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Υπολογισμοί Αθροισμάτων

① Τύπος Cauchy - Γενίκευση

Θεώρημα

$x, y \in \mathbb{R}$; $\nu \geq 0$, $\nu \in \mathbb{Z}$

$$\binom{x+y}{\nu} = \sum_{k=0}^{\nu} \binom{x}{k}\binom{y}{\nu-k}$$

Απόδειξη

Γενικ. Διων. Θεωρ.

$$(1+t)^x = \sum_{k=0}^{\infty} \binom{x}{k} t^k \quad , \quad |t|<1$$

έχουμε για t με $|t|<1$

$$\sum_{\nu=0}^{\infty} \binom{x+y}{\nu} t^\nu = (1+t)^{x+y} = (1+t)^x (1+t)^y$$

$$= \Big(\sum_{k=0}^{\infty} \underbrace{\binom{x}{k}}_{a_k} t^k\Big)\Big(\sum_{k=0}^{\infty} \underbrace{\binom{y}{k}}_{\beta_k} t^k\Big)$$

$$\Rightarrow \gamma_\nu = a_0 \beta_\nu + a_1 \beta_{\nu-1} + \dots + a_\nu \beta_0 = \sum_{k=0}^{\nu} a_k \beta_{\nu-k} \Rightarrow$$

$$\binom{x+y}{\nu} = \sum_{k=0}^{\nu} \binom{x}{k}\binom{y}{\nu-k}$$

② Παράδειγμα

$$S = \sum_{k=0}^{\nu} k \binom{8}{k}\binom{17}{\nu-k} = ;$$

Λύση

$$S = \sum_{k=1}^{\nu} k \frac{8}{k} \binom{7}{k-1}\binom{17}{\nu-k}$$

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$$= 8\sum_{k=1}^{\nu}\binom{7}{k-1}\binom{17}{\nu-k} \overset{j=k-1}{=\!=} 8\sum_{j=0}^{\nu-1}\binom{7}{j}\binom{17}{\nu-1-j}$$

$$\overset{\text{Cauchy}}{=} 8\binom{7+17}{\nu-1} = 8\binom{24}{\nu-1}$$

③ Αθροίσματα τύπου $\sum_{k=0}^{\nu} \underbrace{f(k)}_{\text{πολυώνυμο του } k}\binom{r}{k}\binom{s}{\nu-k}$, $r,s\in\mathbb{R}$

1] Αναπτύσσω το $f(k)$

σε $A_0(k)_0 + A_1(k)_1 + \dots + A_\rho(k)_\rho$

↑ βαθμός πολ.

2] Υπολογίζω

$$\sum_{k=i}^{\nu}(k)_i\binom{r}{k}\binom{s}{\nu-k}$$

$(k)_i = k(k-1)\dots(k-i+1)$

$$= \sum_{k=i}^{\nu} k(k-1)\dots(k-i+1)\,\frac{r}{k}\cdot\frac{r-1}{k-1}\cdots\frac{r-i+1}{k-i+1}\binom{r-i}{k-i}\binom{s}{\nu-k}$$

$$= (r)_i\sum_{k=i}^{\nu}\binom{r-i}{k-i}\binom{s}{\nu-k} \overset{j=k-i}{=\!=} (r)_i\sum_{j=0}^{\nu-i}\binom{r-i}{j}\binom{s}{\nu-i-j}$$

$$= (r)_i\binom{r-i+s}{\nu-i}$$

④ Παράδειγμα

$$S = \sum_{k=0}^{\nu}(k^2+5k-8)\binom{12}{k}\binom{17}{\nu-k}$$

Λύση

$k^2+5k-8 = k(k-1)+6k-8 = (k)_2 + 6(k)_1 - 8(k)_0$

Άρα $S = (12)_2\binom{12+17-2}{\nu-2} + 6\cdot(12)_1\binom{12+17-1}{\nu-1} - 8\binom{12+17}{\nu}$

