cdots + \lambda_m E_m \in \mathbf{M}_n(\mathbf{C}G)$. Then, we have*

$$\sum_{[g]\in\mathcal{C}_f(G)} \frac{|r_g(A)|^2}{[G : C_g]} \leq |\lambda_1|^2\, nr_1(E_1) + \cdots + |\lambda_m|^2\, nr_1(E_m),$$

with equality holding if and only if the matrix A is central in $\mathbf{M}_n(\mathbf{C}G)$.

Proof Let $E'_1, \dots, E'_m$ and A' be the matrices with entries in the von Neumann algebra $\mathcal{N}G$, which are induced from the matrices $E_1, \dots, E_m$ and A respectively. Then, $\tau_n(E'_i) = r_1(E_i)$ for all $i = 1, \dots, m$, where τ_n is the extension of the canonical trace τ of $\mathcal{N}G$ to the matrix algebra $\mathbf{M}_n(\mathcal{N}G)$, and

$$\tau(a^*a) = \|a(\delta_1)\|^2 = \frac{1}{n^2} \sum_{g\in G_f} \frac{|\, r_g(A)\,|^2}{[G : C_g]^2} = \frac{1}{n^2} \sum_{[g]\in\mathcal{C}_f(G)} \frac{|\, r_g(A)\,|^2}{[G : C_g]},$$

where $a = t_{(n)}(A') \in \mathcal{N}G$ is the center-valued trace of A'. Since $A' = \lambda_1 E'_1 + \cdots + \lambda_m E'_m$, the result is an immediate consequence of Corollary A.4. □

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