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INTERACTIONS BETWEEN COMPRESSED SENSING RANDOM MATRICES AND HIGH DIMENSIONAL GEOMETRY

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Abstract. — These notes are an expanded version of short courses given at the occasion of a school held in Université Paris-Est Marne-la-Vallée, 16–20 November 2009, by Djalil Chafaï, Olivier Guédon, Guillaume Lecué, Alain Pajor, and Shahar Mendelson. The central motivation is compressed sensing, involving interactions between empirical processes, high dimensional geometry, and random matrices.

CONTENTS

INTRODUCTION

Compressed sensing also referred to in the literature as compressive sensing or compressive sampling is a framework that enables to get approximate and exact reconstruction of sparse signals from incomplete measurements. The existence of efficient algorithms for the reconstruction, such as the ℓ_1-minimization, and the potential of applications in signal processing and imaging, has led to a fast and wide development of the theory after the seminal articles by D. Donoho [**Don06**], E. Candes, J. Romberg and T. Tao [**CRT06**] and E. Candes and T. Tao [**CT06**].

The ideas and principles underlying the discoveries of these phenomena in high dimensions are related to problems and progresses from Approximation Theory. One significant example of such an interaction is the study of Gelfand and Kolmogorov widths of classical Banach spaces. There is already a large literature on compressed sensing, on both theoretical and numerical aspects. Our aim is not to survey the state of the art of this recent field developing with great speed, but to highlight and to study some interactions with other fields of mathematics, in particular with asymptotic geometric analysis, random matrices and empirical processes.

To introduce the subject, let $T \subset \mathbb{R}^N$ and let A be an $n \times N$ real matrix with rows $Y_1, \dots, Y_n \in \mathbb{R}^N$. Consider the general problem of reconstructing any vector $x \in T$ from the *data* $Ax \in \mathbb{R}^n$, that is from the known *measurements*

$$\langle Y_1, x \rangle, \dots, \langle Y_n, x \rangle.$$

Classical linear algebra suggests that the number n of measurements should be at least as large as its dimension N in order to ensure reconstruction. Compressed sensing provides a way of reconstructing the original signal x from its compression Ax that takes only a small amount of linear measurements, that is with $n \ll N$. Clearly one needs some a priori hypothesis on the subset T of signals that we want to reconstruct and of course, the matrix A should be suitably chosen.

The first point concerns the subset T and is a matter of *complexity*. Many tools within this framework were developed in Approximation Theory and in Geometry of Banach Spaces. This is one of our goal to bring forward these tools.

The second point is concerned with the design of the measurement matrix A. So far the only good matrices are random *sampling* matrices. They are obtained in many examples by sampling $Y_1, \dots, Y_n \in \mathbb{R}^N$ in a suitable way. This is where probability enters. These random sampling matrices will be of Gaussian or Bernoulli (± 1) type or random sub-matrices of the discrete Fourier $N \times N$ matrix (partial Fourier matrices). There is a huge technical difference in the study of unstructured compressive matrices (with i.i.d entries) and other case such as partial Fourier matrices. This is one of our goal to study the main tools from probability theory that fall within this framework. These are tools from classical probabilistic inequalities, concentration of measure and empirical processes as well as from random matrix theory.

This is precisely the purpose of Chapter 1 to present some basic tools and preliminary background. This chapter will look briefly at elementary properties of Orlicz spaces in relation with tail inequalities of random variables. An important connection between high dimensional geometry and the study of empirical processes comes from the behavior of the sum of independent centered random variables with sub-exponential tails. Discretization is an important step in the study of empirical processes. One approach is given by a net argument. The size of the discrete space may be estimated by the covering numbers. The basic tools to estimate covering numbers from above, such as Sudakov inequality, are presented in the last part of Chapter 1.

Chapter 2 is devoted to compressed sensing. The purpose is to provide some of the key mathematical insights underlying this new sampling method. We present first the exact reconstruction problem as introduced above. The a priori hypothesis on the subset of signals T that we investigate is *sparsity.* A vector is said to be m-sparse if it has at most m non-zero coordinates. An important feature of this subset is its peculiar structure: its intersection with the Euclidean unit sphere is the unions of unit spheres supported on m-dimensional coordinate subspaces. This set is highly compact when the degree of compacity is measured in terms of covering numbers. It makes it a *small* subset of the sphere as far as $m \ll N$, which will be the case.

A fundamental feature of compressive sensing is that practical reconstruction can be performed by using efficient algorithms such as the ℓ_1-minimization method which consists, for a given data $y = Ax$, to perform the "linear programming":

$$\min_{t \in \mathbb{R}^N} \sum_{i=1}^N |t_i| \quad \text{subject to} \quad At = y.$$

At this step, the problem comes to find matrices for which this algorithm reconstructs any m-sparse vectors with m relatively large. A study of the cone of constraints to ensure that every m-sparse vector can be reconstructed by the ℓ_1-minimization method leads to a necessary and sufficient condition known as the *null space property* of order m:

$$\forall h \in \ker A,\ h \neq 0,\ \forall I \subset [N],\ |I| \leq m,\ \sum_{i \in I} |h_i| < \sum_{i \in I^c} |h_i|.$$

This property has a nice geometric interpretation on the structure of faces of random polytopes called *neighborliness.* Indeed, if P is the polytope obtained by taking the symmetric convex hull of the columns of A, the *null space property* of order m for

A is equivalent to a *neighborliness* property of order m for P. This means that the matrix A which maps the vertices of the cross-polytope

$$B_1^N = \Big\{ t \in \mathbb{R}^N \,:\, \sum_{i=1}^N |t_i| \le 1 \Big\}$$

onto the vertices of P preserves the structure of k-dimensional faces up to the dimension $k = m$; a remarkable connection between compressed sensing and high dimensional geometry due to D. Donoho [**Don05**].

Unfortunately, the null space property is not easy to verify. An ingenious sufficient condition is the so-called Restricted Isometry Property (RIP) of order m that requires that all column sub-matrices of size m of the matrix are well-conditioned. More precisely, we say that A satisfies the RIP of order p with parameter $\delta = \delta_p$ if

$$1 - \delta_p \leqslant |Ax|_2^2 \leqslant 1 + \delta_p$$

holds for all p-sparse unit vectors $x \in \mathbb{R}^N$. An important feature of this concept is that if A satisfies the RIP of order $2m$ with parameter δ small enough then every m-sparse vector can be reconstructed by the ℓ_1-minimization method. Even if this RIP condition is difficult to check on a given matrix, it actually holds true with high probability for certain models of random matrices and is easily checked for some of them.

Here is the point where probabilistic methods come into play. Among good unstructured sampling matrices we shall study the case of Gaussian and Bernoulli random matrices. The case of partial Fourier matrices, which is more delicate will be studied in Chapter 5. Checking the restricted isometry property for the first two models may be treated along a simple scheme, namely, the ε net argument presented in Chapter 2.

Another angle to tackle the problem of reconstruction by ℓ_1-minimization is to analyse the Euclidean diameter of the section of the cross-polytope B_1^N with the kernel of A. This study leads to the notion of Gelfand widths, particularly for the cross-polytope B_1^N. Its Gelfand widths are defined by the numbers

$$d^n(B_1^N, \ell_2^N) = \inf_{\operatorname{codim} S \leqslant n} \operatorname{rad}(S \cap B_1^N), \quad n = 1, \dots, N$$

where $\operatorname{rad}(S \cap B_1^N) = \max\{\, |x| \,:\, x \in S \cap B_1^N \}$ denotes the half Euclidean diameter of the section of B_1^N.

Significant work was done in this direction in the seventies. This viewpoint from Approximation Theory and Asymptotic Geometric Analysis enlighten a new aspect of the problem and is based on a celebrated result of B. Kashin [**Kaš77**] stating that

$$d^n(B_1^N, \ell_2^N) \leqslant \frac{C}{\sqrt{n}} \log^{O(1)}(N/n)$$

where C is some numerical constant. The relevance of this result in compressed sensing is highlighted by the following connection.

Let $1 \le m \le n$, *if*

$$\operatorname{rad}(\ker A \cap B_1^N) < 1/2\sqrt{m}$$

then every m-sparse vectors can be reconstructed by ℓ_1-minimization.

From this perspective, the goal is to estimate the diameter $\mathrm{rad}\,(\ker A \cap B_1^N)$ from above. This is discussed in detail for some model of random matrices. The connection with the RIP is clarified in the following result.

Assume that A satisfies the RIP of order p with parameter δ, then

$$\mathrm{rad}\,(\ker A \cap B_1^N) \leq \frac{C}{\sqrt{p}} \frac{1}{1-\delta}$$

where C is a numerical constant. Therefore $\mathrm{rad}\,(\ker A \cap B_1^N) < 1/2\sqrt{m}$ *is satisfied with $m = O(p)$.*

The ℓ_1-minimization method extends to the study of approximate reconstruction of vectors which are not far from sparse vectors. Let $x \in \mathbb{R}^N$ and let $x^\sharp$ be a minimizer of

$$\min_{t \in \mathbb{R}^N} \sum_{i=1}^N |t_i| \quad \text{subject to} \quad At = Ax.$$

Again, the notion of width appears very useful. We prove the following:

Assume that $\mathrm{rad}\,(\ker A \cap B_1^N) < 1/4\sqrt{m}$*, then for any $I \subset \{1, \ldots, N\}$ such that $|I| \leqslant m$ and any $x \in \mathbb{R}^N$, we have*

$$|x - x^\sharp|_2 \leq \frac{1}{\sqrt{m}} \sum_{i \notin I} |x_i|.$$

This applies in particular to unit vectors of the space $\ell_{p,\infty}^N$, $0 < p < 1$ for which $\min_{|I| \leqslant m} \sum_{i \notin I} |x_i|$ is $O(m^{1-1/p})$.

In the last section of Chapter 2 we introduce the parameter of complexity $\ell_*(T)$ of a subset $T \subset \mathbb{R}^N$ defined by

$$\ell_*(T) = \mathbb{E} \sup_{t \in T} X_t,$$

where (X_t) is the Gaussian process $X_t = \sum_{i=1}^N g_i t_i$, indexed by $t = (t_i)_{i=1}^N \in T$ and $g_1, ..., g_N$ are independent $N(0,1)$ Gaussian random variables. This kind of parameter plays an important role in empirical processes and in Geometry of Banach spaces ([**Mil86, PTJ86, Tal87**]). It allows to control the size of $\mathrm{rad}\,(\ker A \cap T)$ which as we understand is a crucial issue in approximate reconstruction.

This study will be deepen in Chapter 3. In this Chapter we first present classical results from the Theory of Gaussian processes. To make the link with compressed sensing, observe that if A is a $n \times N$ matrix with row vectors $Y_1, \ldots, Y_n$, then the restricted isometry property of order p with parameter δ_p can be translated in terms of an empirical process property since

$$\delta_p = \sup_{x \in S_2(\Sigma_p)} \Big| \frac{1}{n} \sum_{i=1}^n \langle Y_i, x \rangle^2 - 1 \Big|$$

where $S_2(\Sigma_p)$ is the set of unit one p-sparse vectors of $\mathbb{R}^N$. While Chapter 2 makes use of a simple ε net argument to study such process, we present in Chapter 3 the chaining and generic chaining techniques based on metric complexity measures such as the γ_2 functional. This γ_2 functional is precisely related to the parameter $\ell_*(T)$ by the majorizing measure theorem of M. Talagrand [**Tal87**]. This technique enables to provide a criterion implying a restricted isometry property for unstructured models of random matrices, which include the Bernoulli and Gaussian models.

It is worth noticing that the ε net argument, the chaining argument and the generic mechanism are all techniques sharing the same couple of ideas: one of which is the classical trade-off between complexity and concentration and the other one is an approximation principle. For instance, consider a Gaussian matrix $A = n^{-1/2}(g_{ij})_{1\leqslant i\leqslant n, 1\leqslant j\leqslant N}$ where the g_{ij}'s are i.i.d. standard Gaussian variables. Let T be a subset of the unit sphere S^{N-1} of $\mathbb{R}^N$. A classical problem is to understand how A acts on T. In particular, does A preserve the Euclidean norm on T? In the Compressed Sensing setup, the "input" dimension N is much larger than the number of measurements n, because A is used as a compression matrix. So clearly A cannot preserve the Euclidean norm on the whole sphere S^{N-1}. Hence, it is natural to identify the subsets T of S^{N-1} such that A acts on T in a norm preserving way. Let start with a single point $x \in T$. Then for any $\varepsilon \in (0,1)$, with probability greater than $1 - 2\exp(-c_0 n\varepsilon^2)$,

$$1 - \varepsilon \leqslant |Ax|_2^2 \leqslant 1 + \varepsilon.$$

This result is the one expected since $\mathbb{E}|Ax|_2^2 = |x|_2^2$ (we say that the standard Gaussian measure is isotropic) and the Gaussian measure on $\mathbb{R}^N$ has strong concentration properties. Thus proving that A acts in a norm preserving way on a single vector is only a matter of isotropicity and concentration. Now, we want to see how many points in T share this property simultaneously. This is where the trade-off between complexity and concentration is at stack. A simple union bound tells us that if $\Lambda \subset T$ has a cardinality less than $\exp(c_0 n\varepsilon^2/2)$, then, with probability greater than $1 - 2\exp(-c_0 n\varepsilon^2/2)$,

$$\forall x \in \Lambda \qquad 1 - \varepsilon \leqslant |Ax|_2^2 \leqslant 1 + \varepsilon.$$

This means that A preserves the norm of all the vectors in Λ at the same time as long as $|\Lambda| \leqslant \exp(c_0 n\varepsilon^2/2)$. Have we had other entries in A with other concentration properties, we will have ended up with other cardinality for $|\Lambda|$. As a consequence, it is possible to control the norm of the images by A of $\exp(c_0 n\varepsilon^2/2)$ points in T at the same time. The first way of choosing Λ that may come to mind is to use an ε net of T with respect to ℓ_2^N and then to ask if the norm preserving property of A on Λ extends to T? Indeed, if $m \leq C(\varepsilon) n \log^{-1}\left(N/n\right)$, there exists an ε net Λ of size $\exp(c_0 n\varepsilon^2/2)$ in $S_2(\Sigma_m)$ for the Euclidean metric. And, by what is now called the ε net argument, we can describe all the points in $S_2(\Sigma_m)$ using only the point in Λ:

$$\Lambda \subset S_2(\Sigma_m) \subset (1-\varepsilon)^{-1}\mathrm{conv}(\Lambda).$$

This allows the norm preserving property of A on Λ to be extended to the entire set $S_2(\Sigma_m)$. This was the scheme used in Chapter 2.

But this scheme does not apply to any set T in S^{N-1}. That is why we present in Chapter 3, the chaining and generic chaining methods. Unlike the ε net argument which demanded only to know how A acts on a single ε net of T, these two methods require to study the action of A on a sequence (T_s) of subsets of T with an exponentially increasing cardinality. In the case of the chaining argument, T_s can be chosen as an ε_s net of T where ε_s is chosen so that $|T_s| = 2^s$ and for the generic chaining argument, the choice of (T_s) is recursive: for large values of s, the set T_s is a maximal separated set in T of cardinality 2^{2^s} and for small values of s, the construction of T_s depends on the sequence $(T_r)_{r\geqslant s+1}$. For these methods, the approximation argument follows from the fact that $d_{\ell_2^N}(t, T_s)$ tends to zero when s tends to infinity for any $t \in T$. And, the trade-off complexity/concentration is used at every s stage of the approximation of T by T_s. The metric complexity parameter coming from the chaining method is called the Dudley entropy integral

$$\int_0^\infty \sqrt{\log N(T, d, \varepsilon)}\, d\varepsilon$$

while the one coming out of the generic chaining mechanism is the γ_2 functional

$$\gamma_2(T, \ell_2^N) = \inf_{(T_s)_s} \sup_{t\in T} \sum_{s=0}^{\infty} 2^{s/2} d_{\ell_2^N}(t, T_s)$$

where the infimum is taken over all sequences (T_s) of subsets of T such that $|T_0| \leqslant 1$ and $|T_s| \leqslant 2^{2^s}$ for every $s \geqslant 1$. In Chapter 3, we prove that A acts in a norm preserving way on T with probability exponentially in n close to 1 as long as

$$\gamma_2(T, \ell_2^N) = O(\sqrt{n}).$$

In particular, in the case $T = S_2(\Sigma_m)$ treated in Compressed Sensing, this condition implies $m = O\big(n \log^{-1}\big(N/n\big)\big)$ which is the same condition obtained using the ε net argument in Chapter 2. That is the reason why, as long as norm preserving properties of random operator are concerned, the results obtained in Chapter 3 generalizes the one of Chapter 2. Nevertheless, the norm preserving property of A on a set T implies an exact reconstruction property of A of all m-sparse vectors by the ℓ_1-minimization method only when $T = S_2(\Sigma_m)$. In this case, this norm preserving property is the restricted isometry property of order m.

On the other hand, the Restricted Isometry Property can be translated as a control on the largest and smallest singular values of all sub-matrices of a certain size. The singular values of matrices is precisely the subject of Chapter 4. An $m \times n$ matrix A with $m \leqslant n$ maps the unit sphere to an ellipsoid, and the half lengths of the principle axes of this ellipsoid are precisely the singular values $s_1(A) \geqslant \cdots \geqslant s_m(A)$ of A. In particular,

$$s_1(A) = \max_{|x|_2=1} |Ax|_2 = \|A\|_{2\to 2} \quad \text{and} \quad s_n(A) = \min_{|x|_2=1} |Ax|_2.$$

Geometrically, A is seen as a correspondence–dilation between two orthonormal bases. In matrix form $UAV^* = \mathrm{diag}(s_1(A), \ldots, s_m(A))$ for a couple of unitary matrices U and V of size $m \times m$ and $n \times n$. This is the so called singular value decomposition – SVD for short – which has a tremendous importance in numerical mathematics. One can read on the singular values the rank and the norm of the inverse of the matrix. The singular values are the eigenvalues of the Hermitian matrix $\sqrt{AA^*}$. The largest and smallest singular values enter the definition of the condition number s_1/s_m which allows to control the behavior of linear systems under perturbations of small norm.

The first part of Chapter 4 is a compendium on the singular values of deterministic matrices, including main useful perturbation inequalities, mostly without proofs. The Gram–Schmidt algorithm used for the rows and the columns of A allows to construct a bidiagonal matrix which is unitary equivalent to A. This structural fact is at the heart of most numerical algorithms for the actual computation of the singular values.

The second part of Chapter 4 deals with random matrices with i.i.d. entries and their singular values. The aim is to propose a cultural tour in this vast and growing subject. This tour begins with Gaussian random matrices with i.i.d. entries forming the Ginibre Ensemble. The density is proportional to $G \mapsto \exp(-\mathrm{Tr}(GG^*))$. The matrix $W = GG^*$ follows a Wishart law, a sort of multivariate χ^2. The unitary bidiagonalization allows to compute the density of the singular values of these Gaussian random matrices, which turns out to be proportional to a function of the form

$$s \mapsto \prod_k s_k^\alpha e^{-s_k^2} \prod_{i \neq j} |s_i^2 - s_j^2|^\beta.$$

The change of variable $s_k \mapsto s_k^2$ reveals Laguerre weights in front of the Vandermonde determinant, the starting point of an orthogonal polynomials story. As for most random matrix ensembles, the determinant expresses a logarithmic repulsion. Here it comes from the Jacobian of the SVD. Such Gaussian models allow explicit yet heavy computations. Many large dimensional aspects of random matrices depend only on the first two moments of the entries, and this makes the Gaussian case universal. The most well known universal asymptotic result is indubitably the Marchenko-Pastur theorem. More precisely if M is an $m \times n$ random matrix with i.i.d. entries of variance $n^{-1/2}$ then the empirical counting probability measure of the singular values of M

$$\frac{1}{m} \sum_{k=1}^{m} \delta_{s_k(M)}$$

tends weakly, when $n, m \to \infty$ with $m/n \to \rho \in (0, 1]$, to the Marchenko-Pastur law

$$\frac{1}{\rho \pi x} \sqrt{((x+1)^2 - \rho)(\rho - (x-1)^2)}\, \mathbf{1}_{[1-\sqrt{\rho}, 1+\sqrt{\rho}]}(x) dx.$$

We provide a proof of the Marchenko-Pastur theorem by using the methods of moments. When the entries of M have zero mean and finite fourth moment, the Bai-Yin theorem furnishes the convergence at the edge of the support, in the sense that

$$s_m(M) \to 1 - \sqrt{\rho} \quad \text{and} \quad s_1(M) \to 1 + \sqrt{\rho}.$$

Chapter 4 simply gives some basic aspects of the study of the singular values of random matrices, an immense and fascinating subject still under active development.

As Chapter 2 already pointed out, the study of the radius of the section of the cross-polytope with the kernel of a matrix is central in approximate reconstruction. This approach is pursued in Chapter 5 on the model of partial discrete Fourier matrices or Walsh matrices. The discrete Fourier matrix and the Walsh matrix are particular cases of orthogonal matrices with bounded L_∞ entries. More generally, we consider matrices whose rows are a system of orthogonal vectors $\phi_1, \dots, \phi_N$ such that for any $i = 1, \dots, N$, $|\phi_i|_2 = K$ and $|\phi_i|_\infty \le 1/\sqrt{N}$. Several other models fall into this setting. Let Y be the random vector defined by $Y = \phi_i$ with probability $1/N$ and let $Y_1, \dots, Y_n$ be independent copies of Y. One main result of Chapter 5 indicates that if

$$m \le C_1 \; K^2 \frac{n}{\log N (\log n)^3}$$

then with probability greater than

$$1 - C_2 \exp\left(-C_3 K^2 n/m\right)$$

the matrix $\Phi = (Y_1, \dots, Y_n)^\top$ satisfies

$$\mathrm{rad}\,(\ker \Phi \cap B_1^N) < \frac{1}{2\sqrt{m}}.$$

In Compressed Sensing, n is chosen relatively small with respect to N and the result is that up to logarithmic factors, if m is of the order of n, the matrix Φ has the property that every m-sparse vectors can be exactly reconstructed by the ℓ_1-minimization algorithm. The numbers C_1, C_2 and C_3 are numerical constants and changing C_1 by a smaller constant leads to allow approximate reconstruction by the ℓ_1-minimization algorithm. The randomness introduced here is called the empirical method but it is worth noticing that it can be replaced by the method of selectors, defining Φ with its row vectors $\{\phi_i, i \in I\}$ where $I = \{i, \delta_i = 1\}$ and $\delta_1, \dots, \delta_N$ are independent identically distributed selectors taking values 1 with probability $\delta = n/N$ and 0 with probability $1 - \delta$. In this case the cardinality of I is approximately n with high probability.

