9/11/23

Υπολογισμοί Αθροισμάτων

① Βασικές γεννήτριες

$$\sum_{k=0}^{\infty} t^k = \frac{1}{1-t}, \quad |t|<1$$

$$\sum_{k=0}^{\infty \text{ ή } \nu} \binom{\nu}{k} t^k = (1+t)^\nu \qquad (\nu \in \mathbb{Z}, \nu \geq 0)$$

$$\sum_{k=0}^{\infty} \genfrac{[}{]}{0pt}{}{\nu}{k} t^k = (1-t)^{-\nu}, \quad |t|<1 \qquad (\nu \in \mathbb{Z}, \nu \geq 0)$$

$$\sum_{k=0}^{\infty} \binom{x}{k} t^k = (1+t)^x, \quad x \in \mathbb{R}, \quad |t|<1$$

② Παράδειγμα

$$\sum_{k=0}^{\nu} k(k-1)\binom{\nu}{k} 3^k$$

Λύση:

1ος τρόπος - γεννήτριες

$$\sum_{\kappa=0}^{\nu}\binom{\nu}{\kappa}t^{\kappa}=(1+t)^{\nu}\xrightarrow{\frac{d}{dt}}\sum_{\kappa=0}^{\nu}\binom{\nu}{\kappa}\kappa t^{\kappa-1}=\nu(1+t)^{\nu-1}$$

$$\xrightarrow{\frac{d}{dt}}\sum_{\kappa=0}^{\nu}\binom{\nu}{\kappa}\kappa(\kappa-1)t^{\kappa-2}=\nu(\nu-1)(1+t)^{\nu-2}$$

$$\xrightarrow{\cdot t^{2}}\sum_{\kappa=0}^{\nu}\binom{\nu}{\kappa}\kappa(\kappa-1)t^{\kappa}=\nu(\nu-1)(1+t)^{\nu-2}t^{2}$$

$$\xrightarrow{t=3}\sum_{\kappa=0}^{\nu}\binom{\nu}{\kappa}\kappa(\kappa-1)3^{\kappa}=9\nu(\nu-1)\cdot 4^{\nu-2}$$

2ος τρόπος - ιδιότητα διων. συντ.

$$\binom{\nu}{\kappa}=\frac{\nu}{\kappa}\binom{\nu-1}{\kappa-1}$$

$$\frac{\nu!}{\kappa!(\nu-\kappa)!}=\frac{\nu}{\kappa}\cdot\frac{(\nu-1)!}{(\kappa-1)!(\nu-\kappa)!}$$

$$\sum_{\kappa=0}^{\nu}\kappa(\kappa-1)\binom{\nu}{\kappa}3^{\kappa}=\sum_{\kappa=2}^{\nu}\kappa(\kappa-1)\frac{\nu}{\kappa}\cdot\frac{\nu-1}{\kappa-1}\binom{\nu-2}{\kappa-2}3^{\kappa}$$

$$=\nu(\nu-1)\sum_{\kappa=2}^{\nu}\binom{\nu-2}{\kappa-2}3^{\kappa}\overset{j=\kappa-2}{=}\nu(\nu-1)\sum_{j=0}^{\nu-2}\binom{\nu-2}{j}3^{j+2}$$

$$=9\nu(\nu-1)\underbrace{\sum_{j=0}^{\nu-2}\binom{\nu-2}{j}3^{j}}_{(1+3)^{\nu-2}}$$

③ Αθροίσματα $\sum_{k=0}^{\nu} f(k)\binom{\nu}{k}\rho^k$

↑ πολυώνυμο του k

1) Ανάλυση του $f(k)$ σε καθοδικά παραγοντικά

$(k)_0, (k)_1, (k)_2, \dots$, όσο είναι ο βαθμός του πολυωνύμου

$(k)_0 = 1$, $(k)_1 = k$, $(k)_2 = k(k-1)$

2) Υπολογισμός κάθε όρου από αυτούς που σπάει το άθροισμα με γεννήτριες / ιδιότητα $\binom{\nu}{k} = \frac{\nu}{k}\binom{\nu-1}{k-1}$.

④ Παράδειγμα

$$S = \sum_{k=0}^{\nu} (3k^2-2k+1)\binom{\nu}{k}5^k = ;$$

$$3k^2-2k+1 = A_0(k)_0 + A_1(k)_1 + A_2(k)_2$$

$$\Leftrightarrow 3k^2-2k+1 = A_0 + A_1 k + A_2 k(k-1)$$

$$\Leftrightarrow 3k^2-2k+1 = A_2k^2 + (A_1-A_2)k + A_0$$

$$\Leftrightarrow \left\{\begin{array}{l} A_2 = 3 \\ A_1 - A_2 = -2 \\ A_0 = 1 \end{array}\right\} \Leftrightarrow \begin{array}{l} A_0 = 1 \\ A_1 = 1 \\ A_2 = 3 \end{array}$$

$$S = \sum_{k=0}^{\nu}(1+(k)_1+3(k)_2)\binom{\nu}{k}5^k$$

$$= \sum_{k=0}^{\nu}\binom{\nu}{k}5^k + \sum_{k=0}^{\nu}(k)_1\binom{\nu}{k}5^k + 3\sum_{k=0}^{\nu}(k)_2\binom{\nu}{k}5^k$$