⑤ Υπενθυμίσεις

$$\underbrace{\left[{x \atop k}\right]}_{\frac{x(x+1)\cdots(x+k-1)}{k!}} = \binom{x+k-1}{k} = (-1)^k \binom{-x}{k}, \quad x \in \mathbb{R}$$

$$\binom{x}{k} = \left[{x-k+1 \atop k}\right] = (-1)^k \left[{-x \atop k}\right], \quad x \in \mathbb{R}$$

⑥ Παράδειγμα

$$S = \sum_{k=0}^{v} \binom{r+k}{k} \binom{s-k}{v-k}$$

Λύση

$$S = \sum_{k=0}^{v} \binom{(r+1)+k-1}{k} \binom{(s-v+1)+(v-k)-1}{v-k}$$

$$= \sum_{k=0}^{v} (-1)^k \binom{-(r+1)}{k} (-1)^{v-k} \binom{-(s-v+1)}{v-k}$$

$$= (-1)^v \sum_{k=0}^{v} \binom{-(r+1)}{k} \binom{-(s-v+1)}{v-k}$$

$$\overset{\text{Cauchy}}{=} (-1)^v \binom{-(r+1+s-v+1)}{v} = \binom{r+s-v+2+v-1}{v} = \binom{r+s+1}{v}$$

⑦ Παράδειγμα

$$S = \sum_{k=0}^{v} \frac{1}{(k+1)(k+2)} \binom{15}{k} \binom{35}{v-k}$$

Λύση

$$S = \frac{1}{16 \cdot 17} \sum_{k=0}^{v} \frac{16 \cdot 17}{(k+1)(k+2)} \binom{15}{k} \binom{35}{v-k}$$

$$\frac{v+1}{k+1}\binom{v}{k} = \binom{v+1}{k+1}$$

$$= \frac{1}{16 \cdot 17} \sum_{k=0}^{v} \binom{17}{k+2} \binom{35}{v-k}$$

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$$\overset{j=k+2}{=} \frac{1}{16\cdot 17} \sum_{j=2}^{\nu+2} \binom{17}{j}\binom{35}{\nu+2-j}$$

$$= \frac{1}{16\cdot 17}\left(\sum_{j=0}^{\nu+2} \binom{17}{j}\binom{35}{\nu+2-j} - \binom{17}{0}\binom{35}{\nu+2} - \binom{17}{1}\binom{35}{\nu+1} \right)$$

$$\overset{\text{cauchy}}{=} \frac{1}{16\cdot 17}\left(\binom{17+35}{\nu+2} - \binom{35}{\nu+2} - 17\binom{35}{\nu+1} \right)$$

⑧ Αθροίσματα $\sum_{u=0}^{\nu} \frac{F(u)}{(u+1)(u+2)\cdots(k+i)}\binom{r}{k}\binom{s}{\nu-u}$

$F(u) \to$ πολυώνυμο

1) Ανάπτυξη του $F(k)/(k+1)\cdots(k+i)$

σε $A_{-i}(k)_{-i} + A_{-i+1}(k)_{-i+1} + \dots + A_{\rho}(k)_{\rho}$

2) Υπολογ. κάθε άθροισμα όπως στο παράδειγμα

⑨ Αθροίσματα σε περιττούς ή άρτιους δείκτες

Παράδειγμα

τελευταίος άρτιος $\leq \nu$

$$S_a = \sum_{\substack{u=0 \\ k\ \text{άρτιος}}}^{\nu} \frac{1}{u+1}\binom{\nu}{k} = \frac{1}{1}\binom{\nu}{0} + \frac{1}{3}\binom{\nu}{2} + \frac{1}{5}\binom{\nu}{4} + \dots + \frac{1}{2[\frac{\nu}{2}]+1}\binom{\nu}{2[\frac{\nu}{2}]}$$

Αν ν φυσικός

τελευταίος άρτιος $\leq \nu$ $= 2\left[\frac{\nu}{2}\right]$

τελευταίος περιττός $\leq \nu$ $= \left(\begin{matrix}\text{τελευτ.} \\ \text{άρτιος} \\ \leq \nu-1\end{matrix}\right) + 1 = 2\left[\frac{\nu-1}{2}\right]+1$