Within the framework of selection of characters, the situation is different. A useful observation is that by orthogonality of the system $\{\phi_1, \dots, \phi_N\}$, we have $\ker \Phi = \mathrm{span}\,\{\phi_j\}_{j \in J}$ where $\{Y_i\}_{i=1}^n = \{\phi_i\}_{i \notin J}$. Therefore the previous statement is identical to a result which is to select $|J| \ge N - n$ vectors in $\{\phi_1, \dots, \phi_N\}$ such that the ℓ_1^N norm and the ℓ_2^N norm are comparable on the vectorial span of these vectors. Indeed, the conclusion $\mathrm{rad}\,(\ker \Phi \cap B_1^N) < \frac{1}{2\sqrt{m}}$ is equivalent to the following inequality

$$\forall (\alpha_j)_{j \in J}, \; \left|\sum_{j \in J} \alpha_j \phi_j\right|_2 \le \frac{1}{2\sqrt{m}} \left|\sum_{j \in J} \alpha_j \phi_j\right|_1 .$$

At issue is how large can be the cardinality of J so that the comparison between the ℓ_1^N norm and the ℓ_2^N norm on the subspace spanned by $\{\phi_j\}_{j \in J}$ is better than the trivial Hölder inequality. Choosing n of the order of $N/2$ gives already a remarkable result: there exists a subset J of cardinality greater than $N/2$ such that

$$\forall (\alpha_j)_{j \in J}, \; \frac{1}{\sqrt{N}} \left|\sum_{j \in J} \alpha_j \phi_j\right|_1 \le \left|\sum_{j \in J} \alpha_j \phi_j\right|_2 \le C_4 \, \frac{(\log N)^2}{\sqrt{N}} \left|\sum_{j \in J} \alpha_j \phi_j\right|_1 .$$

This is a Kashin type result. Nevertheless, it is important to remark that in the statement of Dvoretzky [**FLM77**] or Kashin [**Kaš77**] theorem about Euclidean sections of the cross-polytope, the subspace is such that the ℓ_2^N norm and the ℓ_1^N norm are equivalent (without the factor $\log N$) but has no particular structure. In the setting of Harmonic Analysis, the issue is to find a subspace with very strong properties. It should be a coordinates subspace with respect to the basis given by $\{\phi_1, \ldots, \phi_N\}$. J. Bourgain noticed that a factor $\sqrt{\log N}$ is necessary in the last inequality above. Let μ being the discrete probability measure on $\mathbb{R}^N$ with weight $1/N$ on each vectors of the canonical basis, the above inequalities tell that for every scalars $(\alpha_j)_{j\in J}$,

$$\left\| \sum_{j\in J} \alpha_j \phi_j \right\|_{L_1(\mu)} \le \left\| \sum_{j\in J} \alpha_j \phi_j \right\|_{L_2(\mu)} \le C_4 \, (\log N)^2 \left\| \sum_{j\in J} \alpha_j \phi_j \right\|_{L_1(\mu)}.$$

This explains the deep connection between Compressed Sensing and the problem of selecting a large part of a system of characters such that on the vectorial span of this family, the $L_2(\mu)$ and the $L_1(\mu)$ norms are as close as possible, where μ is a probability measure. This subject of Harmonic Analysis goes back to the construction of $\Lambda(p)$ sets which are not $\Lambda(q)$ for $q > p$ where powerful methods of selectors were developed by J. Bourgain [**Bou89**]. M. Talagrand proved in [**Tal98**] that there exists a small numerical constant δ_0 and a subset J of cardinality greater than $\delta_0 N$ such that for every scalars $(\alpha_j)_{j\in J}$,

$$\left\| \sum_{j\in J} \alpha_j \phi_j \right\|_{L_1(\mu)} \le \left\| \sum_{j\in J} \alpha_j \phi_j \right\|_{L_2(\mu)} \le C_5 \, \sqrt{\log N \log\log N} \left\| \sum_{j\in J} \alpha_j \phi_j \right\|_{L_1(\mu)}.$$

It is the purpose of Chapter 5 to emphasize the connections between Compressed Sensing and these problems of Harmonic Analysis. Tools of empirical processes are at the heart of the technics of proof. We will present classical results from the theory of empirical processes that are needed for the proof of the main results of the chapter. We will enlighten about how techniques from Geometry of Banach Spaces are relevant in this setting. We will also present the strategy with regard to extending the result of M. Talagrand [**Tal98**] to a Kashin type setting.

CHAPTER 1

EMPIRICAL METHODS AND HIGH DIMENSIONAL GEOMETRY

This chapter is devoted to the presentation of classical tools that will be used within this book. We present some elementary properties of Orlicz spaces and develop the particular case of ψ_α random variables. Several characterization are given in terms of tail estimate, Laplace transform and moments behavior. One of the important connection between high dimensional geometry and the study of empirical processes comes from the behavior of the sum of (centered) ψ_α random variables. An important part of these preliminaries concentrates on this subject. We illustrate these connections with the presentation of the Johnson-Lindenstrauss lemma. The last part of this chapter is devoted to the study of covering numbers. We focus our attention on some elementary properties and on the presentation of methods to estimate upper bounds of these covering numbers.

1.1. Presentation of the Orlicz spaces

An Orlicz space is a function space which extends naturally the classical L_p spaces when $1 \leq p \leq +\infty$. A function $\psi : [0, \infty) \to [0, \infty)$ is an Orlicz function if it is a convex increasing function such that $\psi(0) = 0$ and $\psi(x) \to \infty$ when $x \to \infty$.

Definition 1.1.1. — *Let ψ be an Orlicz function, for any real random variable X on a measurable space (Ω, σ, μ), we define its L_ψ norm by*

$$\|X\|_\psi = \inf \big\{ c > 0 : \mathbb{E}\psi\big(|X|/c\big) \leqslant \psi(1) \big\}.$$

The space $L_\psi(\Omega, \sigma, \mu) = \{X : \|X\|_\psi < \infty\}$ is called the Orlicz space associated to ψ.

It is well known that L_ψ is a Banach space. Classical examples of Orlicz functions are for $p \geqslant 1$ and $\alpha \geqslant 1$

$$\forall x \geq 0, \quad \phi_p(x) = x^p/p \quad \text{and} \quad \psi_\alpha(x) = \exp(x^\alpha) - 1.$$

The Orlicz space associated to ϕ_p is the classical L_p space. It is also clear by the theorem of monotone convergence that the infimum in the definition of the L_ψ norm of a random variable X, if finite, is attained at $\|X\|_\psi$.

Let ψ be a nonnegative convex function on $[0, \infty)$. Its *convex conjugate* ψ^* (also called the *Legendre transform*) is defined on $[0, \infty)$ by:

$$\forall y \geq 0, \quad \psi^*(y) = \sup_{x>0} \{xy - \psi(x)\}.$$

The convex conjugate of an Orlicz function is also an Orlicz function.

Proposition 1.1.2. — *Let ψ be an Orlicz function and ψ^* be its convex conjugate. For every real random variables $X \in L_\psi$ and $Y \in L_{\psi^*}$,*

$$\mathbb{E}|XY| \leqslant (\psi(1) + \psi^*(1)) \|X\|_\psi \|Y\|_{\psi^*}.$$

Proof. — By homogeneity, we can assume $\|X\|_\psi = \|Y\|_{\psi^*} = 1$. By definition of the convex conjugate, we have

$$|XY| \leqslant \psi(|X|) + \psi^*(|Y|).$$

Taking the expectation, since $\mathbb{E}\psi(|X|) \leq \psi(1)$ and $\mathbb{E}\psi^*(|Y|) \leqslant \psi^*(1)$, we get that $\mathbb{E}|XY| \leq \psi(1) + \psi^*(1)$. □

It is not difficult to observe that if $\phi_p(t) = t^p/p$ then $\phi_p^* = \phi_q$ where $p^{-1} + q^{-1} = 1$ (it is also known as Young's inequality). In this case, Proposition 1.1.2 corresponds to Hölder inequality.

Any information about the ψ_α norm of a random variable is very useful to describe a tail behavior. This will be explained in Theorem 1.1.5. For instance, we say that X is a *sub-Gaussian* random variable when $\|X\|_{\psi_2} < \infty$, we say that X is a *sub-exponential* random variable when $\|X\|_{\psi_1} < \infty$. In general, we say that X is ψ_α when $\|X\|_{\psi_\alpha} < \infty$. It is important to notice (see Corollary 1.1.6 and Proposition 1.1.7) that for any $1 \leq p < +\infty$, for any $\alpha_2 \geqslant \alpha_1 \geq 1$

$$L_\infty \subset L_{\psi_{\alpha_2}} \subset L_{\psi_{\alpha_1}} \subset L_p.$$

One of the main goal of these preliminaries will be to understand the behavior of the maximum of a family of L_ψ-random variables and of the sum and the product of ψ_α random variables. We start with a general maximal inequality.

Proposition 1.1.3. — *Let ψ be an Orlicz function. Then, for any positive integer n and any real valued random variables $X_1, \ldots, X_n$,*

$$\mathbb{E} \max_{1\leq i\leq n} |X_i| \leq \psi^{-1}(n\psi(1)) \max_{1\leq i\leq n} \|X_i\|_\psi,$$

where ψ^{-1} is the inverse function of ψ. Moreover if ψ is such that

$$\exists\, c > 0,\ \forall x,\, y \geq 1/2,\ \psi(x)\psi(y) \leqslant \psi(c\, x\, y) \tag{1.1}$$

then

$$\left\| \max_{1\leqslant i\leqslant n} |X_i| \right\|_\psi \leqslant c\ \max\left\{1/2, \psi^{-1}(2n)\right\} \max_{1\leqslant i\leqslant n} \|X_i\|_\psi.$$

where c is the same as in (1.1).

Remark 1.1.4. — *(i) Since for any* $x, y \geq 1/2$, $(e^x - 1)(e^y - 1) \leq e^{x+y} \leq e^{4xy} \leq (e^{8xy} - 1)$, *we get that for any* $\alpha \geq 1$, ψ_α *satisfies the assumption* (1.1) *with* $c = 8^{1/\alpha}$. *Moreover, the function* ψ_α *is such that* $\psi_\alpha^{-1}(n\psi_\alpha(1)) \leq (1+\log(n))^{1/\alpha}$ *and* $\psi_\alpha^{-1}(2n) = (\log(1+2n))^{1/\alpha}$.
(ii) The assumption (1.1) *may be weakened by* $\limsup_{x,y\to\infty} \psi(x)\psi(y)/\psi(cxy) < \infty$.
(iii) By monotony of ψ, *for* $n \geq \psi(1/2)/2$, $\max\{1/2, \psi^{-1}(2n)\} = \psi^{-1}(2n)$.

Proof. — By homogeneity, we can assume that for any $i = 1, \ldots, n$, $\|X_i\|_\psi \leq 1$.

The first inequality is a simple consequence of Jensen inequality. Indeed,

$$\psi(\mathbb{E} \max_{1\leq i\leq n} |X_i|) \leq \mathbb{E}\psi(\max_{1\leq i\leq n} |X_i|) \leq \sum_{i=1}^n \mathbb{E}\psi(|X_i|) \leq n\psi(1).$$

To prove the second assertion, we define $y = \max\{1/2, \psi^{-1}(2n)\}$. For any $i = 1, \ldots, n$, let $x_i = |X_i|/cy$. We observe that if $x_i \geq 1/2$ then we have by (1.1)

$$\psi\left(|X_i|/cy\right) \leq \frac{\psi(|X_i|)}{\psi(y)}.$$

Also note that by monotony of ψ,

$$\psi(\max_{1\leq i\leq n} x_i) \leq \psi(1/2) + \sum_{i=1}^n \psi(x_i) \mathbb{1}_{\{x_i \geq 1/2\}}.$$

Therefore, we have

$$\begin{aligned}\mathbb{E}\psi\left(\max_{1\leqslant i\leqslant n} |X_i|/cy\right) &\leqslant \psi(1/2) + \sum_{i=1}^n \mathbb{E}\psi\left(|X_i|)/cy\right) \mathbb{1}_{\{(|X_i|)/cy) \geq 1/2\}} \\ &\leq \psi(1/2) + \frac{1}{\psi(y)} \sum_{i=1}^n \mathbb{E}\psi(|X_i|) \leq \psi(1/2) + \frac{n\psi(1)}{\psi(y)}.\end{aligned}$$

From the convexity of ψ and the fact that $\psi(0) = 0$, we have $\psi(1/2) \leq \psi(1)/2$. The proof is finished since by definition of y, $\psi(y) \geq 2n$. □

For every $\alpha \geq 1$, there are very precise connections between the ψ_α norm of a random variable, the behavior of its L_p norms, the tail estimates and the Laplace transform. We sum up these connections in the following Theorem.

Theorem 1.1.5. — *Let* X *be a real valued random variable and* $\alpha \geqslant 1$. *The following assertions are equivalent:*
(1) There exists $K_1 > 0$ *such that* $\|X\|_{\psi_\alpha} \leqslant K_1$.
(2) There exists $K_2 > 0$ *such that for every* $p \geq \alpha$,

$$\left(\mathbb{E}|X|^p\right)^{1/p} \leqslant K_2\, p^{1/\alpha}.$$

(3) There exist $K_3, K_3' > 0$ *such that for every* $t \geq K_3'$,

$$\mathbb{P}\left(|X| \geqslant t\right) \leqslant \exp\left(-t^\alpha/K_3^\alpha\right).$$

Moreover, we have

$$K_2 \leq 2eK_1, K_3 \leq eK_2, K_3' \leq e^2 K_2 \text{ and } K_1 \leq 2\max(K_3, K_3').$$

In the case $\alpha > 1$, let β be such that $1/\alpha + 1/\beta = 1$. The preceding assertions are also equivalent to the following:
(4) There exist $K_4, K_4' > 0$ such that for every $\lambda \geq 1/K_4'$,

$$\mathbb{E}\exp\big(\lambda|X|\big) \leqslant \exp\left(\lambda K_4\right)^{\beta}.$$

Moreover, $K_4 \leq K_1$, $K_4' \leq K_1$, $K_3 \leq 2K_4$ and $K_3' \leq 2K_4^{\beta}/(K_4')^{\alpha-1}$.

Proof. — We start by proving that (1) implies (2). By definition of the L_{ψ_α} norm, we have

$$\mathbb{E}\exp\left(\frac{|X|}{K_1}\right)^{\alpha} \leq e.$$

Moreover, for every positive integer q and every $x \geq 0$, $\exp x \geq x^q/q!$. Hence

$$\mathbb{E}|X|^{\alpha q} \leq e\, q! K_1^{\alpha q} \leq e q^q K_1^{\alpha q}.$$

For any $p \geq \alpha$, let q be the positive integer such that $q\alpha \leq p < (q+1)\alpha$ then

$$\begin{aligned}(\mathbb{E}|X|^p)^{1/p} \leq \left(\mathbb{E}|X|^{(q+1)\alpha}\right)^{1/(q+1)\alpha} &\leq e^{1/(q+1)\alpha} K_1 (q+1)^{1/\alpha} \\ &\leq e^{1/p} K_1 \left(\frac{2p}{\alpha}\right)^{1/\alpha} \leq 2eK_1 p^{1/\alpha}\end{aligned}$$

which means that (2) holds true with $K_2 = 2eK_1$.

We now prove that (2) implies (3). We apply Markov inequality and the estimate of (2) to deduce that for every $t > 0$,

$$\mathbb{P}(|X| \geq t) \leq \inf_{p>0} \frac{\mathbb{E}|X|^p}{t^p} \leq \inf_{p\geq\alpha} \left(\frac{K_2}{t}\right)^p p^{p/\alpha} = \inf_{p\geq\alpha} \exp\left(p \log\left(\frac{K_2 p^{1/\alpha}}{t}\right)\right).$$

Choosing $p = (t/eK_2)^\alpha \geq \alpha$, we get that for $t \geq e\, K_2 \alpha^{1/\alpha}$ then $p \geq \alpha$ and we conclude that

$$\mathbb{P}(|X| \geq t) \leq \exp\left(-t^\alpha/(K_2 e)^\alpha\right).$$

Since $\alpha \geq 1$, $\alpha^{1/\alpha} \leq e$ and (3) holds true with $K_3' = e^2\, K_2$ and $K_3 = eK_2$.

To prove that (3) implies (1), assume that (3) holds true and let $c = 2\max(K_3, K_3')$. Then by integration by parts, we get

$$\begin{aligned}\mathbb{E}\exp\left(\frac{|X|}{c}\right)^{\alpha} - 1 &= \int_0^{+\infty} \alpha u^{\alpha-1} e^{u^\alpha} \mathbb{P}(|X| \geq uc) du \\ &\leq \int_0^{K_3'/c} \alpha u^{\alpha-1} e^{u^\alpha} du + \int_{K_3'/c}^{+\infty} \alpha u^{\alpha-1} \exp\left(u^\alpha\left(1 - \frac{c^\alpha}{K_3^\alpha}\right)\right) du \\ &= \exp\left(\frac{K_3'}{c}\right)^{\alpha} - 1 + \frac{1}{\frac{c^\alpha}{K_3^\alpha} - 1} \exp\left(-\left(\frac{c^\alpha}{K_3^\alpha} - 1\right)\left(\frac{K_3'}{c}\right)^{\alpha}\right) \\ &\leq 2\cosh(K_3'/c)^\alpha - 1 \leq 2\cosh(1/2) - 1 \leq e - 1\end{aligned}$$

by definition of c and the fact that $\alpha \geq 1$. This proves that (1) holds true with $K_1 = 2\max(K_3, K_3')$.

We now assume that $\alpha > 1$ and prove that (4) implies (3). We apply Markov inequality and the estimate of (4) to get that for every $t > 0$,

$$\begin{aligned}\mathbb{P}(|X| > t) &\leqslant \inf_{\lambda>0} \exp(-\lambda t)\mathbb{E}\exp\big(\lambda|X|\big)\\ &\leqslant \inf_{\lambda\geq 1/K_4'} \exp\big((\lambda K_4)^\beta - \lambda t\big).\end{aligned}$$

Choosing $\lambda t = 2(\lambda K_4)^\beta$ we get that if $t \geq 2K_4^\beta/(K_4')^{\alpha-1}$, then $\lambda \geq 1/K_4'$ and we conclude that

$$\mathbb{P}(|X| > t) \leq \exp\left(-t^\alpha/(2K_4)^\alpha\right).$$

This proves that (3) holds true with $K_3 = 2K_4$ and $K_3' = 2K_4^\beta/(K_4')^{\alpha-1}$.

It remains to prove that (1) implies (4). We have already observed that the convex conjugate of the function $\phi_\alpha(t) = t^\alpha/\alpha$ is ϕ_β which implies that for any $x, y > 0$,

$$xy \leq \frac{x^\alpha}{\alpha} + \frac{y^\beta}{\beta}.$$

Hence for every $\lambda > 0$, by convexity of the exponential

$$\exp(\lambda|X|) \leq \frac{1}{\alpha}\exp\left(\frac{|X|}{K_1}\right)^\alpha + \frac{1}{\beta}\exp\left(\lambda K_1\right)^\beta$$

and taking the expectation, we get by definition of the L_{ψ_α} norm that

$$\mathbb{E}\exp(\lambda|X|) \leq \frac{e}{\alpha} + \frac{1}{\beta}\exp\left(\lambda K_1\right)^\beta.$$

We conclude that if $\lambda \geq 1/K_1$ then

$$\mathbb{E}\exp(\lambda|X|) \leq \exp\left(\lambda K_1\right)^\beta$$

which proves that (4) holds true with $K_4 = K_1$ and $K_4' = K_1$. □

A simple corollary of Theorem 1.1.5 is the following connection between the L_p norms of a random variable and its ψ_α norm.

Corollary 1.1.6. — *For every $\alpha \geq 1$, for every real random variable X,*

$$\frac{1}{2e^2}\|X\|_{\psi_\alpha} \leqslant \sup_{p\geqslant\alpha}\frac{(\mathbb{E}|X|^p)^{1/p}}{p^{1/\alpha}} \leqslant 2e\|X\|_{\psi_\alpha}.$$

Moreover for any $\alpha \geq 1$, $L_\infty \subset L_{\psi_\alpha}$ and $\|X\|_{\psi_\alpha} \leq \|X\|_{L_\infty}$.

Proof. — This follows from the implications (1) ⇒ (2) ⇒ (3) ⇒ (1) in Theorem 1.1.5 and from the computations of the constants K_2, K_3, K_3' and K_1. The moreover part is a direct application of the definition of the ψ_α norm. □

We conclude this part with a kind of Hölder inequality for ψ_α random variables.

Proposition 1.1.7. — *Given p, $q \in [1, +\infty]$ be such that $1/p + 1/q = 1$ and two random variables $X \in L_{\psi_p}$, $Y \in L_{\psi_q}$, we have*

$$\|XY\|_{\psi_1} \leq \|X\|_{\psi_p}\|Y\|_{\psi_q}. \tag{1.2}$$

Moreover, if $1 \leq \alpha \leq \beta$ *then for every random variable* X

$$\|X\|_{\psi_1} \leq \|X\|_{\psi_\alpha} \leq \|X\|_{\psi_\beta} .$$

Proof. — By homogeneity, we assume that $\|X\|_{\psi_p} = \|Y\|_{\psi_q} = 1$. Since p and q are conjugate, we know by Young inequality that for every $x, y \in \mathbb{R}$, $|xy| \leq \frac{|x|^p}{p} + \frac{|x|^q}{q}$. By convexity of the exponential, we deduce that

$$\mathbb{E}\exp(|XY|) \leq \frac{1}{p}\mathbb{E}\exp|X|^p + \frac{1}{q}\mathbb{E}\exp|X|^q \leq e$$

which proves that $\|XY\|_{\psi_1} \leq 1$.

The moreover part is a consequence of this result. Indeed, by definition of the ψ_q-norm, the random variable $Y = 1$ satisifes $\|Y\|_{\psi_q} = 1$. Hence applying (1.2) with $p = \alpha$ and q being the conjugate of p, we get that for every $\alpha \geq 1$, $\|X\|_{\psi_1} \leq \|X\|_{\psi_\alpha}$. We also observe that for any $\beta \geq \alpha$, if $\delta \geq 1$ is such that $\beta = \alpha\delta$ then we have

$$\|X\|_{\psi_\alpha}^\alpha = \||X|^\alpha\|_{\psi_1} \leq \||X|^\alpha\|_{\psi_\delta} = \|X\|_{\psi_{\alpha\delta}}^\alpha$$

which proves that $\|X\|_{\psi_\alpha} \leq \|X\|_{\psi_\beta}$. □

1.2. Linear combination of centered Psi-alpha random variables

In this part we will focus on the case of centered ψ_α random variables when $\alpha \geq 1$. We will present several results concerning the linear combination of such random variable. The cases $\alpha = 2$ and $\alpha \neq 2$ are different. We will start by looking at the case $\alpha = 2$. Even if we will prove a sharp estimate for their linear combination, we will also consider the simple and well known example of linear combination of Rademacher. This example will show the limitation of the classification with the ψ_α norm of certain random variables. However in the case $\alpha \neq 2$, different regime will appear in the tail estimates of such sum. This will be of importance in several chapters of this book.

The sub-Gaussian case.— We start by taking a look to sums of ψ_2 random variables. The following proposition can be seen as a generalization of the classical Hoeffding inequality [**Hoe63**] since $L_\infty \subset L_{\psi_2}$.

Theorem 1.2.1. — *Let* $X_1, \ldots, X_n$ *be independent real valued random variables such that for any* $i = 1, \ldots, n$, $\mathbb{E}X_i = 0$. *Then*

$$\left\| \sum_{i=1}^n X_i \right\|_{\psi_2} \leqslant c \left(\sum_{i=1}^n \|X_i\|_{\psi_2}^2 \right)^{1/2}$$

where $c \leq 16$.

Before proving the theorem, we start with the following lemma concerning the Laplace transform of a ψ_2 random variable which is centered. The fact that $\mathbb{E}X = 0$ is crucial to improve the assertion (4) of Theorem 1.1.5.