Πρέπει να υπολογίσω

$$\sum_{k=0}^{\nu}(k)_r\binom{\nu}{k}\rho^k$$

Αυτό γίνεται είτε με r φορές παραγ του $(1+t)^{\nu}$

ή με $\binom{\nu}{u} = \frac{\nu(\nu-1)\dots(\nu-r+1)}{k(u-1)\cdots(u-r+1)}\binom{\nu-r}{u-r}$

Με παραγώγιση:

$$\sum_{u=0}^{\nu}\binom{\nu}{u}t^{u} = (1+t)^{\nu}$$

$$\xrightarrow{\frac{d^r}{dt^r}} \sum_{u=0}^{\nu}\binom{\nu}{u}(k)_r t^{u-r} = (\nu)_r(1+t)^{\nu-r}$$

$$\xrightarrow{\cdot t^r} \sum_{u=0}^{\nu}\binom{\nu}{u}(k)_r t^{u} = (\nu)_r(1+t)^{\nu-r}t^r \xrightarrow{t=\rho} \dots$$

Άρα

$$S = (1+5)^{\nu} + (\nu)_1(1+5)^{\nu-1}5 + 3(\nu)_2(1+5)^{\nu-2}5^2$$

$$= 6^{\nu} + 5\nu 6^{\nu-1} + 75\nu(\nu-1)6^{\nu-2}$$

⑤ Παράδειγμα

$$S = \sum_{u=0}^{\nu}\frac{1}{u+1}\binom{\nu}{u}7^{u} = ;$$

Λύση

1ος τρόπος - γεννήτριες

$$\sum_{u=0}^{\nu}\binom{\nu}{u}t^{u} = (1+t)^{\nu}$$

$$\xrightarrow{\int_0^u dt} \sum_{k=0}^{\nu}\binom{\nu}{k}\left[\frac{t^{k+1}}{k+1}\right]_{t=0}^{u} = \left[\frac{(1+t)^{\nu+1}}{\nu+1}\right]_{t=0}^{u}$$

$$\Rightarrow \sum_{k=0}^{\nu}\binom{\nu}{k}\frac{1}{k+1}u^{k+1} = \frac{(1+u)^{\nu+1}-1}{\nu+1}$$

$$\xRightarrow{u=7} \sum_{u=0}^{\nu}\binom{\nu}{k}\frac{1}{u+1}7^{u} = \frac{8^{\nu+1}-1}{\nu+1}$$

$$\Rightarrow \sum_{k=0}^{\nu} \binom{\nu}{k} \frac{1}{k+1} 7^k = \frac{8^{\nu+1}-1}{7(\nu+1)}$$

$2^{ος}$ τρόπος: Ιδιοτ $\binom{\nu}{k} = \frac{\nu}{k}\binom{\nu-1}{k-1}$

$$\sum_{k=0}^{\nu} \frac{1}{k+1}\binom{\nu}{k}7^k = \frac{1}{\nu+1}\sum_{k=0}^{\nu}\frac{\nu+1}{k+1}\binom{\nu}{k}7^k$$

$$= \frac{1}{\nu+1}\sum_{k=0}^{\nu}\binom{\nu+1}{\underset{j}{\underset{\parallel}{k+1}}}7^k = \frac{1}{\nu+1}\sum_{j=1}^{\nu+1}\binom{\nu+1}{j}7^{j-1}$$

$$= \frac{1}{7(\nu+1)}\left(\sum_{j=0}^{\nu+1}\binom{\nu+1}{j}7^j - \binom{\nu+1}{0}7^0\right)$$

$$= \frac{(1+7)^{\nu+1}-1}{7(\nu+1)}$$

⑥ Αθροίσματα τύπου

$$\sum_{k=0}^{\nu}\frac{F(k)}{(k+1)(k+2)\cdots(k+r)}\binom{\nu}{k}\rho^k$$

πολυων. του k

1) Σπάμε το $\frac{F(k)}{(k+1)\cdots(k+r)} = A_{-r}\underset{\frac{1}{(k+1)\cdots(k+r)}}{\underset{\parallel}{(k)_{r}}} + A_{-r+1}\underset{\frac{1}{(k+1)\cdots(k+r-1)}}{\underset{\parallel}{(k)_{-r+1}}} + \dots \overset{-2}{A}(k)$

Βαθμος του F(k)

2) Κάθε όρος υπολογίζεται χωριστά με

- Γεννήτριες (παραγ + ολοκλ)
- Ιδιοτ $\binom{\nu}{k} = \frac{\nu}{k}\binom{\nu-1}{k-1}$