Τεχνική: Εισάγω το συμπληρωματικό άθροισμα

$$S_n = \sum_{\substack{u=0,\ u\,\text{περ}}}^{v} \frac{1}{u+1}\binom{v}{u}$$

$$S_a + S_n = \sum_{u=0}^{v} \frac{1}{u+1}\binom{v}{u}$$

$$S_a - S_n = \sum_{u=0}^{v} \frac{1}{u+1}\binom{v}{u}(-1)^u$$

Έχω $\sum_{u=0}^{v} \frac{1}{u+1}\binom{v}{u}t^u = \frac{1}{v+1}\sum_{u=0}^{v} \frac{v+1}{u+1}\binom{v}{u}t^u$

$$= \frac{1}{v+1}\sum_{u=0}^{v}\binom{v+1}{u+1}t^u \overset{u+1=j}{=} \frac{1}{v+1}\sum_{j=1}^{v+1}\binom{v+1}{j}t^{j-1}$$

$$= \frac{(1+t)^{v+1}-1}{(v+1)t}$$

Άρα $\sum_{u=0}^{v} \frac{1}{u+1}\binom{v}{u} = \frac{2^{v+1}-1}{v+1}$

$\sum_{u=0}^{v} \frac{1}{u+1}\binom{v}{u}(-1)^u = \frac{-1}{-(v+1)} = \frac{1}{v+1}$

$\Rightarrow S_a + S_n = \frac{2^{v+1}-1}{v+1}$, $S_a - S_n = \frac{1}{v+1}$ $\Rightarrow$

$$\Rightarrow S_a = \frac{2^{v+1}}{2(v+1)} = \frac{2^v}{v+1}$$

(10) Αθροίσματα με χρήση συμμετρίας

$$\binom{v}{u} = \binom{v}{v-u}$$

πχ. $S = \binom{2v}{0} + \binom{2v}{1} + \binom{2v}{2} + \dots + \binom{2v}{v}$

Λύση

$$2S = \binom{2v}{0} + \binom{2v}{1} + \dots + \binom{2v}{v-1} + \binom{2v}{v} + \binom{2v}{2v} + \binom{2v}{2v-1} + \dots + \binom{2v}{v+1} + \binom{2v}{v}$$

$$= \sum_{k=0}^{2\nu} \binom{2\nu}{k} + \binom{2\nu}{\nu} = 2^{2\nu} + \binom{2\nu}{\nu}$$

Άρα $S = \dfrac{2^{2\nu} + \binom{2\nu}{\nu}}{2}$

⑪ Παράδειγμα

$$S = \sum_{k=0}^{\nu} \binom{\nu}{k}^2 = ;$$

Λύση

$$S = \sum_{k=0}^{\nu} \binom{\nu}{k}\binom{\nu}{\nu-k} \underset{\text{Cauchy}}{=} \binom{2\nu}{\nu}$$

⑫ Παράδειγμα

$$\sum_{k=0}^{\nu} (k+1)\binom{\nu}{k}^2 = \sum_{k=0}^{\nu} k\binom{\nu}{k}^2 + \sum_{k=0}^{\nu}\binom{\nu}{k}^2$$

$$= \sum_{k=1}^{\nu} k\binom{\nu}{k}\binom{\nu}{\nu-k} + \binom{2\nu}{\nu}$$

$$= \sum_{k=1}^{\nu} k \frac{\nu}{k}\binom{\nu-1}{k-1}\binom{\nu}{\nu-k} + \binom{2\nu}{\nu}$$

$$= \nu \sum_{k=1}^{\nu} \binom{\nu-1}{k-1}\binom{\nu}{\nu-k} + \binom{2\nu}{\nu}$$

$$\overset{j=k-1}{=} \sum_{j=0}^{\nu-1} \binom{\nu-1}{j}\binom{\nu}{\nu-1-j} + \binom{2\nu}{\nu}$$

$$\overset{\text{cauchy}}{=} \binom{2\nu-1}{\nu-1} + \binom{2\nu}{\nu}$$