Lemma 1.2.2. — *Let X be a ψ_2 centered random variable. Then, for any $\lambda > 0$, the Laplace transform of X satisfies*

$$\mathbb{E}\exp(\lambda X) \leqslant \exp\left(e\lambda^2 \|X\|_{\psi_2}^2\right).$$

Proof. — By homogeneity of the statement, we can assume that $\|X\|_{\psi_2} = 1$. By the definition of the L_{ψ_2} norm, we know that

$$\mathbb{E}\exp\left(X^2\right) \leq e \quad \text{and} \quad \text{for any integer } k, \mathbb{E}X^{2k} \leq ek!$$

Let Y be an independent copy of X. By convexity of the exponential and Jensen inequality, since $\mathbb{E}Y = 0$ we have

$$\mathbb{E}\exp(\lambda X) \leq \mathbb{E}_X\mathbb{E}_Y \exp\lambda(X - Y).$$

Moreover, the random variable $X - Y$ is symmetric hence

$$\mathbb{E}_X\mathbb{E}_Y \exp\lambda(X - Y) = 1 + \frac{\lambda^2}{2}\mathbb{E}_X\mathbb{E}_Y(X - Y)^2 + \sum_{k=2}^{+\infty} \frac{\lambda^{2k}}{(2k)!}\mathbb{E}_X\mathbb{E}_Y(X - Y)^{2k}.$$

Obviously, $\mathbb{E}_X\mathbb{E}_Y(X - Y)^2 = 2\mathbb{E}X^2 \leq 2e$ and $\mathbb{E}_X\mathbb{E}_Y(X - Y)^{2k} \leq 4^k\mathbb{E}X^{2k} \leq e4^k k!$. Since the sequence $v_k = (2k)!/3^k(k!)^2$ is nondecreasing, we know that for $k \geq 2$ $v_k \geq v_2 = 6/3^2$ so that

$$\forall k \geq 2,\ \frac{1}{(2k)!}\mathbb{E}_X\mathbb{E}_Y(X - Y)^{2k} \leq \frac{e4^k k!}{(2k)!} \leq \frac{e3^2}{6}\left(\frac{4}{3}\right)^k \frac{1}{k!} \leq \left(\frac{4\sqrt{e}}{\sqrt{6}}\right)^k \frac{1}{k!} \leq \frac{e^k}{k!}.$$

It follows that for every $\lambda > 0$, $\mathbb{E}\exp(\lambda X) \leq 1 + e\lambda^2 + \sum_{k=2}^{+\infty} \frac{(e\lambda^2)^k}{k!} = \exp(e\lambda^2)$. □

Proof of Theorem 1.2.1. — It is enough to get an upper bound of the Laplace transform of the random variable $\left|\sum_{i=1}^n X_i\right|$. Let $Z = \sum_{i=1}^n X_i$ then by independence of the X_i's, we get from Lemma 1.2.2 that for every $\lambda > 0$,

$$\mathbb{E}\exp(\lambda Z) = \prod_{i=1}^n \mathbb{E}\exp(\lambda X_i) \leqslant \exp\left(e\lambda^2 \sum_{i=1}^n \|X_i\|_{\psi_2}^2\right).$$

For the same reason, $\mathbb{E}\exp(-\lambda Z) \leqslant \exp\left(e\lambda^2 \sum_{i=1}^n \|X_i\|_{\psi_2}^2\right)$. Thus,

$$\mathbb{E}\exp(\lambda|Z|) \leqslant 2\exp\left(3\lambda^2 \sum_{i=1}^n \|X_i\|_{\psi_2}^2\right).$$

We conclude that for any $\lambda \geq 1/\left(\sum_{i=1}^n \|X_i\|_{\psi_2}^2\right)^{1/2}$,

$$\mathbb{E}\exp(\lambda|Z|) \leqslant \exp\left(4\lambda^2 \sum_{i=1}^n \|X_i\|_{\psi_2}^2\right)$$

and using the implication ((4) $\Rightarrow$ (1)) in Theorem 1.1.5 with $\alpha = \beta = 2$ (with the estimates of the constants), we get that $\|Z\|_{\psi_2} \leq c(\sum_{i=1}^n \|X_i\|_{\psi_2}^2)^{1/2}$ with $c \leq 16$. □

Now, we take a particular look to Rademacher processes. Indeed, Rademacher variables are the simplest example of bounded (hence ψ_2) random variables. We denote by $\varepsilon_1, \ldots, \varepsilon_n$ independent random variables taking values ± 1 with probability $1/2$. By definition of L_{ψ_2}, for any $(a_1, \ldots, a_n) \in \mathbb{R}^n$, the random variable $a_i \varepsilon_i$ is centered and has ψ_2 norm equal to $|a_i|$. We apply Theorem 1.2.1 to deduce that

$$\left\| \sum_{i=1}^n a_i \varepsilon_i \right\|_{\psi_2} \leq c|a|_2 = c \left(\mathbb{E} \left| \sum_{i=1}^n a_i \varepsilon_i \right|^2 \right)^{1/2}.$$

Therefore we get from Theorem 1.1.5 that for any $p \geq 2$,

$$\left(\mathbb{E} \left| \sum_{i=1}^n a_i \varepsilon_i \right|^2 \right)^{1/2} \leq \left(\mathbb{E} \left| \sum_{i=1}^n a_i \varepsilon_i \right|^p \right)^{1/p} \leq 2c\sqrt{p} \left(\mathbb{E} \left| \sum_{i=1}^n a_i \varepsilon_i \right|^2 \right)^{1/2}. \tag{1.3}$$

This is the Khinchine's inequality. It is not difficult to extend it to the case $0 < q \leq 2$ by using Hölder inequality: for any random variable Z, if $0 < q \leq 2$ and $\lambda = q/(4-q)$ then

$$\left(\mathbb{E}|Z|^2 \right)^{1/2} \leq \left(\mathbb{E}|Z|^q \right)^{\lambda/q} \left(\mathbb{E}|Z|^4 \right)^{(1-\lambda)/4}.$$

Let $Z = \sum_{i=1}^n a_i \varepsilon_i$, we apply (1.3) to the case $p = 4$ to deduce that for any $0 < q \leq 2$,

$$\left(\mathbb{E} \left| \sum_{i=1}^n a_i \varepsilon_i \right|^q \right)^{1/q} \leq \left(\mathbb{E} \left| \sum_{i=1}^n a_i \varepsilon_i \right|^2 \right)^{1/2} \leq (4c)^{2(2-q)/q} \left(\mathbb{E} \left| \sum_{i=1}^n a_i \varepsilon_i \right|^q \right)^{1/q}.$$

Since for any $x \geq 0$, $e^{x^2} - 1 \geq x^2$, we also observe that

$$(e-1) \left\| \sum_{i=1}^n a_i \varepsilon_i \right\|_{\psi_2} \geq \left(\mathbb{E} \left| \sum_{i=1}^n a_i \varepsilon_i \right|^2 \right)^{1/2}.$$

However the precise knowledge of the ψ_2 norm of the random variable $\sum_{i=1}^n a_i \varepsilon_i$ is not enough to understand correctly the behavior of its L_p norms and consequently of its tail estimate. Indeed, a more precise statement holds true.

Theorem 1.2.3. — *Let $p \geqslant 2$, let $a_1, \ldots, a_n$ be real numbers and let $\varepsilon_1, \ldots, \varepsilon_n$ be independent Rademacher variables. We have*

$$\left(\mathbb{E} \left| \sum_{i=1}^n a_i \varepsilon_i \right|^p \right)^{1/p} \leq \sum_{i \leqslant p} a_i^* + 2c\sqrt{p} \left(\sum_{i > p} a_i^{*2} \right)^{1/2},$$

where $(a_1^*, \ldots, a_n^*)$ is the non-increasing rearrangement of $(|a_1|, \ldots, |a_n|)$. Moreover, the estimate is sharp, up to a multiplicative factor.

Remark 1.2.4. — *We will not present the proof of the lower bound even if it is the difficult part of the Theorem. It is beyond the scope of this chapter.*

Proof. — Since Rademacher random variables are bounded by 1, we also have the trivial upper bound:

$$\left(\mathbb{E}\left|\sum_{i=1}^n a_i\varepsilon_i\right|^p\right)^{1/p} \leq \sum_{i=1}^n |a_i|. \tag{1.4}$$

By independence and by symmetry of the Rademacher we have

$$\left(\mathbb{E}\left|\sum_{i=1}^n a_i\varepsilon_i\right|^p\right)^{1/p} = \left(\mathbb{E}\left|\sum_{i=1}^n a_i^*\varepsilon_i\right|^p\right)^{1/p}.$$

Splitting the sum into two parts, we get that

$$\left(\mathbb{E}\left|\sum_{i=1}^n a_i^*\varepsilon_i\right|^p\right)^{1/p} \leq \left(\mathbb{E}\left|\sum_{i=1}^p a_i^*\varepsilon_i\right|^p\right)^{1/p} + \left(\mathbb{E}\left|\sum_{i>p} a_i^*\varepsilon_i\right|^p\right)^{1/p}.$$

We conclude by applying (1.4) to the first term and (1.3) to the second one. □

This provides a good example of one of the main drawback of this classification with ψ_α-norm. Indeed, being a ψ_α random variable allows only one type of tail estimate. In the sense that, if $Z \in L_{\psi_\alpha}$ then the tail decay of Z behaves like $\exp(-Kt^\alpha)$ for t large enough, but this result is sometimes too weak for a precise study of the L_p norm of Z.

Bernstein's type inequalities, the case $\alpha = 1$. — We start this section with the well known Bernstein's inequalities which hold for an empirical mean of bounded random variables.

Theorem 1.2.5. — *Let $X_1, \ldots, X_n$ be n independent random variables and M be a positive number such that for any $i = 1, \ldots, n$, $\mathbb{E}X_i = 0$ and $|X_i| \leqslant M$ almost surely. Set $\sigma^2 = n^{-1}\sum_{i=1}^n \mathbb{E}X_i^2$. For any $t > 0$, we have*

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^n X_i \geqslant t\right) \leqslant \exp\left(-\frac{n\sigma^2}{M^2} h\left(\frac{Mt}{\sigma^2}\right)\right),$$

where $h(u) = (1+u)\log(1+u) - u$ for all $u > 0$.

Proof. — Let $t > 0$, by Markov inequality and by independence we have

$$\begin{aligned}\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^n X_i \geqslant t\right) &\leqslant \inf_{\lambda>0} \exp(-\lambda t)\mathbb{E}\exp\left(\frac{\lambda}{n}\sum_{i=1}^n X_i\right)\\ &= \inf_{\lambda>0}\exp(-\lambda t)\prod_{i=1}^n \mathbb{E}\exp\left(\frac{\lambda X_i}{n}\right).\end{aligned} \tag{1.5}$$

Since for any $i = 1, \ldots, n$, $\mathbb{E}X_i = 0$ and $|X_i| \leq M$,

$$\mathbb{E}\exp\left(\frac{\lambda X_i}{n}\right) = 1 + \sum_{k\geqslant 2} \frac{\lambda^k \mathbb{E}X_i^k}{n^k k!} \leq 1 + \mathbb{E}X_i^2 \sum_{k\geqslant 2} \frac{\lambda^k M^{k-2}}{n^k k!}$$
$$= 1 + \frac{\mathbb{E}X_i^2}{M^2}\left(\exp\left(\frac{\lambda M}{n}\right) - \left(\frac{\lambda M}{n}\right) - 1\right).$$

Using the fact that $1 + u \leqslant \exp(u)$ for all $u \in \mathbb{R}$, we get that

$$\prod_{i=1}^n \mathbb{E}\exp\left(\frac{\lambda X_i}{n}\right) \leq \exp\left(\frac{\sum_{i=1}^n \mathbb{E}X_i^2}{M^2}\left(\exp\left(\frac{\lambda M}{n}\right) - \left(\frac{\lambda M}{n}\right) - 1\right)\right).$$

By definition of σ and by (1.5), we conclude that for any $t > 0$,

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^n X_i \geqslant t\right) \leq \inf_{\lambda>0} \exp\left(\frac{n\sigma^2}{M^2}\left(\exp\left(\frac{\lambda M}{n}\right) - \left(\frac{\lambda M}{n}\right) - 1\right) - \lambda t\right).$$

The claim follows by choosing λ such that $(1 + tM/\sigma^2) = \exp(\lambda M/n)$. □

Using Taylor expansion, it is not difficult to see that for every $u > 0$ we have $h(u) \geqslant u^2/(2 + 2u/3)$. This proves that if $u \geq 1$, $h(u) \geq 3u/8$ and if $u \leq 1$, $h(u) \geq 3u^2/8$. Therefore the classical Bernstein's inequality for bounded random variables is an immediate corollary of this result.

Theorem 1.2.6. — *Let $X_1, \ldots, X_n$ be n independent random variables such that for all $i = 1, \ldots, n$, $\mathbb{E}X_i = 0$ and $|X_i| \leqslant M$ almost surely. Then, for every $t > 0$,*

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^n X_i \geqslant t\right) \leqslant \exp\left(-\frac{3n}{8}\min\left(\frac{t^2}{\sigma^2}, \frac{t}{M}\right)\right),$$

where $\sigma^2 = \dfrac{1}{n}\displaystyle\sum_{i=1}^n \mathbb{E}X_i^2$.

From Bernstein's inequality, we can deduce that the tail behavior of a sum of centered, bounded random variables has two regimes. There is a sub-exponential regime with respect to M for large values of t ($t \geqslant \sigma^2/M$) and a sub-Gaussian behavior with respect to σ^2 for small values of t ($t \leqslant \sigma^2/M$). Moreover, this inequality is always stronger than the tail estimate that we could deduce from Theorem 1.2.1 (which is only sub-Gaussian with respect to M^2).

Now, we turn to the important case of sum of sub-exponential centered random variables.

Theorem 1.2.7. — *Let $X_1, \ldots, X_n$ be n independent centered ψ_1 random variables. Then, for every $t > 0$,*

$$\mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^n X_i\right| \geqslant t\right) \leqslant 2\exp\left(-c\,n\min\left(\frac{t^2}{\sigma_1^2}, \frac{t}{M_1}\right)\right),$$

where $M_1 = \max_{1\leqslant i\leqslant n} \|X_i\|_{\psi_1}$, $\sigma_1^2 = \dfrac{1}{n}\displaystyle\sum_{i=1}^n \|X_i\|_{\psi_1}^2$ *and* c *is a number that can be taken equal to* $(e-1)/2e(2e-1)$.

Proof. — Since for every $x \geq 0$ and any positive natural integer k, $e^x \geq x^k/k!$, we get by definition of the ψ_1 norm that for any integer $k \geq 1$ and any $i = 1, \ldots, n$,

$$\mathbb{E}|X_i|^k \leqslant ek!\,\|X_i\|_{\psi_1}^k .$$

Moreover $\mathbb{E}X_i = 0$ for any $i = 1, \ldots, n$ and using Taylor expansion of the exponential, we deduce that for every $\lambda > 0$ such that $\lambda \|X_i\|_{\psi_1} \leq \lambda M_1 < n$,

$$\mathbb{E}\exp\left(\frac{\lambda}{n}X_i\right) \leqslant 1 + \sum_{k\geqslant 2}\frac{\lambda^k \mathbb{E}|X_i|^k}{n^k k!} \leqslant 1 + \frac{e\lambda^2 \|X_i\|_{\psi_1}^2}{n^2\left(1-\frac{\lambda}{n}\|X_i\|_{\psi_1}\right)} \leq 1 + \frac{e\lambda^2\|X_i\|_{\psi_1}^2}{n^2\left(1-\frac{\lambda M_1}{n}\right)}.$$

Let $Z = n^{-1}\sum_{i=1}^n X_i$. Since for any real number x, $1+x \leq e^x$, we get by independence of the X_i's that for every $\lambda > 0$ such that $\lambda M_1 < n$

$$\mathbb{E}\exp(\lambda Z) \leq \exp\left(\frac{e\lambda^2}{n^2\left(1-\frac{\lambda M_1}{n}\right)}\sum_{i=1}^n \|X_i\|_{\psi_1}^2\right) = \exp\left(\frac{e\lambda^2\sigma_1^2}{n-\lambda M_1}\right).$$

We conclude by Markov inequality that for every $t > 0$,

$$\mathbb{P}(Z \geq t) \leq \inf_{0<\lambda<n/M_1} \exp\left(-\lambda\, t + \frac{e\lambda^2\sigma_1^2}{n-\lambda M_1}\right).$$

We consider two cases. If $t \leq \sigma_1^2/M_1$, we choose $\lambda = nt/2e\sigma_1^2 \leq n/2eM_1$. A simple computation gives that

$$\mathbb{P}(Z \geq t) \leq \exp\left(-\frac{e-1}{2e(2e-1)}\frac{n\,t^2}{\sigma_1^2}\right).$$

If $t > \sigma_1^2/M_1$, we choose $\lambda = n/2eM_1$. This time, we get

$$\mathbb{P}(Z \geq t) \leq \exp\left(-\frac{e-1}{2e(2e-1)}\frac{n\,t}{M_1}\right).$$

We can do the same argument for $-Z$ and this concludes the proof of the announced result. □

The ψ_α case: $\alpha > 1$. — In this part we will focus on the case $\alpha \neq 2$ and $\alpha > 1$. Our goal is to explain the behavior of the tail estimate of a sum of independent ψ_α centered random variables. As in Bernstein inequalities, there will be two different regimes depending on the level of deviation t.

Theorem 1.2.8. — *Let* $\alpha > 1$ *and* β *be such that* $\alpha^{-1} + \beta^{-1} = 1$. *Given* $X_1, \ldots, X_n$ *be independent mean zero* ψ_α *real-valued random variables, set*

$$A_1 = \left(\sum_{i=1}^n \|X_i\|_{\psi_1}^2\right)^{1/2} \quad \text{and} \quad B_\alpha = \left(\sum_{i=1}^n \|X_i\|_{\psi_\alpha}^\beta\right)^{1/\beta}.$$

Then, for every $t > 0$,

$$\mathbb{P}\left(\left|\sum_{i=1}^n X_i\right| \geqslant t\right) \leq \begin{cases} 2\exp\left(-c_\alpha \min\left(\frac{t^2}{A_1^2}, \frac{t^\alpha}{B_\alpha^\alpha}\right)\right) & \text{if } \alpha < 2, \\ 2\exp\left(-c_\alpha \max\left(\frac{t^2}{B_\alpha^2}, \frac{t^\alpha}{B_\alpha^\alpha}\right)\right) & \text{if } \alpha > 2 \end{cases}$$

where c_α is a number depending only on α.

Remark 1.2.9. — *We can stated the result with the same normalization as in Bernstein inequalities. Let $\sigma_1^2 = \frac{1}{n}\sum_{i=1}^n \|X_i\|_{\psi_1}^2$ and $M_\alpha^\beta = \frac{1}{n}\sum_{i=1}^n \|X_i\|_{\psi_\alpha}^\beta$, then we have*

$$\mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^n X_i\right| \geqslant t\right) \leq \begin{cases} 2\exp\left(-c_\alpha n \min\left(\frac{t^2}{\sigma_1^2}, \frac{t^\alpha}{M_\alpha^\alpha}\right)\right) & \text{if } \alpha < 2, \\ 2\exp\left(-c_\alpha n \max\left(\frac{t^2}{M_\alpha^2}, \frac{t^\alpha}{M_\alpha^\alpha}\right)\right) & \text{if } \alpha > 2. \end{cases}$$

Before proving Theorem 1.2.8, we start by exhibiting a sub-Gaussian behavior of the Laplace transform of any ψ_1 centered random variable.

Lemma 1.2.10. — *Given X a ψ_1 mean-zero random variable, for every λ such that $0 \leq \lambda \leqslant \left(2\|X\|_{\psi_1}\right)^{-1}$ we have*

$$\mathbb{E}\exp(\lambda X) \leqslant \exp\left(4(e-1)\lambda^2\|X\|_{\psi_1}^2\right).$$

Proof. — Let X' be an independent copie of X and denote $Y = X - X'$. Since X is centered, by Jensen inequality,

$$\mathbb{E}\exp\lambda X = \mathbb{E}\exp(\lambda(X - \mathbb{E}X')) \leqslant \mathbb{E}\exp\lambda(X - X') = \mathbb{E}\exp\lambda Y.$$

The random variable Y is symmetric thus, for every λ, $\mathbb{E}\exp\lambda Y = \mathbb{E}\cosh\lambda Y$ and using the Taylor expansion,

$$\mathbb{E}\exp\lambda Y = 1 + \sum_{k\geq 1}\frac{\lambda^{2k}}{(2k)!}\mathbb{E}Y^{2k} = 1 + \lambda^2\sum_{k\geq 1}\frac{\lambda^{2(k-1)}}{(2k)!}\mathbb{E}Y^{2k}.$$

By definition of Y, for every $k \geq 1$, $\mathbb{E}Y^{2k} \leq 2^{2k}\mathbb{E}X^{2k}$. Hence, for every $0 \leq \lambda \leqslant \left(2\|X\|_{\psi_1}\right)^{-1}$, we get that

$$\mathbb{E}\exp\lambda Y \leq 1 + 4\lambda^2\|X\|_{\psi_1}^2\sum_{k\geq 1}\frac{\mathbb{E}X^{2k}}{(2k)!\|X\|_{\psi_1}^{2k}} \leq 1 + 4\lambda^2\|X\|_{\psi_1}^2\left(\mathbb{E}\exp\left(\frac{|X|}{\|X\|_{\psi_1}}\right) - 1\right).$$

By definition of the ψ_1 norm, we conclude that for every $0 \leq \lambda \leqslant \left(2\|X\|_{\psi_1}\right)^{-1}$

$$\mathbb{E}\exp\lambda X \leq 1 + 4(e-1)\lambda^2\|X\|_{\psi_1}^2 \leq \exp\left(4(e-1)\lambda^2\|X\|_{\psi_1}^2\right).$$

$\square$

Proof of Theorem 1.2.8. — We start with the case $1 < \alpha < 2$.
For every $1 = 1, \dots, n$, X_i is a ψ_α random variable with $\alpha > 1$. Then it is a ψ_1 random variable (see Proposition 1.1.7) and from Lemma 1.2.10, we get that

$$\forall\, 0 \leq \lambda \leq 1/2 \left\| X_i \right\|_{\psi_1},\ \mathbb{E} \exp \lambda X_i \leq \exp\left(4(e-1)\lambda^2 \left\| X_i \right\|_{\psi_1}^2 \right).$$

Moreover, from Theorem 1.1.5, we get also that

$$\forall\, \lambda \geq 1/\|X_i\|_{\psi_\alpha},\ \mathbb{E} \exp \lambda X_i \leq \exp\left(\lambda \left\| X_i \right\|_{\psi_\alpha} \right)^{\beta}.$$

Since $1 < \alpha < 2$ then $\beta > 2$ and it is easy to conclude that for $c = 4(e-1)$ we have

$$\forall\, \lambda > 0,\ \mathbb{E} \exp \lambda X_i \leq \exp\left(c \left(\lambda^2 \|X_i\|_{\psi_1}^2 + \lambda^\beta \|X_i\|_{\psi_\alpha}^\beta \right) \right). \tag{1.6}$$

Indeed when $\|X_i\|_{\psi_\alpha} > 2\|X_i\|_{\psi_1}$, we just have to glue the two estimates. Otherwise, we have $\|X_i\|_{\psi_\alpha} \leq 2\|X_i\|_{\psi_1}$ and for every $\lambda \in \left(1/2 \left\| X_i \right\|_{\psi_1}, 1/\|X_i\|_{\psi_\alpha} \right)$, we get by Hölder inequality,

$$\mathbb{E} \exp \lambda X_i \leq \left(\mathbb{E} \exp\left(\frac{X_i}{\|X_i\|_{\psi_\alpha}} \right) \right)^{\lambda \|X_i\|_{\psi_\alpha}} \leq \exp\left(\lambda \|X\|_{\psi_\alpha} \right) \leq \exp\left(\lambda^2 4 \|X_i\|_{\psi_1}^2 \right).$$

Let $Z = \sum_{i=1}^n X_i$, we deduce from (1.6) that for every $\lambda > 0$,

$$\mathbb{E} \exp \lambda Z \leq \exp\left(c \left(A_1^2 \lambda^2 + B_\alpha^\beta \lambda^\beta \right) \right).$$

From Markov inequality, we have

$$\mathbb{P}(Z \geq t) \leq \inf_{\lambda > 0} e^{-\lambda t} \mathbb{E} \exp \lambda Z \leq \inf_{\lambda > 0} \left(c \left(A_1^2 \lambda^2 + B_\alpha^\beta \lambda^\beta \right) - \lambda t \right). \tag{1.7}$$

If $(t/A_1)^2 \geq (t/B_\alpha)^\alpha$ then we have $t^{2-\alpha} \geq A_1^2 / B_\alpha^\alpha$ and we choose $\lambda = \frac{t^{\alpha-1}}{4cB_\alpha^\alpha}$. Therefore,

$$\lambda t = \frac{t^\alpha}{4cB_\alpha^\alpha}, \quad B_\alpha^\beta \lambda^\beta = \frac{t^\alpha}{(4c)^\beta B_\alpha^\alpha} \leq \frac{t^\alpha}{(4c)^2 B_\alpha^\alpha}, \quad \text{and}$$
$$A_1^2 \lambda^2 = \frac{t^\alpha}{(4c)^2 B_\alpha^\alpha} \frac{t^{\alpha-2} A_1^2}{B_\alpha^\alpha} \leq \frac{t^\alpha}{(4c)^2 B_\alpha^\alpha}.$$

We conclude from (1.7) that

$$\mathbb{P}(Z \geq t) \leq \exp\left(-\frac{1}{8c} \frac{t^\alpha}{B_\alpha^\alpha} \right).$$

If $(t/A_1)^2 \leq (t/B_\alpha)^\alpha$ then we have $t^{2-\alpha} \leq A_1^2/B_\alpha^\alpha$ and since $(2-\alpha)\beta/\alpha = (\beta - 2)$ we also have $t^{\beta-2} \leq B_\alpha^\beta / A_1^{2(\beta-1)}$. We choose $\lambda = \frac{t}{4cA_1^2}$ therefore,

$$\lambda t = \frac{t^2}{4cA_1^2}, \quad A_1^2 \lambda^2 = \frac{t^2}{(4c)^2 A_1^2} \quad \text{and} \quad B_\alpha^\beta \lambda^\beta = \frac{t^2}{(4c)^\beta A_1^2} \frac{t^{\beta-2} B_\alpha^\beta}{A_1^{2(\beta-1)}} \leq \frac{t^2}{(4c)^2 A_1^2}.$$

We conclude from (1.7) that

$$\mathbb{P}(Z \geq t) \leq \exp\left(-\frac{1}{8c} \frac{t^2}{A_1^2} \right).$$

The proof is complete with $c_\alpha = 1/4c = 1/16(e-1)$.