⑦ Παράδειγμα

$$S = \sum_{k=0}^{\nu} \frac{k^3+3k^2-2k+5}{(k+1)(k+2)} \binom{\nu}{k} 7^k$$

Λύση

$$\frac{k^3-3k^2-2k+5}{(k+1)(k+2)} = \frac{A_{-2}}{(k+1)(k+2)} + \frac{A_{-1}}{k+1} + A_0 + A_1 k$$

$$\Leftrightarrow k^3-3k^2-2k+5 = A_{-2} + A_{-1}(k+2) + A_0(k+1)(k+2) + A_1 k(k+1)(k+2)$$

$$\Leftrightarrow \left\{ \begin{array}{rl} 5 = & A_{-2} + 2A_{-1} + 2A_0 \\ -2 = & A_{-1} + 3A_0 + 2A_1 \\ -3 = & A_0 + 3A_1 \\ 1 = & A_1 \end{array} \right\}$$

$$\Leftrightarrow A_1 = 1,\ A_0 = -6,\ A_{-1} = 14,\ A_{-2} = -11$$

Άρα $$S = -11 \sum_{k=0}^{\nu} \binom{\nu}{k} \frac{1}{(k+1)(k+2)} 7^k + 14 \sum_{k=0}^{\nu} \binom{\nu}{k} \frac{1}{k+1} 7^k$$

$$- 6 \sum_{k=1}^{\nu} \binom{\nu}{k} 7^k + \sum_{k=0}^{\nu} \binom{\nu}{k} k 7^k$$

Όμως $$\sum_{k=0}^{\nu} \binom{\nu}{k} \frac{1}{(k+1)(k+2)} 7^k = \frac{1}{7^2(\nu+1)(\nu+2)} \underbrace{\sum_{k=0}^{\nu} \binom{\nu+2}{k+2} 7^{k+2}}_{= 8^{\nu-2} - \binom{\nu+2}{1}7 - \binom{\nu+2}{0}}$$

$$= \frac{8^{\nu+2} - 7(\nu+2) - 1}{49(\nu+1)(\nu+2)} \quad \text{κλπ}$$

⑧ Αθροίσματα τύπου $\sum_{k=0}^{\nu \text{ ή } \infty} F(k)\rho^k$

↑ πολυων

ν πεπερ → οποιοδήποτε ρ

ν ∞ → $|\rho|<1$

Ιδέα υπολογισμού:

1) Σπάσιμο $F(k)$ σε καθοδ. παραγοντ.

2) Υπολογ. αθρ. $\sum_{k=0}^{\nu \text{ ή } \infty} (k)_r \rho^k$ με παραγώγιση

Της $\sum_{k=0}^{\infty} t^k = \frac{1}{1-t}$, $|t|<1$ ή της $\sum_{k=0}^{\nu} t^k = \frac{1-t^{\nu+1}}{1-t}$

⑨ Παράδειγμα

$S = 1\cdot 3^1 + 2\cdot 3^2 + 3\cdot 3^3 + \dots + \nu 3^\nu = ;$

Λύση

$S = \sum_{k=0}^{\nu} k 3^k$

Έχω $\sum_{k=0}^{\nu} t^k = \frac{1-t^{\nu+1}}{1-t} \xrightarrow{d/dt} \sum_{k=0}^{\nu} k t^{k-1} = \frac{-(\nu+1)t^\nu(1-t)+(1-t^{\nu+1})}{(1-t)^2}$

Άρα $\sum_{k=0}^{\nu} k 3^k = 3\cdot \frac{-(\nu+1)3^\nu(1-3)+(1-3^{\nu+1})}{(1-3)^2}$

(10) Παράδειγμα

$$S = \sum_{k=0}^{\infty} (k+1)^2 \left(\frac{1}{3}\right)^k = ;$$

Λύση

$$(k+1)^2 = k^2 + 2k + 1 = A_2 (k)_2 + A_1 (k)_1 + A_0 (k)_0$$
$$= A_2 k(k-1) + A_1 k + A_0$$
$$= A_2 k^2 + (A_1 - A_2) k + A_0$$

Άρα $A_0 = 1$, $A_1 - A_2 = 2$, $A_2 = 1$

$$(k+1)^2 = (k)_2 + 3(k)_1 + 1$$

Άρα

$$S = \sum_{k=0}^{\infty} (k)_2 \left(\frac{1}{3}\right)^k + 3 \sum_{k=0}^{\infty} (k)_1 \left(\frac{1}{3}\right)^k + \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^k$$

Έχω $\sum_{k=0}^{\infty} t^k = (1-t)^{-1}$

$$\xRightarrow{d/dt} \sum_{k=0}^{\infty} k t^{k-1} = (1-t)^{-2}$$

$$\xRightarrow{d/dt} \sum_{k=0}^{\infty} k(k-1) t^{k-2} = 2(1-t)^{-3}$$

Άρα

$$S = \left[\frac{2t^2}{(1-t)^3} + 3\frac{t}{(1-t)^2} + \frac{1}{1+t} \right]_{t=\frac{1}{3}}$$

$$= \frac{\frac{2}{9}}{\frac{8}{27}} + \frac{1}{\frac{4}{9}} + \frac{1}{\frac{2}{3}} = \ldots$$