In the case $\alpha > 2$, we have $1 < \beta < 2$ and the estimate (1.6) for the Laplace transform of each random variable X_i has to be replaced by

$$\forall\,\lambda > 0,\ \mathbb{E}\exp\lambda X_i \le \exp\left(c\lambda^2\|X_i\|_{\psi_\alpha}^2\right) \text{ and } \mathbb{E}\exp\lambda X_i \le \exp\left(c\lambda^\beta\|X_i\|_{\psi_\alpha}^\beta\right).$$

Indeed, when $\lambda\,\|X\|_{\psi_1} \le 1/2$ then $(\lambda\,\|X\|_{\psi_1})^2 \le (\lambda\,\|X\|_{\psi_1})^\beta \le (\lambda\,\|X\|_{\psi_\alpha})^\beta$ and when $\lambda\,\|X\|_{\psi_\alpha} \ge 1$ then $(\lambda\,\|X\|_{\psi_\alpha})^\beta \le (\lambda\,\|X\|_{\psi_\alpha})^2$. Therefore both inequalities hold true. We conclude as before that for $Z = \sum_{i=1}^n X_i$, for every $\lambda > 0$,

$$\mathbb{E}\exp\lambda Z \le \exp\left(c\min\left(B_\alpha^2\lambda^2, B_\alpha^\beta\lambda^\beta\right)\right).$$

The rest is identical to the preceding proof. □

1.3. A geometric application: the Johnson-Lindenstrauss lemma

The Johnson-Lindenstrauss lemma [**JL84**] is a result concerning low-distortion embeddings of points from a high-dimensional Euclidean space into low-dimensional Euclidean space. The lemma states that a finite number of points in a high-dimensional space can be embedded into a space of much lower dimension (which depends of the cardinality of the set) in such a way that distances between the points are nearly preserved. The map used for the embedding is a linear map and can even be taken to be an orthogonal projection. We present here an approach using random Gaussian matrices.

Let $G_1, \ldots, G_k$ be k independent Gaussian vectors in $\mathbb{R}^n$ distributed according to the normal law $\mathcal{N}(0, \mathrm{I}d)$. Let $\Gamma : \mathbb{R}^n \to \mathbb{R}^k$ be the random operator defined for every $x \in \mathbb{R}^n$ by

$$\Gamma x = \begin{pmatrix} \langle G_1, x\rangle \\ \vdots \\ \langle G_k, x\rangle \end{pmatrix} \in \mathbb{R}^k. \tag{1.8}$$

We will prove that with high probability, this Gaussian random matrix satisfies the desired property in the Johnson-Lindenstrauss lemma.

Lemma 1.3.1. — *There exists a numerical constant C such that, given $0 < \varepsilon < 1$, a set T of N distinct points in $\mathbb{R}^n$ and an integer $k > k_0 = C\log(N)/\varepsilon^2$ then there exists a linear operator $A : \mathbb{R}^n \to \mathbb{R}^k$ such that for every x, $y \in T$,*

$$\sqrt{1-\varepsilon}\ |x-y|_2 \le |A(x-y)|_2 \le \sqrt{1+\varepsilon}\ |x-y|_2.$$

Proof. — Let Γ be defined by (1.8). For any vector $z \in \mathbb{R}^n$ and every $i = 1, \ldots, k$, we have $\mathbb{E}\langle G_i, z\rangle^2 = |z|_2^2$. Therefore, for every x, $y \in T$,

$$\left|\frac{\Gamma(x-y)}{\sqrt{k}}\right|_2^2 - |x-y|_2^2 = \frac{1}{k}\sum_{i=1}^k \langle G_i, x-y\rangle^2 - \mathbb{E}\langle G_i, x-y\rangle^2.$$

For every $i = 1, \ldots, k$, we define the random variable X_i by $X_i = \langle G_i, x-y\rangle^2 - \mathbb{E}\langle G_i, x-y\rangle^2$. It is a centered random variable. Since $e^u \geq 1+u$, we know that $\mathbb{E}\langle G_i, x-y\rangle^2 \leq (e-1)\left\|\langle G_i, x-y\rangle^2\right\|_{\psi_1}$. Hence by definition of the ψ_2 norm,

$$\|X_i\|_{\psi_1} \leq 2(e-1)\left\|\langle G_i, x-y\rangle^2\right\|_{\psi_1} = 2(e-1)\left\|\langle G_i, x-y\rangle\right\|_{\psi_2}^2. \tag{1.9}$$

By definition of the Gaussian law, $\langle G_i, x-y\rangle$ is distributed like $|x-y|_2\, g$ where g is a standard real Gaussian variable. It is not difficult to check that with our definition of the ψ_2 norm, $\|g\|_{\psi_2}^2 = 2e^2/(e^2-1)$. We will call c_0^2 this number and $c_1^2 = 2(e-1)c_0^2$. We conclude that $\|\langle G_i, x-y\rangle\|_{\psi_2}^2 = c_0^2\,|x-y|_2^2$ and that $\|X_i\|_{\psi_1} \leq c_1^2\,|x-y|_2^2$. We apply Theorem 1.2.7. In this case, $M_1 = \sigma_1 = c_1^2\,|x-y|_2^2$ and we get that for $t = \varepsilon|x-y|_2^2$ with $0 < \varepsilon < 1$,

$$\mathbb{P}\left(\left|\frac{1}{k}\sum_{i=1}^{k}\langle G_i, x-y\rangle^2 - \mathbb{E}\langle G_i, x-y\rangle^2\right| > \varepsilon|x-y|_2^2\right) \leq 2\exp(-c'\,k\,\varepsilon^2)$$

since $t \leq |x-y|_2^2 \leq c_1^2\,|x-y|_2^2 \leq \sigma_1^2/M_1$. The constant c' is defined by $c' = c/c_1^4$ where c comes from Theorem 1.2.7. Since the cardinality of the set $\{(x,y) : x \in T, y \in T\}$ is less than N^2, we get by the union bound that

$$\mathbb{P}\left(\exists\, x,\, y \in T : \left|\left|\frac{\Gamma(x-y)}{\sqrt{k}}\right|_2^2 - |x-y|_2^2\right| > \varepsilon|x-y|_2^2\right) \leq N^2\exp(-c'\,k\,\varepsilon^2)$$

and if $k > k_0 = \log(N^2)/c'\varepsilon^2$ then the probability of this event is strictly less than one. This means that there exists a realization of the matrix $\Gamma/\sqrt{k}$ that defines A and that satisfies the contrary i.e.

$$\forall\, x,\, y \in T,\ \sqrt{1-\varepsilon}\ |x-y|_2 \leq |A(x-y)|_2 \leq \sqrt{1+\varepsilon}\ |x-y|_2.$$

□

Remark 1.3.2. — *The value of C is less than 1800.*
In fact, the proof uses only the ψ_2 behavior of $\langle G_i, x\rangle$. We could replace the Gaussian vectors by any copies of an isotropic vector Y with independent entries with bounded ψ_2 norms, like e.g. a random vector with independent Rademacher coordinates. Indeed, by Theorem 1.2.1, $\|\langle Y, x-y\rangle\|_{\psi_2} \leq c|x-y|_2$ which gives an estimate of (1.9). *The rest of the proof is identical.*

1.4. Complexity and covering numbers

The study of covering and packing numbers is a wide subject. We will only present some basic estimates needed for the purpose of this book.

In approximation theory, in compressed sensing, in statistics, it is of importance to measure the complexity of a set. An important notion is the entropy number which measures the compactness of a set. Given U and V two sets of $\mathbb{R}^n$, we define the covering number $N(U, V)$ to be the minimum of translates of V needed to cover U. The formal definition is

$$N(U,V) = \inf\left\{ N : \exists x_1, \ldots, x_N \in \mathbb{R}^n, U \subset \bigcup_{i=1}^{N} (x_i + V) \right\}.$$

If moreover V is a symmetric convex set, the packing number $M(U, V)$ is the maximal number of points in U that are 1-separated for the norm induced by the convex set V. Formally, for every sets $U, V \subset \mathbb{R}^n$,

$$M(U,V) = \sup\left\{ N : \exists x_1, \ldots, x_N \in U, \forall i \neq j, x_i - x_j \notin V \right\}.$$

If V is a symmetric convex set (we always mean symmetric with respect to the origin), we can define the norm associated to V: for every $x \in \mathbb{R}^n$

$$\|x\|_V = \inf\{t > 0, x \in tV\}.$$

Hence $x_i - x_j \notin V$ is equivalent to say that $\|x_i - x_j\|_V > 1$. For any positive number ε, we will also use the notation

$$N(U, \varepsilon, \|\cdot\|_V)$$

for $N(U, \varepsilon V)$. Moreover, the family $x_1, \ldots, x_N$ is said to be an ε-net if it is such that $U \subset \bigcup_{i=1}^{N} (x_i + \varepsilon V)$. Also if we define the polar of V by

$$V^o = \{y \in \mathbb{R}^n : \forall x \in V, \langle x, y\rangle \leq 1\}$$

then the dual of the vectorial normed space $(\mathbb{R}^n, \|\cdot\|_V)$ is isometric to $(\mathbb{R}^n, \|\cdot\|_{V^o})$.

In the case V being a symmetric convex set, the notions of packing and covering numbers are closely related.

Proposition 1.4.1. — *If $U, V \subset \mathbb{R}^n$ and $0 \in V$ then $N(U, V) \leq M(U, V)$.*
If U is a convex set and V is a symmetric convex set then $M(U, V) \leq N(U, V/2)$.

Proof. — Let $N = M(U, V)$ be the maximal number of points $x_1, \ldots, x_N$ in U such that for every $i \neq j$, $x_i - x_j \notin V$. Let $u \in U \setminus \{x_1, \ldots, x_N\}$ then $\{x_1, \ldots, x_N, u\}$ is not 1-separated in V and this means that there exists $i \in \{1, \ldots, N\}$ such that $u - x_i \in V$. Consequently $U \subset \bigcup_{i=1}^{N} (x_i + V)$, since $0 \in V$, and $N(U, V) \leq M(U, V)$.

Let $x_1, \ldots, x_M$ be a family of vectors of U that are 1-separated. Let $z_1, \ldots, z_N$ be a family of vectors such that $U \subset \bigcup_{i=1}^{N} (z_i + V/2)$. Since for every $i = 1, \ldots, M$, $x_i \in U$, we can define an application $j : \{1, \ldots, M\} \to \{1, \ldots, N\}$ where $j(i)$ is such that $x_i \in z_{j(i)} + V/2$. If $j(i_1) = j(i_2)$ then $x_{i_1} - x_{i_2} \in V/2 - V/2$. By convexity and symmetry of V, $V/2 - V/2 = V$ hence $x_{i_1} - x_{i_2} \in V$. But the family $x_1, \ldots, x_M$ is 1-separated in V hence necessarily $i_1 = i_2$. This proves that the map j is injective and this implies that $M(U, V) \leq N(U, V/2)$. $\square$

Moreover, it is not difficult to check that for any U, V, W convex bodies $N(U, W) \leq N(U, V)N(V, W)$. We have the following simple and important volumetric estimate.

Lemma 1.4.2. — *Given V a symmetric convex set in $\mathbb{R}^n$, for every $\varepsilon > 0$,*

$$N(V, \varepsilon V) \leq \left(1 + \frac{2}{\varepsilon}\right)^n.$$

Proof. — By Proposition 1.4.1, $N(V, \varepsilon V) \leq M(V, \varepsilon V)$. Let $M = M(V, \varepsilon V)$ be the maximal number of points $x_1, \ldots, x_M$ in V such that for every $i \neq j$, $x_i - x_j \notin \varepsilon V$. Since V is a symmetric convex set, the sets $x_i + \varepsilon V/2$ are disjoint and

$$\bigcup_{i=1}^{M} (x_i + \varepsilon V/2) \subset V + \varepsilon V/2 = \left(1 + \frac{\varepsilon}{2}\right) V.$$

By taking the volume, we get that

$$M \left(\frac{\varepsilon}{2}\right)^n \leq \left(1 + \frac{\varepsilon}{2}\right)^n$$

which proves the desired estimate. □

Another important parameter that will be used to measure the size of a subset T of $\mathbb{R}^n$ is $\ell_*(T)$, defined by

$$\ell_*(T) = \mathbb{E} \sup_{t \in T} \langle G, t \rangle$$

where G is a Gaussian vector in $\mathbb{R}^n$ distributed according to the normal law $\mathcal{N}(0, \mathrm{I}d)$. By definition, $\ell_*(T) = \ell_*(\operatorname{conv} T)$ where $\operatorname{conv} T$ denotes the convex hull of T.

We will present some classical tools to estimate the covering numbers of the unit ball of ℓ_1^n by parallelepipeds and some classical estimates relating covering numbers of T by a multiple of the Euclidean ball and $\ell_*(T)$ or $\ell_*(T^o)$.

The empirical method. — We will introduce this method through a concrete example. Let d be a positive integer and Φ be an $d \times d$ matrix. We assume that the entries of Φ are such that for all $i, j \in \{1, \ldots, d\}$,

$$|\Phi_{ij}| \leqslant \frac{K}{\sqrt{d}} \tag{1.10}$$

where $K > 0$ is an absolute constant.

We denote by $\Phi_1, \ldots, \Phi_d$ the row vectors of Φ and we define for all $p \in \{1, \ldots, d\}$ the semi-norm $\|\cdot\|_{\infty,p}$ for all $x \in \mathbb{R}^d$ by

$$\|x\|_{\infty,p} = \max_{1 \leqslant j \leqslant p} |\langle \Phi_j, x \rangle|.$$

Its unit ball is denoted by $B_{\infty,p}$. If $E = \operatorname{span}\{\Phi_1, \ldots, \Phi_p\}$ and P_E is the orthogonal projection on E then we have $B_{\infty,p} = P_E B_{\infty,p} + E^{\perp}$. Moreover, $P_E B_{\infty,p}$ is a parallelepiped in E. In the next theorem, we obtain an upper bound of the logarithm of the covering numbers of the unit ball of ℓ_1^d, denoted by B_1^d, by a multiple of $B_{\infty,p}$. Observe that from the hypothesis (1.10) on the entries of the matrix Φ, we get that for any $x \in B_1^d$ and any $j = 1, \ldots, p$, $|\langle \Phi_j, x \rangle| \leq |\Phi_j|_\infty |x|_1 \leq K/\sqrt{d}$. Therefore

$$B_1^d \subset \frac{K}{\sqrt{d}} B_{\infty,p} \tag{1.11}$$

and for any $\varepsilon \geq K/\sqrt{d}$, $N(B_1^d, \varepsilon B_{\infty,p}) = 1$.

Theorem 1.4.3. — *With the preceding notations, we have for any* $0 < t < 1$,

$$\log N\left(B_1^d, \frac{tK}{\sqrt{d}} B_{\infty,p}\right) \leqslant \min\left\{c_0 \frac{\log(p)\log(2d+1)}{t^2},\ p\log\left(1+\frac{2}{t}\right)\right\}$$

where c_0 *is an absolute constant.*

The first estimate is proven using an empirical method, while the second one is based on the volumetric estimate.

Proof. — Let x be in B_1^d and define the random variable Z by

$$\mathbb{P}\big(Z = \mathrm{Sign}(x_i)e_i\big) = |x_i| \text{ for all } i = 1, \dots, d \text{ and } \mathbb{P}(Z = 0) = 1 - |x|_1$$

where $(e_1, \dots, e_d)$ is the canonical basis of $\mathbb{R}^d$. Observe that we have $\mathbb{E}Z = x$.

We use a well known symmetrization argument, see Chapter 5. Take m to be chosen later and $Z_1, \dots, Z_m, Z_1', \dots, Z_m'$ be i.i.d. copies of Z. We have by Jensen inequality

$$\mathbb{E}\left\|x - \frac{1}{m}\sum_{i=1}^m Z_i\right\|_{\infty,p} = \left\|\frac{1}{m}\sum_{i=1}^m \mathbb{E}'Z_i' - Z_i\right\|_{\infty,p} \le \mathbb{E}\mathbb{E}'\left\|\frac{1}{m}\sum_{i=1}^m Z_i' - Z_i\right\|_{\infty,p}.$$

The random variable $Z_i' - Z_i$ is symmetric hence it has the same law than $\varepsilon_i(Z_i' - Z_i)$ where $\varepsilon_1, \dots, \varepsilon_m$ are i.i.d. Rademacher random variables. Therefore, by the triangle inequality

$$\mathbb{E}\mathbb{E}'\left\|\frac{1}{m}\sum_{i=1}^m Z_i' - Z_i\right\|_{\infty,p} = \frac{1}{m}\mathbb{E}\mathbb{E}'\mathbb{E}_\varepsilon\left\|\sum_{i=1}^m \varepsilon_i(Z_i' - Z_i)\right\|_{\infty,p} \le \frac{2}{m}\mathbb{E}\mathbb{E}_\varepsilon\left\|\sum_{i=1}^m \varepsilon_i Z_i\right\|_{\infty,p}.$$

We conclude that

$$\begin{aligned}\mathbb{E}\left\|x - \frac{1}{m}\sum_{i=1}^m Z_i\right\|_{\infty,p} &\le \frac{2}{m}\mathbb{E}\mathbb{E}_\varepsilon\left\|\sum_{i=1}^m \varepsilon_i Z_i\right\|_{\infty,p} \\ &= \frac{2}{m}\mathbb{E}\mathbb{E}_\varepsilon \max_{1\le j\le p}\left|\sum_{i=1}^m \varepsilon_i\langle Z_i, \Phi_j\rangle\right|. \qquad (1.12)\end{aligned}$$

By definition of Z and by (1.10), we know that $|\langle Z_i, \Phi_j\rangle| \le K/\sqrt{d}$. Let a_{ij} be a sequence of real number such that $|a_{ij}| \le K/\sqrt{d}$. For any j, let X_j be the random variable $X_j = \sum_{i=1}^m a_{ij}\varepsilon_i$. From Theorem 1.2.1, we deduce that

$$\forall j = 1, \dots, p,\ \|X_j\|_{\psi_2} \le c\left(\sum_{i=1}^m a_{ij}^2\right)^{1/2} \le c\frac{K\sqrt{m}}{\sqrt{d}}.$$

Therefore, by Proposition 1.1.3 (and the remark after it), we get

$$\mathbb{E}\max_{1\le j\le p}|X_j| \le c\sqrt{(1+\log p)}\ \frac{K\sqrt{m}}{\sqrt{d}}.$$

From (1.12) and the preceding argument, we conclude that

$$\mathbb{E}\left\| x - \frac{1}{m}\sum_{i=1}^{m} Z_i \right\|_{\infty,p} \leq \frac{2\,c\,K\sqrt{(1+\log p)}}{\sqrt{md}}$$

Let m be the integer such that

$$\frac{4c^2(1+\log p)}{t^2} \leq m \leq \frac{4c^2(1+\log p)}{t^2} + 1$$

For this choice of m we have

$$\mathbb{E}\left\| x - \frac{1}{m}\sum_{i=1}^{m} Z_i \right\|_{\infty,p} \leqslant \frac{tK}{\sqrt{d}}.$$

In particular, there exists $\omega \in \Omega$ such that

$$\left\| x - \frac{1}{m}\sum_{i=1}^{m} Z_i(\omega) \right\|_{\infty,p} \leqslant \frac{tK}{\sqrt{d}}$$

and so the set

$$\left\{ \frac{1}{m}\sum_{i=1}^{m} z_i : z_1, \ldots, z_m \in \{\pm e_1, \ldots, \pm e_d\} \cup \{0\} \right\}$$

is a $tK/\sqrt{d}$-net of B_1^d with respect to $\|\cdot\|_{\infty,p}$. Since its cardinality is less than $(2d+1)^m$, we get the first estimate:

$$\log N\left(B_1^d, \frac{tK}{\sqrt{d}} B_{\infty,p}\right) \leqslant \frac{c_0(1+\log p)\log(2d+1)}{t^2}$$

where c_0 is an absolute constant.

To prove the second estimate, we recall by (1.11) that $B_1^d \subset K/\sqrt{d} B_{\infty,p}$. Hence

$$N\left(B_1^d, \frac{tK}{\sqrt{d}} B_{\infty,p}\right) \leq N\left(\frac{K}{\sqrt{d}} B_{\infty,p}, \frac{tK}{\sqrt{d}} B_{\infty,p}\right) = N\left(B_{\infty,p}, tB_{\infty,p}\right).$$

Moreover, we have already observed that $B_{\infty,p} = P_E B_{\infty,p} + E^{\perp}$ which means that

$$N\left(B_{\infty,p}, tB_{\infty,p}\right) = N(V, tV)$$

where V is the symmetric convex set $P_E B_{\infty,p}$. Since $\dim E \leq p$, we apply Lemma 1.4.2 to conclude that

$$N\left(B_1^d, \frac{tK}{\sqrt{d}} B_{\infty,p}\right) \leq \left(1 + \frac{2}{t}\right)^p.$$

□

Sudakov's inequality and dual Sudakov's inequality. — Classical tools for the computation of the covering numbers of a set by Euclidean balls or in a dual situation, covering numbers of a Euclidean ball by translates of a symmetric convex set are the Sudakov and dual Sudakov inequalities. They relate these covering numbers with the complexity ℓ_* of the sets.

Theorem 1.4.4. — *Let T be a subset of $\mathbb{R}^N$ and V be a symmetric convex set in $\mathbb{R}^N$. Then, the following inequalities hold:*

$$\sup_{\varepsilon>0} \varepsilon\sqrt{\log N(T, \varepsilon B_2^N)} \leqslant c\,\ell_*(T) \tag{1.13}$$

and

$$\sup_{\varepsilon>0} \varepsilon\sqrt{\log N(B_2^N, \varepsilon V)} \leqslant c\,\ell_*(V^o) \tag{1.14}$$

where for a normal Gaussian vector $G \in \mathbb{R}^N$, $\ell_*(T) = \mathbb{E}\sup_{t\in T}\langle G, t\rangle$ and $\ell_*(V^o) = \mathbb{E}\sup_{t\in V^o}\langle G, t\rangle = \mathbb{E}\|G\|_V$.

The proof of the Sudakov inequality (1.13) is based on comparison properties between Gaussian processes. We recall the Slepian comparison lemma without proving it.

Lemma 1.4.5. — *Let $X_1, \dots, X_M$ and $Y_1, \dots, Y_M$ be Gaussian random variables such that for all $i, j = 1, \dots, M$*

$$\mathbb{E}|Y_i - Y_j|^2 \leq \mathbb{E}|X_i - X_j|^2$$

then

$$\mathbb{E}\max_{1\leq k\leq M} Y_k \leq 2\,\mathbb{E}\max_{1\leq k\leq M} X_k.$$

Proof of Theorem 1.4.4. — We start by proving (1.13). Let $x_1, \dots, x_M$ be M points of T that are ε-separated with respect to the Euclidean norm $|\cdot|_2$ and define for every $i = 1, \dots, M$, the Gaussian variables $X_i = \langle x_i, G\rangle$ where G is a standard Gaussian vector in $\mathbb{R}^N$. We have

$$\mathbb{E}|X_i - X_j|^2 = |x_i - x_j|_2^2 \geq \varepsilon^2 \quad \text{for all } i \neq j.$$

Let $g_1, \dots, g_M$ be M standard independent Gaussian random variables and for every $i = 1, \dots, M$ let Y_i be defined by $Y_i = \frac{\varepsilon}{\sqrt{2}} g_i$. We have for all $i \neq j$

$$\mathbb{E}|Y_i - Y_j|^2 = \varepsilon^2$$

and we conclude from Lemma 1.4.5 that

$$\frac{\varepsilon}{\sqrt{2}}\mathbb{E}\max_{1\leq k\leq M} g_k \leq 2\,\mathbb{E}\max_{1\leq k\leq M}\langle x_k, G\rangle \leq 2\ell(T).$$

Moreover there exists a constant $c > 0$ such that for every positive integer M

$$\mathbb{E}\max_{1\leq k\leq M} g_k \geq \sqrt{\log M}/c \tag{1.15}$$

and this proves that $\varepsilon\sqrt{\log M} \leq 2c\sqrt{2}\ell(T)$. By Proposition 1.4.1, the proof of inequality (1.13) is complete. The lower bound (1.15) is a classical exercise about Gaussian random variables. First, we observe that $\mathbb{E}\max(g_1, g_2)$ is computable, it is equal to

$1/\sqrt{\pi}$. Hence we can assume that M is large enough (say greater than 10^4). In this case, we observe that

$$2\,\mathbb{E}\max_{1\le k\le M} g_k \ge \mathbb{E}\max_{1\le k\le M}|g_k| - \mathbb{E}|g_1|.$$

Indeed,

$$\mathbb{E}\max_{1\le k\le M} g_k = \mathbb{E}\max_{1\le k\le M}(g_k-g_1) = \mathbb{E}\max_{1\le k\le M}\max((g_k-g_1),0)$$

and by symmetry of the g_i's,

$$\begin{aligned}\mathbb{E}\max_{1\le k\le M}|g_k-g_1| &\le \mathbb{E}\max_{1\le k\le M}\max((g_k-g_1),0) + \mathbb{E}\max_{1\le k\le M}\max((g_1-g_k),0)\\ &= 2\,\mathbb{E}\max_{1\le k\le M}(g_k-g_1) = 2\mathbb{E}\max_{1\le k\le M} g_k.\end{aligned}$$

But, by independence of the g_i's

$$\begin{aligned}\mathbb{E}\max_{1\le k\le M}|g_k| &= \int_0^{+\infty}\mathbb{P}\left(\max_{1\le k\le M}|g_k|>t\right)dt = \int_0^{+\infty}\left(1-\mathbb{P}\left(\max_{1\le k\le M}|g_k|\le t\right)\right)dt\\ &= \int_0^{+\infty}\left(1-\left(1-\sqrt{\frac{2}{\pi}}\int_t^{+\infty}e^{-u^2/2}du\right)^M\right)dt\end{aligned}$$

and it is easy to see that for every $t>0$,

$$\int_t^{+\infty}e^{-u^2/2}du \ge e^{-(t+1)^2/2}.$$

Let $t_0+1=\sqrt{2\log M}$ then

$$\begin{aligned}\mathbb{E}\max_{1\le k\le M}|g_k| &\ge \int_0^{t_0}\left(1-\left(1-\sqrt{\frac{2}{\pi}}\int_t^{+\infty}e^{-u^2/2}du\right)^M\right)dt\\ &\ge t_0\left(1-\left(1-\frac{\sqrt{2}}{M\sqrt{\pi}}\right)^M\right) \ge t_0(1-e^{-\sqrt{2/\pi}})\end{aligned}$$

which concludes the proof of (1.15).

We will now prove the dual Sudakov inequality (1.14). The argument is very similar to the volumetric argument introduced in Lemma 1.4.2, replacing the Lebesgue measure by the Gaussian measure. Let $r>0$ to be chosen later. Observe that $N(B_2^N,\varepsilon V)=N(rB_2^N,r\varepsilon V)$ and let $x_1,\dots,x_M$ be M points in rB_2^N that are $r\varepsilon$ separated for the norm induced by the symmetric convex set V. By Proposition 1.4.1, it is enough to prove that

$$\varepsilon\sqrt{\log M}\le c\,\ell_*(V^o).$$

The balls centered at the points x_i and of radius $r\varepsilon/2$ are disjoints and by taking the Gaussian measure of the union of these sets, we get that

$$\gamma_N\left(\bigcup_{i=1}^M (x_i+r\varepsilon/2\,V)\right) = \sum_{i=1}^M\int_{\|z-x_i\|_V\le r\varepsilon/2} e^{-|z|_2^2/2}\frac{dz}{(2\pi)^{N/2}}\le 1.$$

However, by the change of variable $z - x_i = u_i$, we have

$$\int_{\|z-x_i\|_V \le r\varepsilon/2} e^{-|z|_2^2/2} \frac{dz}{(2\pi)^{N/2}} = e^{-|x_i|_2^2/2} \int_{\|u_i\|_V \le r\varepsilon/2} e^{-|u_i|_2^2/2} e^{-\langle u_i, x_i\rangle} \frac{du_i}{(2\pi)^{N/2}}$$

and from Jensen inequality and the fact that V has barycenter at the origin,

$$\frac{1}{\gamma_N\left(\frac{r\varepsilon}{2}V\right)} \int_{\|z-x_i\|_V \le r\varepsilon/2} e^{-|z|_2^2/2} \frac{dz}{(2\pi)^{N/2}} \ge e^{-|x_i|_2^2/2}.$$

Since $x_i \in rB_2^N$, we have proved that

$$M\, e^{-r^2/2}\, \gamma_N\left(\frac{r\varepsilon}{2}V\right) \le 1.$$

To conclude, we choose r such that $r\varepsilon/2 = 2\ell_*(V^o)$. Hence by Markov inequality, $\gamma_N\left(\frac{r\varepsilon}{2}V\right) \ge 1/2$ and we have proved that $M \le 2e^{r^2/2}$ which means that for a constant c,

$$\varepsilon\sqrt{\log M} \le c\,\ell_*(V^o).$$

□

The metric entropy of the Schatten balls. — To finish this chapter, we show how to apply Sudakov and dual Sudakov inequalities to compute the metric entropy of Schatten balls with respect to Schatten norms. We denote by $B_p^{m,n}$ the unit ball of the Banach spaces of matrices in $\mathcal{M}_{m,n}$ endowed with the Schatten norm $\|\cdot\|_{S_p}$ defined for any $A \in \mathcal{M}_{m,n}$ by

$$\|A\|_{S_p} = \left(\mathrm{tr}(A^*A)^{p/2}\right)^{1/p}.$$

It is also the ℓ_p-norm of the singular values of A and we refer to Chapter 4 for more informations about the singular values of a matrix.

Proposition 1.4.6. — *For every $m \ge n \geqslant 1$, $p, q \in [1, +\infty]$ and $\varepsilon > 0$,*

$$\varepsilon\sqrt{\log N(B_p^{m,n}, \varepsilon B_2^{m,n})} \leqslant c_1\sqrt{m}\; n^{(1-1/p)} \tag{1.16}$$

and

$$\varepsilon\sqrt{\log N(B_2^{m,n}, \varepsilon B_q^{m,n})} \leqslant c_2\sqrt{m}\; n^{1/q} \tag{1.17}$$

where c_1 and c_2 are numerical constants. Moreover, for $n \ge m \ge 1$ the same result holds by exchanging m and n.

Proof. — We start by proving a rough upper bound of the operator norm of a Gaussian random matrix $\Gamma \in \mathcal{M}_{m,n}$ i.e. a matrix with independent standard Gaussian entries:

$$\mathbb{E}\,\|\Gamma\|_{S_\infty} \le c(\sqrt{n} + \sqrt{m}) \tag{1.18}$$

for some numerical constant C. Let $X_{u,v}$ be the Gaussian process defined for any $u \in B_2^m, v \in B_2^n$ by

$$X_{u,v} = \langle \Gamma v, u\rangle.$$

It is defined such that

$$\mathbb{E}\,\|\Gamma\|_{S_\infty} = \mathbb{E} \sup_{u\in B_2^m, v\in B_2^n} X_{u,v}.$$

From Lemma 1.4.2, there exist $\Lambda \subset B_2^m$ and $\Lambda' \subset B_2^n$, $(1/4)$-net of B_2^m and B_2^n respectively, for their own metric, such that $|\Lambda| \leqslant 9^m$ and $|\Lambda'| \leqslant 9^n$. Let $u \in B_2^m$ and $u' \in \Lambda$ such that $|u - u'|_2 < 1/4$ and similarly, let $v \in B_2^n$ and $v' \in \Lambda'$ such that $|v - v'|_2 < 1/4$, then we have

$$|X_{u,v} - X_{u',v'}| = |\langle \Gamma v, u - u'\rangle + \langle \Gamma(v - v'), u'\rangle| \leq \|\Gamma\|_{S_\infty} |u - u'|_2 + \|\Gamma\|_{S_\infty} |v - v'|_2.$$

We deduce that $\|\Gamma\|_{S_\infty} \leqslant \sup_{u'\in\Lambda, v'\in\Lambda'} |X_{u',v'}| + (1/2)\|\Gamma\|_{S_\infty}$ and therefore

$$\|\Gamma\|_{S_\infty} \leqslant 2 \sup_{u'\in\Lambda, v'\in\Lambda'} |X_{u',v'}|.$$

Now $X_{u',v'}$ is a Gaussian centered random variable with variance $|u'|_2^2|v'|_2^2 \leqslant 1$. By Lemma 1.1.3,

$$\mathbb{E} \sup_{u'\in\Lambda, v'\in\Lambda'} |X_{u',v'}| \leqslant c\sqrt{\log|\Lambda||\Lambda'|} \leq c\sqrt{\log 9}\,(\sqrt{m} + \sqrt{n})$$

and Equation 1.18 follows.

We first prove (1.16) in the case $m \geq n \geq 1$. Using Sudakov inequality (1.13), we have for all $\varepsilon > 0$,

$$\varepsilon\sqrt{\log N(B_p^{m,n}, \varepsilon B_2^{m,n})} \leqslant c\ell_*(B_p^{m,n}).$$

However

$$\ell_*(B_p^{m,n}) = \mathbb{E} \sup_{A\in B_p^{m,n}} \langle \Gamma, A\rangle$$

where $\langle \Gamma, A\rangle = \operatorname{Tr}(\Gamma\, A^*)$. If p' is such that $1/p + 1/p' = 1$ then we have by the trace duality

$$\langle \Gamma, A\rangle \leq \|\Gamma\|_{S_{p'}} \|A\|_{S_p} \leqslant n^{1/p'} \|\Gamma\|_{S_\infty} \|A\|_{S_p}.$$

By taking the supremum over $A \in B_p^{m,n}$, the expectation and using (1.18), we deduce that

$$\ell_*(B_p^{m,n}) \leq n^{1/p'}\mathbb{E}\,\|\Gamma\|_{S_\infty} \leq c\sqrt{m}\; n^{1/p'}$$

which ends the proof of (1.16)

We prove (1.17) in the case $m \geq n \geq 1$. Using the dual Sudakov inequality (1.14) and (1.18) we get that for any $q \in [1, +\infty]$:

$$\varepsilon\sqrt{\log N(B_2^{m,n}, \varepsilon B_q^{m,n})} \leqslant c\,\mathbb{E}\,\|\Gamma\|_{S_q} \leqslant c\,n^{1/q}\mathbb{E}\,\|\Gamma\|_{S_\infty} \leqslant c' n^{1/q}\sqrt{m}.$$

The proof of the case $n \geq m$ is completely similar.

□

Concentration of norms of Gaussian vectors. — We finish this chapter by an other important property of Gaussian processes, a concentration of measure inequality which will be used in the next chapter. It is stated without proof. The reader is referred to the book [**Pis89**] to learn more about this.

Theorem 1.4.7. — *Let $G \in \mathbb{R}^n$ be a Gaussian vector distributed according to the normal law $\mathcal{N}(0, \mathrm{Id})$. Let $T \subset \mathbb{R}^n$ and let*

$$\sigma(T) = \sup_{t\in T}\{\left(\mathbb{E}\langle G, t\rangle|^2\right)^{1/2}\}.$$

We have

$$\forall u > 0 \qquad \mathbb{P}\left(\left|\sup_{t\in T}\langle G, t\rangle - \mathbb{E}\sup_{t\in T}\langle G, t\rangle\right| > u\right) \le 2\exp\left(-cu^2/\sigma^2(T)\right) \tag{1.19}$$

where c is a numerical constant.

1.5. Notes and comments

We focused in this chapter on the study of some very particular concentration inequalities. Of course, there exist different and powerful other type of concentration inequalities. Several books and surveys are devoted to this subject and we refer for example to [**LT91, vdVW96, Led01, BBL04, Mas07**] for the interested reader. The classical references for the study of Orlicz spaces are [**KR61, LT77, LT79, RR91, RR02**].

Tail and moment estimates for Rademacher averages are well understood. Theorem 1.2.3 is due to Montgomery-Smith [**MS90**] and several extensions to the vector valued case are known [**DMS93, MS95**]. The case of sum of independent random variables with logarithmically concave tails has been studied by Gluskin and Kwapien [**GK95**]. For the proof of Theorem 1.2.8, we could have followed a classical probabilistic trick which reduces the proof of the result to the case of Weibull random variables. These variables are defined such that the tails are equals to e^{-t^α}. Hence, the tails are logarithmically concave and the result is a corollary of the results of Gluskin and Kwapien [**GK95**]. We have presented here an approach which follows the line of [**Tal94**]. The results are only written for random variables with densities $c_\alpha e^{-t^\alpha}$, but the proofs work in the general context of ψ_α random variables.

Originally, Lemma 1.3.1 is proved in [**JL84**] and the operator is chosen at random in the set of orthogonal projections onto a random k-dimensional subspace of ℓ_2, uniformly according to the Haar measure on the Grassman manifold $\mathcal{G}_{n,k}$.

The classical references for the study of entropy numbers are [**Pie72, Pie80, Pis89, CS90**]. The method of proof of Theorem 1.4.3 has been introduced by Maurey, in particular for studying entropy numbers of operators from ℓ_1^d into a Banach space of type p. This was published in [**Pis81**]. The method was extended and developed by Carl in [**Car85**]. Sudakov inequality 1.13 is due to Sudakov [**Sud71**] while the dual Sudakov inequality 1.14 is due to Pajor and Tomczak-Jaegermann [**PTJ86**]. The proof that we presented follows the lines of Ledoux-Talagrand [**LT91**]. We have made the choice to speak only about Slepian inequality, Lemma 1.4.5. The result of Slepian [**Sle62**] is more general, it tells about distribution inequality. In the context of Lemma 1.4.5, Fernique [**Fer74**] proved that the constant 2 can be replaced by 1 and Gordon [**Gor85, Gor87**] extended these results to min-max of some Gaussian processes. About the covering numbers of the Schatten balls, Proposition 1.4.6 is due to Pajor [**Paj99**]. Theorem 1.4.7 is due to Maurey and Pisier (see the book [**Pis89**] and [**Pis86**] for variations on the same theme).

CHAPTER 2

COMPRESSED SENSING AND GELFAND WIDTHS

2.1. Short introduction to compressed sensing

Compressed Sensing is a quite new framework that enables to get approximate and exact reconstruction of sparse signals from incomplete measurements. The ideas and principles are strongly related to other problems coming from different fields such as approximation theory, in particular to the study of Gelfand and Kolmogorov width of classical Banach spaces (diameter of sections). Since the seventies an important work was done in that direction, in Approximation Theory and in Asymptotic Geometric Analysis (called Geometry of Banach spaces at that time).

It is not in our aim to give here an introduction to compressed sensing, there are many good references for that, but mainly to emphasize some interactions with other fields of mathematics, in particular with asymptotic geometric analysis, random matrices and empirical processes. The possibility of reconstructing any vector from a given subset is highly related to the *complexity* of this subset and in the field of Geometry of Banach spaces, many tools were developed to analyze various concepts of complexity.

In this introduction to compressive sensing, for simplicity, we will consider the real case, real vectors and real matrices. Let $1 \leq n \leq N$ be integers. We are given a rectangular $n \times N$ real matrix A. One should think of $N \gg n$; we have in mind to compress some vectors from $\mathbb{R}^N$ for large N into vectors in $\mathbb{R}^n$. Let $X_1, \dots, X_N \in \mathbb{R}^n$ be the columns of A and let $Y_1, .., Y_n \in \mathbb{R}^N$ its rows. One has

$$A = \Big(X_1 \cdots\cdots\cdots X_N \Big) = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix}.$$

We are also given a subset $T \subset \mathbb{R}^N$ of vectors. Now let $x \in T$ be an *unknown* vector. The data one is given are n linear measurements of x (again, think of $N \gg n$)

$$\langle Y_1, x \rangle, \dots, \langle Y_n, x \rangle$$

or equivalently

$$y = Ax.$$

We wish to *recover* x or more precisely to *reconstruct* x, exactly or approximately, within a given accuracy and in an efficient way (fast algorithm).

2.2. The exact reconstruction problem

Let us first discuss the exact reconstruction question. Let $x \in T$ be unknown and recall that the given data is $y = Ax$. When $N \gg n$, the problem is *ill-posed* because the system $At = y$, $t \in \mathbb{R}^N$ is highly under-determined. Thus if we want to recover x we need some more information on its *nature*. Moreover if we want to recover any x from T, one should have some a priori information on the set T, on its *complexity* whatever it means at this stage. We shall see various parameters of complexity in these notes. The a priori hypothesis that we investigate now is *sparsity*.

Sparsity. — We first introduce some notation. We equip $\mathbb{R}^n$ and $\mathbb{R}^N$ with the canonical scalar product $\langle \cdot, \cdot \rangle$ and Euclidean norm $|\cdot|_2$. We use the notation $|\cdot|$ to denote the cardinality of a set. By B_2^N we denote the unit Euclidean ball and by S^{n-1} its unit sphere.

Definition 2.2.1. — *Let $0 \le m \le N$ be integers. For any $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$, denote by $\operatorname{supp} x = \{k : 1 \le k \le N, x_k \neq 0\}$ the subset of non-zero coordinate of x. The vector x is said m-sparse if $|\operatorname{supp} x| \le m$. The subset of m-sparse vectors of $\mathbb{R}^N$ is denoted by $\Sigma_m = \Sigma_m(\mathbb{R}^N)$ and its unit sphere by*

$$S_2(\Sigma_m) = \{x \in \mathbb{R}^N : |x|_2 = 1 \,\text{and}\, |\operatorname{supp} x| \le m\}.$$

Similarly let

$$B_2(\Sigma_m) = \{x \in \mathbb{R}^N : |x|_2 \le 1 \,\text{and}\, |\operatorname{supp} x| \le m\}.$$

Note that Σ_m is not a linear subspace and that $B_2(\Sigma_m)$ is not convex.

Problem 2.2.2. — **The exact reconstruction problem.** *We wish to reconstruct exactly any m-sparse vector $x \in \Sigma_m$ from the given data $y = Ax$. Thus we are looking for a* decoder *Δ such that*

$$\forall x \in \Sigma_m, \qquad \Delta(A, Ax) = x.$$

Claim 2.2.3. — *Linear algebra tells us that such a decoder Δ exists iff*

$$\ker A \cap \Sigma_{2m} = \{0\}.$$

Example 2.2.4. — *Let $m \ge 1$, $N \ge 2m$ and $0 < a_1 < \cdots < a_N = 1$. Let $n = 2m$ and build the Vandermonde matrix $A = (a_j^{i-1}), 1 \le i \le n, 1 \le j \le N$. Clearly all the $2m \times 2m$ minors of A are non singular Vandermonde matrices. Unfortunately it is known that such matrices are ill-conditioned. Therefore reconstructing $x \in \Sigma_m$ from $y = Ax$ is numerically unstable.*

Metric entropy. — As already said, there are many different approaches to seize and measure complexity of a metric space. The most simple is probably to estimate a degree of compactness via the so-called covering and packing numbers.

Since all the metric spaces we will consider here are subsets of normed spaces, we restrict to this setting.

Definition 2.2.5. — *Let B and C be subsets of a vector space and let $\varepsilon > 0$. An ε-net of B by translates of $\varepsilon\, C$ is a subset Λ of B such that for every $x \in B$, there exits $y \in \Lambda$ and $z \in C$ such that $x = y + \varepsilon z$. In other words, one has*

$$B \subset \Lambda + \varepsilon C = \bigcup_{x\in\Lambda} (x + \varepsilon C)\,,$$

where $\Lambda + \varepsilon C := \{a + \varepsilon c : a \in \Lambda, c \in C\}$ is the Minkowski sum of the sets Λ and εC. The covering number of B by εC is the smallest cardinality of such an ε-net and is denoted by $N(B, \varepsilon C)$. The function $\varepsilon \to \log N(B, \varepsilon C)$ is called the metric entropy of B by C.

Remark 2.2.6. — *If (B, d) is a metric space, an ε-net of (B, d) is a covering of B by balls of radius ε for the metric d. The covering number is the smallest cardinality of an ε-net and is denoted by $N(B, d, \varepsilon)$. In our setting, the metric d will be defined by a norm with unit ball say C. Then $x + \varepsilon\, C$ is a ball of radius ε centered at x.*

Let us start by an easy but important fact. Let $C \subset \mathbb{R}^N$ be a symmetric convex body, that is a symmetric convex compact subset of $\mathbb{R}^N$, with non-empty interior (that is, the unit ball of a norm on $\mathbb{R}^N$). Consider a subset $\Lambda \subset C$ of maximal cardinality such that the points of Λ are εC-apart in the sense that:

$$\forall x \neq y, x, y \in \Lambda,\ \text{one has } x - y \notin \varepsilon C$$

(recall that $C = -C$). It is clear that Λ is an ε-net of C by εC. Moreover the balls

$$(x + (\varepsilon/2)C)_{x\in\Lambda}$$

of radius $(\varepsilon/2)$ centered at the points of Λ are pairwise disjoint and their union is a subset of $(1 + (\varepsilon/2))\, C$ (this is where convexity is involved). Taking volume of this union, we get that $N(C, \varepsilon C) \leq (1 + (2/\varepsilon))^N$. Let us conclude:

Proposition 2.2.7. — *Let $\varepsilon \in (0, 1)$. Let $C \subset \mathbb{R}^N$ be a symmetric convex body (the unit ball of a norm). There exists an ε-net Λ of C by translates of εC such that $|\Lambda| \leq (1 + 2/\varepsilon)^N$. Moreover $\Lambda \subset C \subset (1 - \varepsilon)^{-1}\,\mathrm{conv}\,(\Lambda)$.*

Let us prove the moreover part of the Proposition by successive approximation.

Proof. — Since Λ is an ε-net of C by translates of εC, every $z \in C$ can be written as $z = x_0 + \varepsilon z_1$, where $x_0 \in \Lambda$ and $z_1 \in C$. Iterating, it follows that $z = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots$, with $x_i \in \Lambda$, which implies by convexity that $C \subset (1 - \varepsilon)^{-1}\mathrm{conv}\,(\Lambda)$. □

This gives the first claim:

Claim 2.2.8. — *Covering the unit Euclidean sphere by Euclidean balls of radius ε. One has*

$$\forall \varepsilon \in (0,1), \qquad N(S^{N-1}, \varepsilon B_2^N) \le \left(\frac{3}{\varepsilon}\right)^N.$$

Now, since $S_2(\Sigma_m)$ is the union of spheres of dimension m,

$$N(S_2(\Sigma_m), \varepsilon B_2^N) \le \binom{N}{m} N(S^{m-1}, \varepsilon B_2^m).$$

Using $\binom{N}{m} \le (eN/m)^m$, we get:

Claim 2.2.9. — *Covering the set of sparse unit vectors by Euclidean balls of radius ε: let $1 \le m \le N$ and $\varepsilon \in (0,1)$, then*

$$N(S_2(\Sigma_m), \varepsilon B_2^N) \le \left(\frac{3eN}{m\varepsilon}\right)^m.$$

The ℓ_1-minimization method. — Coming back to the exact reconstruction problem, if we want to solve the system

$$At = y$$

where $y = Ax$ is given and x is m-sparse, it is tempting to test all possible support of the unknown vector x. This is the so-called ℓ_0-method. But there are $\binom{N}{m}$ possible supports, too many to answer the request of a fast algorithm. A more clever approach was proposed, namely the convex relaxation of the ℓ_0-method. Let x be the unknown vector. The given data is $y = Ax$. For $t = (t_i) \in \mathbb{R}^N$ denote by

$$|t|_1 = \sum_{i=1}^{N} |t_i|$$

its ℓ_1 norm. The ℓ_1-minimization method (also called *basis-pursuit*) is the following program:

$$(P) \qquad \min_{t \in \mathbb{R}^N} |t|_1 \quad \text{subject to} \quad At = y.$$

This program may be recast as a linear programming by

$$\min \sum_{i=1}^{N} s_i, \ \text{subject to } s \ge 0, -s \le t \le s, At = y.$$

Definition 2.2.10. — **Exact reconstruction by ℓ_1-minimization.** *We say that the matrix A has the exact reconstruction property of order m by ℓ_1-minimization if for every $x \in \Sigma_m$ the problem*

$$(P) \ \min_{t \in \mathbb{R}^N} |t|_1 \quad \textit{subject to} \quad At = Ax \textit{ has a unique solution equal to } x. \tag{2.1}$$

Note that the above property is not specific to the matrix A but rather a property of its null space. In order to emphasize this point, let us introduce some notation.

For any subset $I \subset [N]$ where $[N] = \{1, \dots, N\}$, let I^c be its complement and for any $x \in \mathbb{R}^N$, let us write x_I for the vector in $\mathbb{R}^N$ with the same coordinates as x for indices in I and 0 for indices in I^c. We are ready for a criterium on the null space.

Proposition 2.2.11. — **The null space property.** *The following are equivalent*

i) *For any $x \in \Sigma_m$, the problem*

$$(P) \qquad \min_{t \in \mathbb{R}^N} |t|_1 \quad \textit{subject to} \quad At = Ax$$

has a unique solution equal to x (that is A has the exact reconstruction property of order m by ℓ_1-minimization)

ii)

$$\forall h \in \ker A, h \neq 0, \forall I \subset [N], |I| \leq m, |h_I|_1 < |h_{I^c}|_1. \tag{2.2}$$

Proof. — On one side, let $h \in \ker A, h \neq 0$ and $I \subset [N], |I| \leq m$. Put $x = -h_I$. Then $x \in \Sigma_m$ and (2.1) implies that $|x + h|_1 > |x|_1$, that is $|h_{I^c}|_1 > |h_I|_1$.

For the reverse implication, suppose that

$$\forall h \in \ker A, h \neq 0, \forall I \subset [N], |I| \leq m, \ |h_I|_1 < |h_{I^c}|_1.$$

Let $x \in \Sigma_m$ and let $I = \mathrm{supp}(x)$. Then $|I| \leq m$ and

$$|x + h|_1 = |x_I + h_I|_1 + |h_{I^c}|_1 > |x_I + h_I|_1 + |h_I|_1 \geq |x|_1.$$

□

Definition 2.2.12. — *We say that A satisfies the null space property of order m if (2.2) is satisfied.*

This property has a nice geometric interpretation. To introduce it, we need some more notation. Let $(e_i)_{1 \leq i \leq N}$ be the canonical basis of $\mathbb{R}^N$. Let ℓ_1^N be the N-dimensional space $\mathbb{R}^N$ equipped with the ℓ_1-norm and B_1^N be its unit ball. Denote also

$$S_1(\Sigma_m) = \{x \in \Sigma_m \, : \, |x|_1 = 1\} \quad \text{and} \quad B_1(\Sigma_m) = \{x \in \Sigma_m \, : \, |x|_1 \leq 1\}.$$

Let $1 \leq m \leq N$. Any $(m-1)$-dimensional face of B_1^N is of the form $\mathrm{conv}(\{\varepsilon_i e_i : i \in I\})$ with $I \subset [N], |I| = m$ and $(\varepsilon_i) \in \{-1, 1\}^I$, where we denoted by $\mathrm{conv}(\,\cdot\,)$ the convex hull. ¿From the geometric point of view, $S_1(\Sigma_m)$ is the union of all the $(m-1)$-dimensional faces of B_1^N.

Let A be an $n \times N$ matrix and $X_1, \dots, X_N \in \mathbb{R}^n$ be its columns. A polytope $P \subset \mathbb{R}^n$ is said centrally symmetric if $P = -P$. Observe that

$$A(B_1^N) = \mathrm{conv}(\pm X_1, \dots, \pm X_N).$$

Proposition 2.2.11 can be reformulated in the following geometric language:

Proposition 2.2.13. — **The geometry of faces of $A(B_1^N)$.** *Let $1 \le m \le n \le N$. Let A be an $n \times N$ matrix with columns $X_1, \dots, X_N \in \mathbb{R}^n$. Then A satisfies the null space property (2.2) iff one has*

$$\forall I \subset [N],\ 1 \le |I| \le m, \forall (\varepsilon_i) \in \{-1,1\}^I,$$
$$\operatorname{conv}(\{\varepsilon_i X_i \,:\, i \in I\}) \cap \operatorname{conv}(\{\pm X_j \,:\, j \notin I\}) = \emptyset \tag{2.3}$$

Taking advantage of the symmetries, property (2.3) is equivalent to the following:

$$\forall I \subset [N],\ 1 \le |I| \le m, \forall (\varepsilon_i) \in \{-1,1\}^I,$$
$$\operatorname{Aff}(\{\varepsilon_i X_i \,:\, i \in I\}) \cap \operatorname{conv}(\{\pm X_j \,:\, j \notin I\}) = \emptyset \tag{2.4}$$

where $\operatorname{Aff}(\{\varepsilon_i X_i \,:\, i \in I\})$ denotes the affine subspace generated by $\{\varepsilon_i X_i \,:\, i \in I\}$.

The proof of this equivalence is left as exercise.

Definition 2.2.14. — *Let $1 \le m \le n$. A centrally symmetric polytope $P \subset \mathbb{R}^n$ is said to be symmetric m-neighborly if every set of m of its vertices, containing no antipodal pair, is the set of all vertices of some face of P.*

Note that any centrally symmetric polytope is symmetric 1-neighborly. Neighborliness property becomes non-trivial when $m \ge 2$. In that case, observe that $A(B_1^N) = \operatorname{conv}(\pm X_1, \dots, \pm X_N)$ has $\{\pm X_1, \dots, \pm X_N\}$ as vertices AND is symmetric m-neighborly iff A maps every $(m-1)$-dimensional face of B_1^N onto a $(m-1)$ dimensional face of $\operatorname{conv}(\pm X_1, \dots, \pm X_N)$. From (2.3) and (2.4), we deduce the following criterium.

Proposition 2.2.15. — *Let $1 \le m \le N$. The matrix A has the null space property of order m iff its columns $\pm X_1, \dots, \pm X_N$ are the $2N$ vertices of $A(B_1^N)$ and $A(B_1^N)$ is m-neighborly.*

Consider the quotient map

$$Q : \ell_1^N \longrightarrow \ell_1^N / \ker A$$

If A has maximum rank n, then $\ell_1^N / \ker A$ is n-dimensional. Denote by $\|\,.\,\|$ the quotient norm on $\ell_1^N / \ker A$ defined by

$$\|Qx\| = \min_{h \in \ker A} |x + h|_1.$$

Property (2.1) implies that Q is norm preserving on Σ_m. Since $\Sigma_{\lfloor m/2 \rfloor} - \Sigma_{\lfloor m/2 \rfloor} \subset \Sigma_m$, Q is an isometry on $\Sigma_{\lfloor m/2 \rfloor}$ equipped with the ℓ_1 metric. In other words,

$$\forall x, y \in \Sigma_{\lfloor m/2 \rfloor} \quad \|Qx - Qy\| = |x - y|_1.$$

As it is classical in approximation theory, we can take benefit of such an isometric embedding to bound the complexity by comparing the metric entropy of the source space $(\Sigma_{\lfloor m/2 \rfloor}, \ell_1^N)$ with the target space, which lives in a much lower dimension.

The following lemma is a well known fact on packing.

Lemma 2.2.16. — *There exists a family Λ of subset of $[N]$ with cardinality $m \leqslant N/2$ such that for every $I, J \in \Lambda, I \ne J, |I \cap J| \le \lfloor m/2 \rfloor$ and $|\Lambda| \ge \left\lfloor \frac{N}{32em} \right\rfloor^{\lfloor m/2 \rfloor}$.*

Proof. — We use successive enumeration of the subsets of cardinality m and exclusion of wrong items. Without loss of generality, assume that $m/2$ is an integer. Pick any subset I_1 of $\{1, ..., N\}$ of cardinality m and throw away all subsets J of $\{1, ..., N\}$ of size m such that the Hamming distance $|I_1 \vartriangle J| \leq m$, where Δ stands for the symmetrical difference. There are at most

$$\sum_{k=m/2}^{m} \binom{m}{k}\binom{N-m}{m-k}$$

such subsets and since $m \leq N/2$ we have

$$\sum_{k=m/2}^{m} \binom{m}{k}\binom{N-m}{m-k} \leq 2^m \max_{m/2\leq k\leq m} \binom{N-m}{m-k} \leq 2^m \binom{N}{m/2}.$$

Now, select a new subset I_2 of size m from the remaining subsets. Repeating this argument, we obtain a family $\Lambda = \{I_1, I_2, \dots, I_p\}$, $p = |\Lambda|$, of subsets of cardinality m which are $(m/2)$-separated in the Hamming metric and such that

$$|\Lambda| \geq \left\lfloor \binom{N}{m} \Big/ 2^m \binom{N}{m/2} \right\rfloor.$$

Since for $m \leq N/2$ we have $\left(\frac{N}{2m}\right)^m \leq \binom{N}{m} \leq \left(\frac{eN}{m}\right)^m$, we get that

$$|\Lambda| \geq \left\lfloor \frac{(N/2m)^m}{2^m (Ne/(m/2))^{(m/2)}} \right\rfloor \geq \left\lfloor \left(\frac{N}{32em}\right)^{m/2} \right\rfloor \geq \left\lfloor \frac{N}{32em} \right\rfloor^{\lfloor m/2 \rfloor}$$

which concludes the proof. □

Let Λ be the family constructed in the previous lemma. For every $I \in \Lambda$, define $x(I) = \frac{1}{m}\sum_{i\in I} e_i$. Then $x(I) \in S_1(\Sigma_m)$ and for every $I, J \in \Lambda, I \neq J$

$$|x(I) - x(J)|_1 = 2\left(1 - \frac{|I \cap J|}{m}\right) \geqslant 2\left(1 - \frac{[m/2]}{m}\right) \geq 1.$$

If the matrix A has the exact reconstruction property of order m, then

$$\forall I, J \in \Lambda\ I \neq J, \quad \|Q(x(I)) - Q(x(J))\| = \|Q(x(I) - x(J))\| = |x(I) - x(J)|_1 \geq 1.$$

On one side $|\Lambda| \geq \left\lfloor C\frac{N}{[m/2]} \right\rfloor^{[m/2]}$, but on the other side, the cardinality of the set $(Q(x(I)))_{I\in\Lambda}$ cannot be too big. Indeed, it is a subset of the unit ball $Q(B_1^N)$ of the quotient space and we already saw that the maximum cardinality of a set of points of a unit ball which are 1-apart is less than 3^n. It follows that

$$\lfloor N/32em \rfloor^{\lfloor m/2 \rfloor} \leq 3^n.$$

Proposition 2.2.17. — *If the matrix A has the exact reconstruction property of order m by ℓ_1-minimization, then*

$$m \log(cN/m) \leq Cn.$$

where $C, c > 0$ are universal constants.

Whatever is the matrix A, this proposition gives an upper bound on the size m of sparsity such that any vectors from Σ_m can be exactly reconstructed by ℓ_1-minimization method.

2.3. The restricted isometry property

So far, we do not know of any "simple" condition in order to check whether a matrix A satisfies the exact reconstruction property (2.1). Let us start with the following definition which plays an important role in compressed sensing.

Definition 2.3.1. — *Let A be a $n \times N$ matrix. For any $0 \leq p \leq N$, the restricted isometry constant of order p of A is the smallest number $\delta_p = \delta_p(A)$ such that*

$$(1-\delta_p)|x|_2^2 \leq |Ax|_2^2 \leq (1+\delta_p)|x|_2^2$$

for all p-sparse vectors $x \in \mathbb{R}^N$. Let $\delta \in (0,1)$. We say that the matrix A satisfies the Restricted Isometry Property of order p with parameter δ, shortly $\mathrm{RIP}_p(\delta)$, if $0 \leqslant \delta_p(A) < \delta$.

The relevance of the Restricted Isometry parameter is revealed in the following result:

Theorem 2.3.2. — *Let $1 \leqslant m \leqslant N/2$. Let A be an $n \times N$ matrix. If*

$$\delta_{2m}(A) < \sqrt{2}-1.$$

then A satisfies the exact reconstruction property of order m by ℓ_1-minimization.

For simplicity, we shall discuss an other parameter involving a more general concept. The aim is to relax the constraint $\delta_{2m}(A) < \sqrt{2}-1$, in Theorem 2.3.2 and still get an exact reconstruction property of a certain order by ℓ_1-minimization.

Definition 2.3.3. — *Let $0 \leq p \leq n$ be integers and let A be an $n \times N$ matrix. Define $\alpha_p = \alpha_p(A)$ and $\beta_p = \beta_p(A)$ as the best constants such that*

$$\forall x \in \Sigma_p, \quad \alpha_p|x|_2 \leq |Ax|_2 \leq \beta_p|x|_2.$$

Thus $\beta_p = \max\{|Ax|_2 : x \in \Sigma_p\ |x|_2 = 1\}$ and $\alpha_p = \min\{|Ax|_2 : x \in \Sigma_p\ |x|_2 = 1\}$. Now we define the parameter $\gamma_p = \gamma_p(A)$ by

$$\gamma_p(A) := \frac{\beta_p(A)}{\alpha_p(A)}.$$

In other words, let $I \subset [N]$ with $|I| = p$. Denote by A^I the $n \times p$ matrix with columns $(X_i)_{i \in I}$ obtained by extracting from A the columns X_i with index $i \in I$. Then α_p is the smallest singular value among all the block matrices A^I with $|I| = p$, and β_p is the largest. In other words, denoting by $B^\top$ the transposed matrix of a matrix B and $\lambda_{min}((A^I)^\top A^I)$, respectively $\lambda_{max}((A^I)^\top A^I)$, the smallest and largest eigenvalues of $(A^I)^\top A^I$, then

$$\alpha_p^2 = \alpha_p^2(A) = \min_{I \subset [N], |I|=p} \lambda_{min}((A^I)^\top A^I)$$

whereas

$$\beta_p^2 = \beta_p^2(A) = \max_{I \subset [N], |I|=p} \lambda_{max}((A^I)^\top A^I).$$

Of course, if A satisfies $\mathrm{RIP}_p(\delta)$, then $\gamma_p(A)^2 \leq \frac{1+\delta}{1-\delta}$. The concept of RIP is not homogenous, in the sense that A may satisfy $\mathrm{RIP}_p(\delta)$ but not a multiple of A. One can "rescale" the matrix to satisfy a Restricted Isometry Property. This does not ensure that the new matrix, say A' will satisfy $\delta_{2m}(A') < \sqrt{2} - 1$ and will not allow us to conclude to an exact reconstruction from Theorem 2.3.2 (compare with Corollary 2.4.3 in the next section). Also note that the Restricted Isometry Property for A can be written

$$\forall x \in S_2(\Sigma_p) \quad \big| |Ax|_2^2 - 1 \big| \leq \delta$$

expressing a form of concentration property of $|Ax|_2$. Such a property may not be satisfied despite the fact that A does satisfy the exact reconstruction property of order p by ℓ_1-minimization (see Example 2.6.6).

2.4. The geometry of the null space

Let $1 \leqslant m \leqslant p \leqslant N$. Let $h \in \mathbb{R}^N$ and let $\varphi = \varphi_h : [N] \to [N]$ be a one-to-one mapping associated to a non-increasing rearrangement of $(|h_i|)$; in others words $|h_{\varphi(1)}| \geq |h_{\varphi(2)}| \geq \cdots \geq |h_{\varphi(N)}|$. Denote by $I_1 = \varphi_h(\{1, \ldots, m\})$ (a subset of indices of the largest m coordinates of $(|h_i|)$) then by $I_2 = \varphi_h(\{m+1, \ldots, m+p\})$ (a subset of indices of the next p largest coordinates of $(|h_i|)$) and iterate $I_{k+1} = \varphi_h(\{m+(k-1)p+1, \ldots, m+kp\})$, for $k \geq 2$, as far as $m + kp \leq N$, in order to partition $[N]$ in subsets of cardinality p, except the first one, I_1 which has cardinality m and the last one, which may have cardinality not greater than p.

Claim 2.4.1. — *Let $h \in \mathbb{R}^N$. Suppose that $1 \leqslant m \leqslant p \leqslant N$ and $N \geq m + p$. With the previous notation, we have*

$$\forall k \geq 2, \quad |h_{I_{k+1}}|_2 \leq \frac{1}{\sqrt{p}} |h_{I_k}|_1$$

and

$$\sum_{k \geq 3} |h_{I_k}|_2 \leq \frac{1}{\sqrt{p}} \, |h_{I_1^c}|_1.$$

Proof. — Let $k \geq 1$. We have

$$|h_{I_{k+1}}|_2 \leq \sqrt{|I_{k+1}|} \, \max\{|h_i| \, : \, i \in I_{k+1}\}$$

and

$$\max\{|h_i| \, : \, i \in I_{k+1}\} \leq \min\{|h_i| \, : \, i \in I_k\} \leq |h_{I_k}|_1 / |I_k|.$$

We deduce that

$$\forall k \geq 1 \quad |h_{I_{k+1}}|_2 \leq \frac{\sqrt{|I_{k+1}|}}{|I_k|} |h_{I_k}|_1.$$

Adding up these inequalities for all $k \geq 2$, for which $\sqrt{|I_{k+1}|}/|I_k| \leqslant 1/\sqrt{p}$, this prove the claim. □

We are ready for the main Theorem of this section

Theorem 2.4.2. — *Let $1 \leqslant m \leqslant p \leqslant N$ and $N \geq m+p$. Let A be an $n \times N$ matrix. Then*

$$\forall h \in \ker A, h \neq 0, \quad \forall I \subset [N], |I| \leq m, \quad |h_I|_1 < \sqrt{\frac{m}{p}}\, \gamma_{2p}(A)\ |h_{I^c}|_1 \tag{2.5}$$

and $\forall h \in \ker A, h \neq 0,\ \forall I \subset [N], |I| \leq m,$

$$|h|_2 \leq \sqrt{\frac{1+\gamma_{2p}^2(A)}{p}}\ |h_{I_1^c}|_1 \leq \sqrt{\frac{1+\gamma_{2p}^2(A)}{p}}\ |h|_1. \tag{2.6}$$

In particular,

$$\mathrm{rad}\,(\ker A \cap B_1^N) \leq \sqrt{\frac{1+\gamma_{2p}^2(A)}{p}}$$

where $\mathrm{rad}\,(B) = \sup_{x \in B} |x|_2$.

Proof. — Let $h \in \ker A$, $h \neq 0$ and organize the coordinates of h as above. By definition of α_{2p} (see 2.3.3), one has

$$|h_{I_1} + h_{I_2}|_2 \leq \frac{1}{\alpha_{2p}} |A(h_{I_1} + h_{I_2})|_2.$$

Using that $h \in \ker A$ we obtain

$$|h_{I_1} + h_{I_2}|_2 \leq \frac{1}{\alpha_{2p}} |A(h_{I_1} + h_{I_2} - h)|_2 = \frac{1}{\alpha_{2p}} |A(-\sum_{k\geq 3} h_{I_k})|_2.$$

Then from the definition of β_p and γ_p (2.3.3), using Claim 2.4.1, we get

$$|h_{I_1}|_2 < |h_{I_1} + h_{I_2}|_2 \leq \frac{\beta_p}{\alpha_{2p}} \sum_{k\geq 3} |h_{I_k}|_2 \leq \frac{\gamma_{2p}(A)}{\sqrt{p}} |h_{I_1^c}|_1. \tag{2.7}$$

This inequality is strict because $h_{I_2} = 0$ would imply $h_{I_1^c} = 0$ and subsequently $h_{I_1} = 0$. To conclude the proof of (2.5), note that for any subset $I \subset [N], |I| \leq m$, $|h_{I_1^c}|_1 \leq |h_{I^c}|_1$ and $|h_I|_1 \leq |h_{I_1}|_1 \leq \sqrt{m}|h_{I_1}|_2$.

To prove (2.6), we start from

$$|h|_2^2 = |h - h_{I_1} - h_{I_2}|_2^2 + |h_{I_1} + h_{I_2}|_2^2$$

Using Claim (2.4.1), the first term satisfies

$$|h - h_{I_1} - h_{I_2}|_2 \leq \sum_{k\geq 3} |h_{I_k}|_2 \leq \frac{1}{\sqrt{p}}\, |h_{I_1^c}|_1.$$

From(2.7), $|h_{I_1} + h_{I_2}|_2 \leq \frac{\gamma_{2p}(A)}{\sqrt{p}}\ |h_{I_1^c}|_1$ and putting things together, we derive that

$$|h|_2 \leq \sqrt{\frac{1+\gamma_{2p}^2(A)}{p}}\ |h_{I_1^c}|_1 \leq \sqrt{\frac{1+\gamma_{2p}^2(A)}{p}}\ |h|_1.$$

□

¿From relation (2.5) and the null space property (Proposition 2.2.11) we derive the following corollary.

Corollary 2.4.3. — *Let $1 \leqslant p \leqslant N/2$. Let A be a $n \times N$ matrix. If $\gamma_{2p}(A) \le \sqrt{p}$, then A satisfies the exact reconstruction property of order m by ℓ_1-minimization with*

$$m = \left\lfloor p/\gamma_{2p}^2(A) \right\rfloor .$$

Our main goal now is to find p such that γ_{2p} is bounded by some numerical constant. This means that we need a uniform control of the smallest and largest singular values of all block matrices of A with $2p$ columns. By Corollary 2.4.3 this is a sufficient condition for the exact reconstruction of m-sparse vectors by ℓ_1-minimization with $m \sim p$. When $|Ax|_2$ satisfies good concentration properties, the restricted isometry property is more adapted. In this situation, $\gamma_{2p} \sim 1$. When the isometry constant δ_{2p} is sufficiently small, A satisfies the exact reconstruction of m-sparse vectors with $m = p$ (see Theorem 2.3.2).

Similarly, an estimate of $\mathrm{rad}\,(\ker A \cap B_1^N)$ gives an estimate of the size of sparsity of vectors which can be reconstructed by ℓ_1-minimization.

Proposition 2.4.4. — *Let A be an $n \times N$ matrix and let $1 \le m$. If*

$$\mathrm{rad}\,(\ker A \cap B_1^N) < \frac{1}{2\sqrt{m}}$$

then the matrix A satisfies the exact reconstruction property of order m by ℓ_1-minimization.

Proof. — Let $h \in \ker A$ and $I \subset [N], |I| \le m$. By our assumption, we have that

$$\forall h \in \ker A, h \neq 0 \quad |h|_2 < |h|_1/2\sqrt{m}.$$

Thus $|h_I|_1 \le \sqrt{m}\ |h_I|_2 \le \sqrt{m}\ |h|_2 < |h|_1/2$ and $|h_I|_1 < |h_{I^c}|_1$. We conclude using the null space property (Proposition 2.2.11). □

To conclude the section, note that 2.6 implies that if a $n \times N$ matrix A satisfies a restricted isometry property of order $m \ge 1$, then $\mathrm{rad}\,(\ker A \cap B_1^N) = \frac{O(1)}{\sqrt{m}}$.

2.5. Gelfand widths

The study of the previous section leads to the notion of Gelfand widths.

Definition 2.5.1. — *Let T be a bounded subset of a normed space E. Let $k \ge 0$ be an integer. Its k-th Gelfand width is defined as*

$$d^k(T, E) := \inf_S \sup_{x \in S \cap T} \|x\|_E,$$

where $\|\,.\,\|_E$ denotes the norm of E and where the infimum is taken over all linear subspaces S of codimension $\le k$.

A different notation is used in Banach space and Operator Theory. Let $u : X \longrightarrow Y$ be an operator between two normed spaces X and Y. The k-th Gelfand number is defined by

$$c_k(u) = \inf\{\| \, u_{|S} \, \| : S \subset X, \operatorname{codim} S < k\}$$

where $u_{|S}$ denotes the restriction of the operator u under S. This reads equivalently as

$$c_k(u) = \inf_S \sup_{x \in S \cap B_X} \|u(x)\|_Y,$$

where B_X denotes the unit ball of X and the infimum is taken over all subspaces S of X with codimension $< k$. Thus, the different notation are related by

$$c_{k+1}(u) = d^k(u(B_X), Y).$$

If F is a linear space ($\mathbb{R}^N$ for instance) equipped with two norms defining two normed spaces X and Y and if $id : X \to Y$ is the identity mapping of F considered from the normed spaces X to Y, then

$$d^k(B_X, Y) = c_{k+1}(id : X \to Y).$$

As a particular but important instance, we have

$$d^k(B_1^N, \ell_2^N) = c_{k+1}(id : \ell_1^N \to \ell_2^N) = \inf_{\operatorname{codim} S \leqslant k} \operatorname{rad}(S \cap B_1^N).$$

The study of these numbers attracted a lot of attention during the seventies and the eighties. An important result is the following

Theorem 2.5.2. — *There exist $c, C > 0$ such that for any integers $1 \leq k \leq N$,*

$$c \min\left\{1, \sqrt{\frac{\log(N/k)}{k}}\right\} \leq c_k(id : \ell_1^N \to \ell_2^N) \leq C \min\left\{1, \sqrt{\frac{\log(N/k)}{k}}\right\}.$$

Moreover, if $\mathbb{P}$ is the rotation invariant probability measure on the Grassmann manifold of subspaces S of $\mathbb{R}^N$ with $\operatorname{codim}(S) = k - 1$, *then*

$$\mathbb{P}\left(\operatorname{rad}(S \cap B_1^N) \leq C \min\left\{1, \sqrt{\frac{\log(N/k)}{k}}\right\}\right) \geq 1 - \exp(-ck).$$

Coming back to compressed sensing, let $1 \leqslant m \leqslant n$ and let us assume that

$$d^n(B_1^N, \ell_2^N) < \frac{1}{2\sqrt{m}}.$$

In other words, we assume that there is a subspace $S \subset \mathbb{R}^N$ of codimension $\leqslant n$ such that $\operatorname{rad}(S \cap B_1^N) < \frac{1}{2\sqrt{m}}$. Choose any $n \times N$ matrix A such that $\ker A = S$, then

$$\operatorname{rad}(\ker A \cap B_1^N) < \frac{1}{2\sqrt{m}}.$$

Proposition 2.4.4 shows A satisfies the exact reconstruction property of order m by ℓ_1-minimization.

We deduce from Theorem 2.5.2 that there exists a matrix A satisfying the exact reconstruction property of order

$$\lfloor c_1 n / \log(c_2 N/n) \rfloor$$

where c_1, c_2 are universal constants. From Proposition 2.2.17 it is the optimal order.

2.6. Gaussian random matrices satisfy a RIP

So far, we did not give yet any example of matrices satisfying the exact reconstruction property of order m with large m. It is known that with high probability Gaussian matrices do satisfy this property.

The subgaussian Ensemble. — We consider a probability $\mathbb{P}$ on the space of real $n \times N$ matrices $M(n,N)$ satisfying the following concentration inequality: there exists an absolute constant c_0 such that for every $x \in \mathbb{R}^N$ we have

$$\mathbb{P}\left(\{A : \left||Ax|_2^2 - |x|_2^2\right| \geq t|x|_2^2\}\right) \leq 2e^{-c_0 t^2 n} \qquad \text{for all } 0 < t \leq 1. \tag{2.8}$$

Definition 2.6.1. — *For a real random variable Z we define the ψ_2-norm by*

$$\|Z\|_{\psi_2} = \inf\left\{s > 0 \,:\, \mathbb{E}\exp\left(|Z|/s\right)^2 \leqslant e\right\}.$$

We say that a random vector $Y \in \mathbb{R}^N$ is isotropic if it is centered and

$$\forall y \in \mathbb{R}^N, \quad \mathbb{E}|\langle Y, y\rangle|^2 = |y|_2^2.$$

A random vector $Y \in \mathbb{R}^N$ satisfies a ψ_2 estimate with constant α (shortly Y is ψ_2 with constant α) if

$$\forall y \in \mathbb{R}^N, \quad \|\langle Y, y\rangle\|_{\psi_2} \leqslant \alpha |y|_2.$$

It is well-known that a real random variable Z is ψ_2 (with some constant) if and only if it satisfies a subgaussian tail estimate. In particular if Z is a real random variable with $\|Z\|_{\psi_2} \leq \alpha$, then for every $t \geq 0$,

$$\mathbb{P}(|Z| \geq t) \leq e^{-(t/\alpha)^2 + 1}$$

This ψ_2 property can also be characterized by the growth of moments. Well known examples are Gaussian random variables and bounded centered random variables (see Chapter 1 for details).

Let us consider $Y_1, \ldots, Y_n \in \mathbb{R}^N$ be i.i.d. isotropic random vectors which are ψ_2 with the same constant α. Let A be the matrix with $Y_1, \ldots, Y_n \in \mathbb{R}^N$ as rows. We consider the probability $\mathbb{P}$ on the space of matrices $M(n,N)$ induced by the mapping $(Y_1, \ldots, Y_n) \to A$.

Let us recall Bernstein's inequality (see Chapter 1). For $y \in S^{N-1}$ consider the average of n independent copies of the random variable $\langle Y_1, y\rangle^2$. Then for every $t > 0$,

$$\mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^n \langle Y_i, y\rangle^2 - 1\right| > t\right) \leqslant 2\exp\left(-cn\min\left\{\frac{t^2}{\alpha^4}, \frac{t}{\alpha^2}\right\}\right),$$

where c is an absolute constant. Note that since $\mathbb{E}\langle Y_1, y\rangle^2 = 1$, one has $\alpha \geqslant 1$ and that $\left|\frac{Ay}{\sqrt{n}}\right|_2^2 = \frac{1}{n}\sum_{i=1}^n \langle Y_i, y\rangle^2$. This shows the next claim:

Claim 2.6.2. — *Let $Y_1, \ldots, Y_n \in \mathbb{R}^N$ be i.i.d. isotropic random vectors that are ψ_2 with constant α. Let $\mathbb{P}$ be the probability induced on $M(n, N)$. Then for every $x \in \mathbb{R}^N$ we have*

$$\mathbb{P}\left(\left|\left|\frac{A}{\sqrt{n}}x\right|_2^2 - |x|_2^2\right| \geq t|x|_2^2\right) \leq 2e^{-\frac{c}{\alpha^4}t^2 n} \qquad \text{for all } 0 < t \leq 1$$

where $c > 0$ is an absolute constant.

The most important examples for us of model of random matrices satisfying (2.8) are matrices with independent subgaussian rows, normalized in the right way.

Example 2.6.3. — *Some classical examples:*

- *$Y_1, \ldots, Y_n \in \mathbb{R}^N$ are independent copies of the Gaussian vector $Y = (g_1, \ldots, g_N)$ where the g_i's are independent $\mathcal{N}(0,1)$ Gaussian variables*
- *$Y_1, \ldots, Y_n \in \mathbb{R}^N$ are independent copies of the random sign vector $Y = (\varepsilon_1, \ldots, \varepsilon_N)$ where the ε_i's are independent, symmetric ± 1 (Bernoulli) random variables*
- *$Y_1, \ldots, Y_n \in \mathbb{R}^N$ are independent copies of a random vector uniformly distributed on the Euclidean sphere of radius $\sqrt{N}$.*

In all these cases the (Y_i) are isotropic with a ψ_2 constant α, for a suitable $\alpha \geqslant 1$. For the last case see e.g. [**LT91**]. For more details on Orlicz norm and probabilistic inequalities used here see Chapter 1.

Sub-Gaussian matrices are almost norm preserving on Σ_m. — An important feature of Σ_m and its subsets $S_2(\Sigma_m)$ and $B_2(\Sigma_m)$ is their peculiar structure: the two last are the unions of the unit spheres, and unit balls, respectively, supported on m-dimensional coordinate subspaces of $\mathbb{R}^N$.

We begin with the following well known lemma (see Chapter 1) which allows to step up from an ε-net to the whole unit sphere.

Lemma 2.6.4. — *Let $m \geq 1$ be an integer, $\|\,.\,\|$ be a semi-norm in $\mathbb{R}^m$ and $\varepsilon \in (0, 1/3)$. Let $\Lambda \subset S^{m-1}$ be an ε-net of S^{m-1} by εB_2^m. If*

$$\forall y \in \Lambda \quad 1 - \varepsilon \leq \|y\| \leq 1 + \varepsilon,$$

then

$$\forall y \in S^{m-1} \quad \frac{1 - 3\varepsilon}{1 - \varepsilon} \leq \|y\| \leq \frac{1 + \varepsilon}{1 - \varepsilon}.$$

Proof. — Proposition 2.2.7 implies that $S^{m-1} \subset (1-\varepsilon)^{-1}\mathrm{conv}\,\Lambda$. Therefore we have

$$\sup_{y \in S^{m-1}} \|y\| \leq (1 + \varepsilon)(1 - \varepsilon)^{-1}.$$

To get a lower estimate, write any $y \in S^{m-1}$ as $y = y_1 + \varepsilon y_2$, with $y_1 \in \Lambda$ and $y_2 \in B_2^m$. Then $\|y\| \geq \|y_1\| - \varepsilon\|y_2\| \geq (1 - \varepsilon) - \varepsilon(1 + \varepsilon)(1 - \varepsilon)^{-1} = (1 - 3\varepsilon)/(1 - \varepsilon)$ which proves the claim. $\square$

We can give now a simple proof that subgaussian matrices satisfy the exact reconstruction property of order m by ℓ_1-minimization with large m.

Theorem 2.6.5. — *Let $\mathbb{P}$ be a probability on $M(n, N)$ satisfying (2.8). Then there exist positive constants c_1, c_2 and c_3 depending only on c_0 from (2.8), for which the following holds: with probability at least $1 - 2\exp(-c_3 n)$, A satisfies the exact reconstruction property of order m by ℓ_1-minimization with*

$$m = \left\lfloor \frac{c_1 n}{\log(c_2 N/n)} \right\rfloor.$$

Moreover, A satisfies $RIP_m(\delta)$ for any $\delta \in (0,1)$ with $m \sim c\delta^2 n/\log(CN/\delta^3 n)$ where c and C depend only on c_0.

Proof. — Let $\varepsilon \in (0, 1/3)$ to be fixed later. Let $1 \le p \le N/2$. Let y_i, $i = 1, 2, \ldots, n$, be the rows of A. For every subset I of $[N]$ of cardinality $2p$ let Λ_I be an ε-net of the unit sphere of $\mathbb{R}^I$ by εB_2^I satisfying Claim 2.2.8, that is with $|\Lambda_I| \le \left(\frac{3}{\varepsilon}\right)^{2p}$. Apply Lemma 2.6.4 to the semi-norm

$$||y|| := \left(\frac{1}{n}\sum_{i=1}^n \langle y_i, y\rangle^2\right)^{1/2}$$

on the unit sphere of $\mathbb{R}^I$. Let $\Lambda \subset \mathbb{R}^N$ be the union of all these Λ_I for $|I| = 2p$.

Suppose that

$$\sup_{y\in\Lambda} \Big| \frac{1}{n}\sum_{i=1}^n (\langle y_i, y\rangle^2 - 1) \Big| \le \varepsilon,$$

then

$$\forall y \in S_2(\Sigma_{2p}) \qquad \frac{1-3\varepsilon}{1-\varepsilon} \le \frac{1}{\sqrt{n}} \big(\sum_{i=1}^n \langle y_i, y\rangle^2\big)^{1/2} \le \frac{1+\varepsilon}{1-\varepsilon}.$$

Note that there is nothing random in that relation. This is why we change the notation of the rows from (Y_i) to (y_i). Thus checking how well the matrix A defined by the rows (y_i) is acting on Σ_{2p} is reduced to checking that on the finite set Λ. Now recall that $|\Lambda| \le \binom{N}{2p}\left(\frac{3}{\varepsilon}\right)^{2p} \le \exp\left(2p\log\left(\frac{3eN}{2p\varepsilon}\right)\right)$.

Given a probability $\mathbb{P}$ on $M(n, N)$ satisfying (2.8), and using a union bound estimate, we get that the inequalities

$$\forall x \in S_2(\Sigma_{2p}) \quad \frac{1-3\varepsilon}{1-\varepsilon} \le |Ax|_2 \le \frac{1+\varepsilon}{1-\varepsilon}$$

hold with probability at least

$$1 - 2|\Lambda|e^{-c_0\varepsilon^2 n} \ge 1 - 2\exp\left(2p\log\left(\frac{3eN}{2p\varepsilon}\right)\right)e^{-c_0\varepsilon^2 n} \ge 1 - 2e^{-c_0\varepsilon^2 n/2}$$

whenever

$$2p\log\left(\frac{3eN}{2p\varepsilon}\right) \le c_0\varepsilon^2 n/2.$$

Assuming these inequalities, we get

$$\gamma_{2p}(A) \le (1+\varepsilon)/(1-3\varepsilon)$$

with probability larger than $1-\exp(-c_0\varepsilon^2 n/2)$. ¿From Corollary 2.4.3, we deduce that A satisfies the exact reconstruction property of order m by ℓ_1-minimization with

$$m = \lfloor p/\gamma_{2p}(A)^2 \rfloor.$$

This gives the announced result by fixing ε (say $\varepsilon = 1/4$) and solving $2p \log\left(\frac{3eN}{2p\varepsilon}\right) \le c_0\varepsilon^2 n/2$. □

The strategy that we used in the proof of Theorem 2.6.5 is the following:

- *discretization*: discretization of the set Σ_{2p}, it is a net argument
- *concentration*: $|Ax|_2$ concentrates around its mean for each individual x of the net
- *union bound*: concentration should be good enough to balance the cardinality of the net and to conclude to a uniform concentration on the net of $|Ax|_2$ around its mean
- *from the net to the whole set*, that is checking RIP, is obtained by Lemma 2.6.4.

We conclude this section by an example of an $n \times N$ matrix A which is a good compressed sensing matrix but none of the $n \times N$ matrices with the same kernel as A satisfy a restricted isometry property of any order ≥ 1 with good parameter. As we already noticed, if A has parameter γ_p, one can find $t_0 > 0$ and rescale the matrix so that $\delta_p(t_0 A) = \gamma_p^2 - 1/\gamma_p^2 + 1 \in [0,1)$. In this example, γ_p is large, $\delta_p(t_0 A) \sim 1$ and one cannot deduce any result about exact reconstruction from Theorem 2.3.2.

Example 2.6.6. — *Let $1 \le n \le N$. Let $\delta \in (0,1)$. There exists an $n \times N$ matrix A such that for any $p \le cn/\log(CN/n)$, one has $\gamma_{2p}(A)^2 \le c'(1-\delta)^{-1}$. Thus, for any $m \le c"(1-\delta)n/\log(CN/n)$, the matrix A satisfies the exact reconstruction property of m-sparse vectors by ℓ_1-minimization. Nevertheless, for any $n \times n$ matrix U, the restricted isometry constant of order 1 of UA satisfies, $\delta_1(UA) \ge \delta$ (think of $\delta \ge 1/2$). Here, $C, c, c', c" > 0$ are universal constants.*

The proof is left as exercise.

2.7. RIP for other "simple" subsets: almost sparse vectors

As already mentioned, various "random projection" operators may act as "almost norm preserving" on "thin" subsets of the sphere. We analyze a simple structure of the metric entropy of a set $T \subset \mathbb{R}^N$ in order that, with high probability, (a multiple of) Gaussian or subgaussian matrices act almost like an isometry on T. This will apply to a more general case than sparse vectors.

Theorem 2.7.1. — *Consider a probability on the space of $n \times N$ matrices satisfying*

$$\forall x \in \mathbb{R}^N \qquad \mathbb{P}\left(\left||Ax|_2^2 - |x|_2^2\right| \ge t|x|_2^2\right) \le 2e^{-c_0 t^2 n} \qquad \text{for all } 0 < t \le 1.$$

Let $T \subset S^{N-1}$ and $0 < \varepsilon < 1/15$. Assume the following:

(i) *There exists an ε-net $\Lambda \subset S^{N-1}$ of T satisfying $|\Lambda| \leq \exp(c_0\varepsilon^2 n/2)$*
(ii) *There exists a subset Λ' of εB_2^N such that $(T-T) \cap \varepsilon B_2^N \subset 2 \operatorname{conv} \Lambda'$ and $|\Lambda'| \leq \exp(c_0 n/2)$.*

Then with probability at least $1 - 3\exp(-c_0\varepsilon^2 n/2)$, one has that for all $x \in T$,

$$1 - 15\varepsilon \leq |Ax|_2^2 \leq 1 + 15\varepsilon. \tag{2.9}$$

Proof. — The idea is to show that A acts on Λ in an almost norm preserving way. This is the case because the degree of concentration of each variable $|Ax|_2^2$ around its mean defeats the cardinality of Λ. Then one shows that $A(\operatorname{conv}\Lambda')$ is contained in a small ball - thanks to a similar argument.

Consider the set Ω of matrices A such that

$$||Ax_0|_2 - 1| \leq \left||Ax_0|_2^2 - 1\right| \leq \varepsilon \qquad \text{for all } \ x_0 \in \Lambda, \tag{2.10}$$

and

$$|Az|_2 \leq 2\varepsilon \qquad \text{for all } \ z \in \Lambda'. \tag{2.11}$$

¿From our assumption (2.8), i) and ii)

$$\mathbb{P}(\Omega) \geq 1 - 2\exp(-c_0\varepsilon^2 n/2) - \exp(-c_0 n/2) \geq 1 - 3\exp(-c_0\varepsilon^2 n/2).$$

Let $x \in T$ and consider $x_0 \in \Lambda$ such that $|x - x_0|_2 \leq \varepsilon$. Then for every $A \in \Omega$

$$|Ax_0|_2 - |A(x-x_0)|_2 \leq |Ax|_2 \leq |Ax_0|_2 + |A(x-x_0)|_2\,.$$

Since $x - x_0 \in (T-T) \cap \varepsilon B_2^N$, property ii) and (2.11) give that

$$|A(x-x_0)|_2 \leq 2 \sup_{z \in \operatorname{conv}\Lambda'} |Az|_2 = 2\sup_{z\in\Lambda'} |Az|_2 \leq 4\varepsilon. \tag{2.12}$$

Combining this with (2.10) implies that $1 - 5\varepsilon \leq |Ax|_2 \leq 1 + 5\varepsilon$. The proof is completed by squaring. □

Approximate reconstruction of almost sparse vectors. — After analyzing the restricted isometry property for thin sets of the type of Σ_m, we look again at the ℓ_1-minimization method in order to get approximate reconstruction of vectors which are not far from sparse vectors. As well as for the exact reconstruction, approximate reconstruction depends on a null-space property.

Proposition 2.7.2. — *Let A be a $n \times N$ matrix and $\lambda \in (0,1)$. Assume that*

$$\forall h \in \ker A, h \neq 0, \forall I \subset [N], |I| \leq m, |h_I|_1 \leqslant \lambda |h_{I^c}|_1. \tag{2.13}$$

Let $x \in \mathbb{R}^N$ and let $x^\sharp$ be a minimizer of

$$(P) \qquad \min_{t\in\mathbb{R}^N} |t|_1 \quad \textit{subject to} \quad At = Ax.$$

Then for any $I \subset [N], |I| \leqslant m$,

$$|x - x^\sharp|_1 \leq 2\,\frac{1+\lambda}{1-\lambda}\,|x - x_I|_1.$$

Proof. — Let $x^\sharp$ be a minimizer of (P) and set $h = x^\sharp - x \in \ker A$. Let $m \geqslant 1$ and $I \subset [N]$ such that $|I| \leqslant m$. Observe that

$$|x|_1 \geq |x + h|_1 = |x_I + h_I|_1 + |x_{I^c} + h_{I^c}|_1 \geq |x_I|_1 - |h_I|_1 + |h_{I^c}|_1 - |x_{I^c}|_1$$

and thus

$$|h_{I^c}|_1 \leq |h_I|_1 + 2|x_{I^c}|_1.$$

On the other hand, from the null space assumption, we get

$$|h_{I^c}|_1 \leq |h_I|_1 + 2|x_{I^c}|_1 \leq \lambda |h_{I^c}|_1 + 2|x_{I^c}|_1.$$

Therefore

$$|h_{I^c}|_1 \leq \frac{2}{1-\lambda}\, |x_{I^c}|_1.$$

Since the null space assumption reads equivalently $|h|_1 \leqslant (1+\lambda)\, |h_{I^c}|_1$, we can conclude the proof. □

Note that the minimum of $|x - x_I|_1$ over all subsets I such that $|I| \leq m$, is obtained when I is the support of the m-largest coordinates of x. The vector x_I is henceforth the best m-sparse approximation of x (in the ℓ_1 norm). Note also that if x is m-sparse then we go back to the exact reconstruction scheme.

Property (2.13), which is a strong form of the null space property, may be studied by means of parameters such as the Gelfand diameters. This gives us the next proposition.

Proposition 2.7.3. — *Let A be a $n \times N$ matrix and $1 \leq m \leq n$. Let $x \in \mathbb{R}^N$ and let $x^\sharp$ be a minimizer of*

$$(P) \qquad \min_{t \in \mathbb{R}^N} |t|_1 \quad \textit{subject to} \quad At = Ax.$$

Let $\rho = \operatorname{rad}(B_1^N \cap \ker A) = \sup_{x \in B_1^N \cap \ker A} |x|_2$. Assume that $\rho \leqslant 1/4\sqrt{m}$ then for any $I \subset [N], |I| \leqslant m$,

$$|x - x^\sharp|_1 \leq 4\, |x - x_I|_1$$

and

$$|x - x^\sharp|_2 \leq \frac{1}{\sqrt{m}}\, |x - x_I|_1.$$

Proof. — Let $h \in \ker A$. We have

$$|h_I|_1 \leq \sqrt{m}\, |h_I|_2 \leqslant \sqrt{m}\, |h|_2 \leqslant \sqrt{m}\rho |h|_1.$$

Therefore

$$|h_I|_1 \leq \frac{\rho\sqrt{m}}{1 - \rho\sqrt{m}}\, |h_{I^c}|_1$$

whenever $\rho\sqrt{m} < 1$. We deduce that Property (2.13) is satisfied with $\lambda = \frac{\rho\sqrt{m}}{1-\rho\sqrt{m}}$. The inequality $|x - x^\sharp|_1 \leq 4\, |x - x_I|_1$ follows directly from Proposition 2.7.2 and the assumption $\rho \leqslant 1/4\sqrt{m}$. The relation $|h|_2 \leqslant \rho |h|_1 \leq 4\rho |x - x_I|_1$ concludes the proof of the last inequality. □

Let $1 \leq m \leq p \leq n$ and $N \geqslant m+p$. The last proposition could be reformulated in terms of the constant of the restricted isometry property or in terms of the parameter γ_p, since from (2.6),

$$\rho \leq \sqrt{\frac{1+\gamma_{2p}^2(A)}{p}},$$

but we shall not go any further ahead.

Remark 2.7.4. — *To sum up, Theorem 2.4.2 shows that if a $n\times N$ matrix A satisfies a restricted isometry property of order $m \geq 1$, then*

$$\operatorname{rad}(\ker A \cap B_1^N) = \frac{O(1)}{\sqrt{m}}. \tag{2.14}$$

On the other hand, Propositions 2.4.4 and 2.7.3 show that if a $n\times N$ matrix A satisfies (2.14), then the matrix A satisfies the exact reconstruction property of order $O(m)$ by ℓ_1-minimization as well as an approximate reconstruction property.

Based on this remark, we could focus on estimates of the diameters, but the example of Gaussian matrices shows that it may be easier to prove a restricted isometry property than computing widths. We conclude this section by an application of Proposition 2.7.3.

Corollary 2.7.5. — *Let $0 < p < 1$ and consider*

$$T = B_{p,\infty}^N = \left\{x = (x_1, \dots, x_N) \in \mathbb{R}^N \,:\, |\{i \,:\, |x_i| \geq s\}| \leq s^{-p} \text{ for all } \; s > 0\right\}$$

the unit ball of $\ell_{p,\infty}^N$. Let A be a $n \times N$ matrix and $1 \leq m \leq n$. Let $x \in T$ and let $x^\sharp$ be a minimizer of

$$(P) \qquad \min_{t\in\mathbb{R}^N} |t|_1 \quad \text{subject to} \quad At = Ax.$$

Let $\rho = \operatorname{rad}(B_1^N \cap \ker A) = \sup_{x\in B_1^N\cap\ker A} |x|_2$ and assume that $\rho \leqslant 1/4\sqrt{m}$, then

$$|x - x^\sharp|_2 \leqslant ((1/p) - 1)^{-1} m^{1/2-1/p}.$$

Proof. — Observe that for any $x \in B_{p,\infty}^N$, one has $x_i^* \leq 1/i^{1/p}$, for every $i \geq 1$, where $(x_i^*)_{i=1}^N$ is a non-increasing rearrangement of $(|x_i|)_{i=1}^N$. Let $I \subset [N]$, such that $|I| = m$ and let x_I be the best m-sparse approximation of x. Note that $\sum_{i>m} i^{-1/p} \leq (1/p - 1)^{-1} m^{1-1/p}$. ¿From Proposition 2.7.3, we get that if $\rho \leqslant 1/4\sqrt{m}$ and if $x^\sharp$ is a minimizer of (P), then

$$|x - x^\sharp|_2 \leq \frac{1}{\sqrt{m}} |x - x_I|_1 \leqslant ((1/p) - 1)^{-1} m^{1/2-1/p}.$$

$\square$

Reducing the computation of Gelfand widths by truncation. — We begin with a simple principle which reduces the computation of Gelfand widths to the width of a truncated set.

Definition 2.7.6. — *We say that a subset $T \subset \mathbb{R}^N$ is star-shaped in* 0 *or shortly, star-shaped, if $\lambda T \subset T$ for every $0 \leq \lambda \leq 1$. Let $\rho > 0$ and let $T \subset \mathbb{R}^N$ be star-shaped, we denote by T_ρ the subset*

$$T_\rho = T \cap \rho S^{N-1}.$$

Recall that $\operatorname{rad}(S) = \sup_{x \in S} |x|_2$.

Lemma 2.7.7. — *Let $\rho > 0$ and let $T \subset \mathbb{R}^N$ be star-shaped. Then for any linear subspace $E \subset \mathbb{R}^N$ such that $E \cap T_\rho = \emptyset$ we have* $\operatorname{rad}(E \cap T) < \rho$.

Proof. — If $\operatorname{rad}(E \cap T) \geqslant \rho$, there would be $x \in E \cap T$ of norm greater or equal to ρ. Since T is star-shaped, so is $E \cap T$ and thus $x/|x|_2 \in E \cap T_\rho$; a contradiction. □

This easy lemma will be a useful tool in the next sections and in Chapter 5. The subspace E will be the kernel of our matrix A, ρ a parameter that we try to estimate as small as possible such that $\ker A \cap T_\rho = \emptyset$, that is such that $Ax \neq 0$ for all $x \in T$ with $|x|_2 = \rho$. This will be in particular the case if A or a multiple of A acts on T_ρ in an almost norm preserving way.

With Theorem 2.7.1 in mind, we apply this plan to subsets T like Σ_m.

Corollary 2.7.8. — *Let $\mathbb{P}$ be a probability on $M(n, N)$ satisfying (2.8). Consider a star-shaped set $T \subset \mathbb{R}^N$ and let $\rho > 0$. Assume that $\frac{1}{\rho} T_\rho \subset S^{N-1}$ satisfies the hypothesis of Theorem 2.7.1 for some $0 < \varepsilon < 1/15$. Then* $\operatorname{rad}(\ker A \cap T) < \rho$, *with probability at least $1 - 2\exp(-cn)$ where $c > 0$ is an absolute constant.*

Application to subsets related to ℓ_p unit balls. — To illustrate this method, we consider some examples of set T, for $0 < p < 2$:

- the unit ball of ℓ_1^N, denoted by B_1^N
- the unit ball $B_p^N = \{x \in \mathbb{R}^N : \sum_1^N |x_i|^p \leq 1\}$ of ℓ_p^N, $0 < p < 1$
- the unit ball $B_{p,\infty}^N = \left\{x \in \mathbb{R}^N : |\{i : |x_i| \geq s\}| \leq s^{-p} \text{ for all } s > 0\right\}$ of $\ell_{p,\infty}^N$ (weak ℓ_p^N), for $0 < p < 1$.

Note that for $0 < p < 1$, the "unit ball" B_p^N is not convex and that $B_p^N \subset B_{p,\infty}^N$, so that for estimating Gelfand widths, we can restrict to the balls $B_{p,\infty}^N$.

We need two lemmas. The first uses the following classical fact:

Claim 2.7.9. — *Let $(a_i), (b_i)$ two sequences of positive numbers such that (a_i) is non-increasing. Then the sum $\sum a_i b_{\pi(i)}$ is maximized over all permutations π of the index set, if $b_{\pi(1)} \geq b_{\pi(2)} \geq \ldots$.*

Lemma 2.7.10. — *Let $0 < p < 1$, $1 \leqslant m \leqslant N$ and set $r = (1/p - 1)m^{1/p-1/2}$. Then, for every $x \in \mathbb{R}^N$,*

$$\sup_{z \in rB_{p,\infty}^N \cap B_2^N} \langle x, z\rangle \leqslant 2\left(\sum_{i=1}^m x_i^{*2}\right)^{1/2},$$

where $(x_i^*)_{i=1}^N$ is a non-increasing rearrangement of $(|x_i|)_{i=1}^N$. Equivalently,

$$rB_{p,\infty}^N \cap B_2^N \subset 2\,\mathrm{conv}\,(S_2(\Sigma_m)). \tag{2.15}$$

Moreover one has

$$\sqrt{m}B_1^N \cap B_2^N \subset 2\,\mathrm{conv}\,(S_2(\Sigma_m)). \tag{2.16}$$

Proof. — We treat only the case of $B_{p,\infty}^N$, $0 < p < 1$. The case of B_1^N is similar. Note first that if $z \in B_{p,\infty}^N$, then for any $i \geq 1$, $z_i^* \leq 1/i^{1/p}$, where $(z_i^*)_{i=1}^N$ is a non-increasing rearrangement of $(|z_i|)_{i=1}^N$. Using Claim 2.7.9 we get that for any $r > 0, m \geq 1$ and $z \in rB_{p,\infty}^N \cap B_2^N$,

$$\begin{aligned}
\langle x, z\rangle &\leqslant \left(\sum_{i=1}^m x_i^{*2}\right)^{1/2} + \sum_{i>m} \frac{rx_i^*}{i^{1/p}} \\
&\leqslant \left(\sum_{i=1}^m x_i^{*2}\right)^{1/2} \left(1 + \frac{r}{\sqrt{m}} \sum_{i>m} \frac{1}{i^{1/p}}\right) \\
&\leqslant \left(\sum_{i=1}^m x_i^{*2}\right)^{1/2} \left(1 + \left(\frac{1}{p} - 1\right)^{-1} \frac{r}{m^{1/p-1/2}}\right).
\end{aligned}$$

By the definition of r, this completes the proof. □

The second lemma shows that $m^{1/p-1/2}B_{p,\infty}^N \cap S^{N-1}$ is well approximated by vectors on the sphere with short support.

Lemma 2.7.11. — *Let $0 < p < 2$ and $\delta > 0$, and set $\varepsilon = 2(2/p-1)^{-1/2}\delta^{1/p-1/2}$. Let $1 \leq m \leq N$. Then $S_2(\Sigma_{\lceil m/\delta\rceil})$ is an ε-net of $m^{1/p-1/2}B_{p,\infty}^N \cap S^{N-1}$ with respect to the Euclidean metric.*

Proof. — Let $x \in m^{1/p-1/2}B_{p,\infty}^N \cap S^{N-1}$ and assume without loss of generality that $x_1 \geq x_2 \geq \ldots \geq x_n \geq 0$. Define z' by $z_i' = x_i$ for $1 \leq i \leq \lceil m/\delta\rceil$ and $z_i' = 0$ otherwise. Then

$$|x - z'|_2^2 = \sum_{i>m/\delta} |x_i|^2 \leq m^{2/p-1} \sum_{i>m/\delta} 1/i^{2/p} < (2/p-1)^{-1}\,\delta^{2/p-1}.$$

Thus $1 \geq |z'|_2 \geq 1 - (2/p-1)^{-1/2}\,\delta^{1/p-1/2}$. Put $z = z'/|z'|_2$. Then $z \in S_2(\Sigma_{\lceil m/\delta\rceil})$ and

$$|z - z'|_2 = 1 - |z'|_2 \leq (2/p-1)^{-1/2}\,\delta^{1/p-1/2}.$$

By the triangle inequality $|x - z|_2 < \varepsilon$, completing the proof. □

The preceding lemmas are used to show that the hypothesis of Theorem 2.7.1 are satisfied for an appropriate choice of T and ρ. Before that, property ii) of Theorem 2.7.1, brings us to the following definition.

Definition 2.7.12. — *We say that a subset T of $\mathbb{R}^N$ is quasi-convex with constant $a \geq 1$, if T is star-shaped and $T + T \subset 2aT$.*

Let us note the following easy fact.

Claim 2.7.13. — *Let $0 < p < 1$, then $B_{p,\infty}^N$ and B_p^N are quasi-convex with constant $2^{(1/p)-1}$.*

We come up now with the main claim:

Claim 2.7.14. — *Let $0 < p < 1$ and $T = B_{p,\infty}^N$. Then $(1/\rho)T_\rho$ satisfies properties i) and ii) of Theorem 2.7.1 with*

$$\rho = C_p \left(\frac{n}{\log(cN/n)} \right)^{1/p-1/2}$$

where C_p depends only on p and $c > 0$ is an absolute constant. Moreover if $T = B_1^N$, then $(1/\rho)T_\rho$ satisfies properties i) and ii) of Theorem 2.7.1 with

$$\rho = \left(\frac{c_1 n}{\log(c_2 N/n)} \right)^{1/2}$$

where c_1, c_2 are positive absolute constants.

Proof. — We consider only the case of $T = B_{p,\infty}^N$, $0 < p < 1$. The case of B_1^N is similar. Since the mechanism has already been developed in details, we will only indicate the different steps. Fix $\varepsilon_0 = 1/20$. To get i) we use Lemma 2.7.11 with $\varepsilon = \varepsilon_0/2$ and δ obtained from the equation $\varepsilon_0/2 = 2(2/p-1)^{-1/2}\delta^{1/p-1/2}$. Let $1 \leq m \leq N$. We get that $S_2(\Sigma_{\lceil m/\delta \rceil})$ is an $(\varepsilon_0/2)$-net of $m^{1/p-1/2}B_{p,\infty}^N \cap S^{N-1}$ with respect to the Euclidean metric. Set $m' = \lceil m/\delta \rceil$. By Claim 2.2.9, we have

$$N(S_2(\Sigma_{m'}), \frac{\varepsilon_0}{2}\, B_2^N) \leq \left(\frac{3eN}{m'(\varepsilon_0/2)} \right)^{m'} = \left(\frac{6eN}{m'\varepsilon_0} \right)^{m'}.$$

Thus, by the triangle inequality, we have

$$N(m^{1/p-1/2}B_{p,\infty}^N \cap S^{N-1}, \varepsilon_0 B_2^N) \leq \left(\frac{6eN}{m'\varepsilon_0} \right)^{m'}$$

so that

$$N(m^{1/p-1/2}B_{p,\infty}^N \cap S^{N-1}, \varepsilon_0 B_2^N) \leq \exp(c_0 n/2)$$

whenever

$$\left(\frac{6eN}{m'\varepsilon_0} \right)^{m'} \leq \exp(c_0 n/2).$$

This shows that under this condition on m' (that is on m), the set $m^{1/p-1/2}B_{p,\infty}^N \cap S^{N-1}$ satisfies i).

In order to tackle ii), recall that $B_{p,\infty}$ is quasi-convex with constant $2^{1/p-1}$ (Claim 2.7.13. By symmetry, we have

$$B^N_{p,\infty} - B^N_{p,\infty} \subset 2^{1/p} B^N_{p,\infty}.$$

Let $r = (1/p-1)m^{1/p-1/2}$. From Lemma 2.7.10, one has

$$rB^N_{p,\infty} \cap B^N_2 \subset 2\,\mathrm{conv}\, S_2(\Sigma_m).$$

As we saw previously,

$$N(S_2(\Sigma_m), \frac{1}{2}\, B^N_2) \leq \left(\frac{3eN}{m(1/2)}\right)^m = \left(\frac{6eN}{m}\right)^m$$

and by Claim 2.2.9 there exists a subset $\Lambda' \subset S^{N-1}$ with $|\Lambda'| \leq N(S_2(\Sigma_m), \frac{1}{2}\, B^N_2)$ such that $S_2(\Sigma_m) \subset 2\,\mathrm{conv}\,\Lambda'$. We arrive at

$$\varepsilon_0 2^{-1/p}\left(rB^N_{p,\infty} - rB^N_{p,\infty}\right) \cap \varepsilon_0 B^N_2 \subset \varepsilon_0\left(rB^N_{p,\infty} \cap B^N_2\right) \subset$$
$$\subset 4\varepsilon_0\,\mathrm{conv}\,\Lambda' \subset 2\,\mathrm{conv}\,(\Lambda' \cup -\Lambda').$$

Therefore $\varepsilon_0 2^{-1/p} rB^N_{p,\infty} \cap S^{N-1}$ satisfies ii) whenever $2\,(6eN/m)^m \leq \exp(c_0 n/2)$.

Finally $\varepsilon_0 2^{-1/p} rB^N_{p,\infty} \cap S^{N-1}$ satisfies i) and ii) whenever the two conditions on m are verified, that is when $cm\log(CN/m) \leq c_0 n/2$ where $c, C > 0$ are absolute constants. We compute m and r and set $\rho = \varepsilon_0 2^{-1/p} r$ to conclude. □

Now we can apply Corollary 2.7.8, to conclude

Theorem 2.7.15. — *Let $\mathbb{P}$ be a probability satisfying (2.8) on the space of $n \times N$ matrices and let $0 < p < 1$. There exist c_p depending only on p, c' depending on c_0 and an absolute constant c such that the set Ω of $n \times N$ matrices A satisfying*

$$\mathrm{rad}\,\left(\ker A \cap B^N_p\right) \leqslant \mathrm{rad}\,\left(\ker A \cap B^N_{p,\infty}\right) \leqslant c_p\left(\frac{\log(cN/n)}{n}\right)^{1/p-1/2}.$$

has probability at least $1 - \exp(-c'n)$.

In particular, if $A \in \Omega$ and if $x', x \in B^n_{p,\infty}$ are such that $Ax' = Ax$ then

$$|x' - x|_2 \leqslant c'_p\left(\frac{\log(c_1 N/n)}{n}\right)^{1/p-1/2}.$$

An analogous result holds for the ball B^N_1.

2.8. An other complexity measure

In the last section, we introduce a new parameter $\ell_*(T)$ which is a *complexity measure* of a set $T \subset \mathbb{R}^N$. We define

$$\ell_*(T) = \mathbb{E}\sup_{t\in T}\left|\sum_{i=1}^N g_i t_i\right|, \qquad (2.17)$$

where $t = (t_i)_{i=1}^N \in \mathbb{R}^N$ and $g_1, ..., g_N$ are independent $N(0,1)$ Gaussian random variables. This kind of parameter plays an important role in empirical processes (see Chapter 1) and in Geometry of Banach spaces.

Theorem 2.8.1. — *There exist absolute constants $c, c' > 0$ for which the following holds. Let $1 \leq n \leq N$. Let A be a Gaussian matrix with i.i.d. entries that are centered and variance one Gaussian random variables. Let $T \subset S^{N-1}$ be a star-shaped set. Then, with probability at least $1 - \exp(-c'n)$,*

$$\mathrm{rad}\,(\ker A \cap T) \leq c\,\ell_*(T)/\sqrt{n}.$$

Proof. — The plan of the proof consists first to prove a restricted isometry property, then to argue as in Lemma 2.7.7. Let $\delta \in (0, 1)$. The restricted isometry property is proved using a discretization by a net argument and an approximation argument. For any $\theta > 0$, let $\Lambda(\theta) \subset T$ be a θ-net of T for the Euclidean metric. Let $\pi_\theta : T \to \Lambda(\theta)$ be a mapping such that for every $t \in T$, $|t - \pi_\theta(t)|_2 \leqslant \theta$. By the triangular inequality, we have

$$\sup_{t\in T}\left|\frac{|At|_2^2}{n} - \mathbb{E}|\langle Y, t\rangle|^2\right| \leqslant \sup_{s\in\Lambda(\theta)}\left|\frac{|As|_2^2}{n} - \mathbb{E}|\langle Y, s\rangle|^2\right| + \sup_{t\in T}\left|\frac{|At|_2^2}{n} - \frac{|A\pi_\theta(t)|_2^2}{n}\right|.$$

– First step. *Entropy estimate via Sudakov minoration.* Let $s \in T$. Let (Y_i) be the rows of A and let Y be a Gaussian random vector with the identity as covariance matrix. Thus $\langle Y, s\rangle$ is ψ_1 with respect to some absolute constant. Thus Bernstein inequality from Theorem 1.2.7 applies and gives that

$$\left|\frac{|As|_2^2}{n} - \mathbb{E}|\langle Y, s\rangle|^2\right| = \left|\frac{1}{n}\sum_1^n(\langle Y_i, s\rangle^2 - \mathbb{E}|\langle Y, s\rangle|^2)\right| \leqslant \delta/2$$

with probability larger than $1 - 2\exp(-cn\delta^2)$, where $c > 0$ is a numerical constant. Let $\theta > 0$. A union bound principle ensures that

$$\sup_{s\in\Lambda(\theta)}\left|\frac{|As|_2^2}{n} - \mathbb{E}|\langle Y, s\rangle|^2\right| \leqslant \delta/2$$

holds with probability larger than $1 - 2\exp(-cn\delta^2 + \log|\Lambda(\theta)|)$.

From Sudakov inequality (1.13) (Theorem 1.4.4), there exists $c' > 0$ such that, if

$$\theta = c'\frac{\ell_*(T)}{\delta\sqrt{n}}$$

then $\log|\Lambda(\theta)| \leqslant c\,n\delta^2/2$. Therefore we get that

$$\sup_{s\in\Lambda(\theta)}\left|\frac{|As|_2^2}{n} - 1\right| \leqslant \delta/2$$

holds with probability larger than $1 - 2\exp(-c\,n\delta^2/2)$.

– Second step. *The approximation term.* To begin with, observe that for any $s, t \in T$, $\big||As|_2^2 - |At|_2^2\big| \leqslant |A(s-t)|_2\,|A(s+t)|_2$. Thus

$$\sup_{t\in T}\big||At|_2^2 - |A\pi_\theta(t)|_2^2\big| \leqslant 2\sup_{t\in T(\theta)}|At|_2\sup_{t\in T}|At|_2$$

where $T(\theta) = \{s - t\,;\ s, t \in T, |s - t|_2 \leqslant \theta\}$. In order to estimate these two norms of the matrix A, we consider a $(1/2)$-net of the unit Euclidean sphere of $\mathbb{R}^n$. According to Proposition 2.2.7, there exists such a net $\mathcal{N}$ with cardinality not larger than 5^n and such that $B_2^n \subset 2\,\mathrm{conv}(\mathcal{N})$. Therefore

$$\sup_{t\in T} |At|_2 = \sup_{t\in T}\ \sup_{|u|_2\leqslant 1} |\langle At, u\rangle| \leqslant 2 \sup_{u\in\mathcal{N}} \sup_{t\in T} |\langle t, A^\top u\rangle|.$$

Since A is a standard Gaussian matrix with i.i.d. entries, centered and with variance one, for every $u \in \mathcal{N}$, $A^\top u$ is a standard Gaussian vector and

$$\mathbb{E}\sup_{t\in T}\langle t, A^\top u\rangle = \ell_*(T).$$

It follows from Theorem 1.4.7 of Chapter 1 that for any fixed $u \in \mathcal{N}$,

$$\forall z > 0 \qquad \mathbb{P}\left(\left|\sup_{t\in T}\langle t, A^\top u\rangle - \mathbb{E}\sup_{t\in T}\langle t, A^\top u\rangle\right| > z\right) \le 2\exp\left(-c'' z^2/\sigma^2(T)\right)$$

for some numerical constant c'', where $\sigma(T) = \sup_{t\in T}\{\left(\mathbb{E}\langle t, A^\top u\rangle|^2\right)^{1/2}\}$.

Combining a union bound inequality and the estimate on the cardinality of the net, we get

$$\forall z > 0 \quad \mathbb{P}\left(\sup_{u\in\mathcal{N}}\sup_{t\in T}\langle A^\top u, t\rangle \ge \ell_*(T) + z\sigma(T)\sqrt{n}\right) \le 2\exp\left(-c''n(\,z^2 - \log 5\,)\right).$$

We deduce that

$$\sup_{t\in T}|At|_2 \leqslant 2\left(\ell_*(T) + z\sigma(T)\sqrt{n}\right)$$

with probability larger than $1 - 2\exp\left(-c''n(\,z^2 - \log 5\,)\right)$.

This reasoning applies as well to $T(\theta)$. In our case, $\sigma(T) = 1$ and $\sigma(T(\theta)) \leqslant \theta$. Therefore,

$$\sup_{t\in T}\left||At|_2^2 - |A\pi_\theta(t)|_2^2\right| \le 8\left(\ell_*(T) + z\sqrt{n}\right)\left(\ell_*(T) + z\theta\sqrt{n}\right)$$

with probability larger than $1 - 4\exp\left(-c''n(\,z^2 - \log 5\,)\right)$.

– Third step. *The restricted isometry property.* Set $z = \sqrt{\log 5}/\sqrt{2}$, say. Plugging the value of θ, we get that with high probability

$$\sup_{t\in T}\left||At|_2^2 - |A\pi_\theta(t)|_2^2\right| \leqslant 8\left(\ell_*(T) + z\sqrt{n}\right)\left(\ell_*(T) + zc'\frac{\ell_*(T)}{\delta}\right)$$

and

$$\sup_{t\in T}\left|\frac{|At|_2^2}{n} - 1\right| \leqslant \frac{\delta}{2} + c'''\left(\frac{\ell_*(T)^2}{n} + z^2\frac{\ell_*(T)}{\delta\sqrt{n}}\right)$$

for some new constant c'''. It is clear that one can choose c'''' such that, whenever

$$\ell_*(T) \leqslant c''''\delta^2\sqrt{n}$$

then

$$\sup_{t\in T}\left|\frac{|At|_2^2}{n} - 1\right| \leqslant \delta$$

with probability larger than $1 - 2\exp(-c\,n\delta^2/2) - 4\exp(-c''nz^2/2)$.

– Last step. *Estimating the width.* Applying the previous estimate with $\delta = 1/2$ to the subset $\frac{1}{\rho}T \cap S^{N-1}$, we get that with high probability, $\ker A \cap \left(\frac{1}{\rho}T \cap S^{N-1}\right) = \emptyset$

whenever $\ell_*\left(\frac{1}{\rho}T\cap S^{N-1}\right)\leqslant\frac{\ell_*(T)}{\rho}<c''''\delta^2\sqrt{n}$. The conclusion follows from Lemma 2.7.7.

□

Remark 2.8.2. — *The proof of Theorem 2.8.1 generalizes to the case of a matrix with independent sub-Gaussian rows. Only the second step has to be modified by using the majorizing measure theorem which precisely allows to compare deviation inequalities of supremum of sub-Gaussian processes to their equivalent in the Gaussian case. We will not give here the proof of this result, see Theorem 3.2.1 in Chapter 3, where an other approach will be developed.*

We show now how Theorem 2.8.1 applies to some sets T.

Corollary 2.8.3. — *There exist absolute constants* $c, c' > 0$ *such that the following holds. Let* $1\leqslant n\leqslant N$ *and let* A *be as in Theorem 2.8.1. Let* $\lambda>0$*. Let* $T\subset S^{N-1}$ *and assume that* $T\subset 2\,\mathrm{conv}\,\Lambda$ *for some* $\Lambda\subset B_2^N$ *with* $|\Lambda|\leq\exp(\lambda^2 n)$*. Then with probability at least* $1-\exp(-c'n)$,

$$\mathrm{rad}\,(\ker A\cap T)\leq c\lambda.$$

Remark 2.8.4. — *Constant 2 in the inclusion* $T\subset 2\,\mathrm{conv}\,\Lambda$ *is not significant.*

Proof. — The main point in the proof is that if $T\subset 2\,\mathrm{conv}\,\Lambda$, $\Lambda\subset B_2^N$ and we have a reasonable control of $|\Lambda|$, then $\ell_*(T)$ can be bounded from above. The rest is a direct application of Theorem 2.8.1. Let $c, c'>0$ be constants from Theorem 2.8.1. It is well known (see Chapter 3) that there exists an absolute constant $c''>0$ such that for every $\Lambda\subset B_2^N$,

$$\ell_*(\mathrm{conv}\,\Lambda)=\ell_*(\Lambda)\leqslant c''\sqrt{\log(|\Lambda|)}\,,$$

and since $T\subset 2\,\mathrm{conv}\,\Lambda$,

$$\ell_*(T)\leq 2\ell_*(\mathrm{conv}\,\Lambda)\leqslant 2c''\left(\lambda^2 n\right)^{1/2}.$$

The conclusion follows from Theorem 2.8.1. □

2.9. Notes and comments

For further information on the origin and the genesis of compressed sensing and on the ℓ_1-minimization method, the reader may consult the articles by D. Donoho [**Don06**], E. Candes, J. Romberg and T. Tao [**CRT06**] and E. Candes and T. Tao [**CT06**]. For further and more advanced studies on compressed sensing, see the book [**FR11**].

Proposition 2.2.15 is due to D. Donoho [**Don05**]. Proposition 2.2.17 and its proof is a particular case of a more general result from [**FPRU10**]. See also [**LN06**] where the analogous problem for neighborliness is studied.

The definition 2.3.1 of the Restricted Isometry Property was introduced in [**CT05**] and plays an important role in compressed sensing. The relevance of the Restricted

Isometry parameter for the reconstruction property was for instance revealed in [**CT06**], [**CT05**], where it was shown that if

$$\delta_m(A) + \delta_{2m}(A) + \delta_{3m}(A) < 1$$

then the encoding matrix A has the exact reconstruction property of order m. This result was improved in [**Can08**] to $\delta_{2m}(A) < \sqrt{2} - 1$ as stated in Theorem 2.3.2. This constant $\sqrt{2} - 1$ was recently improved in [**FL09**]. In the same paper these authors introduced the parameter γ_p from Definition 2.3.3.

The proofs of results of Section 2.4 are following lines from [**CT05**], [**CDD09**], [**FL09**], [**FPRU10**] and [**KT07**]. Relation (2.5) was proved in [**FL09**] with a better numerical constant. Theorem 2.5.2 from [**GG84**] gives the optimal behavior of the Gelfand widths of the cross-polytope. This completes a celebrated result from [**Kaš77**]. The result of [**Kaš77**] was proved using Kolmogorov widths (dual to the Gelfand widths) and with a non-optimal power of the logarithm (power 3/2 instead of 1/2 later improved in [**GG84**]). The upper bound of Kolmogorov widths was obtained via random matrices with i.i.d. Bernoulli entries, whereas [**Glu83**] and [**GG84**] used properties of random Gaussian matrices.

The simple proof of Theorem 2.6.5 stating that subgaussian matrices satisfy the exact reconstruction property of order m by ℓ_1-minimization with large m is taken from [**BDDW08**] and [**MPTJ08**]. The strategy of this proof is very classical in Approximation Theory, see [**Kaš77**] and in Banach space theory where it has played an important role in quantitative version of Dvoretsky's theorem on almost spherical sections, see [**FLM77**] and [**MS86**].

Section 2.7 follows the lines of [**MPTJ08**]. Proposition 2.7.3 from [**KT07**] is stated in terms of Gelfand width rather than in terms of constants of isometry as in [**Can08**] and [**CDD09**]. The principle of reducing the computation of Gelfand widths by truncation as stated in Subsection 2.7 goes back to [**Glu83**]. The parameter $\ell_*(T)$ defined in Section 2.8 plays an important role in Geometry of Banach spaces (see [**Pis89**]). Theorem 2.8.1 is from [**PTJ86**].

The restricted isometry property for the model of partial discrete Fourier matrices will be studied in Chapter 5. There exists many other interesting model of random sensing matrices (see [**FR11**]). Random matrices with i.i.d. entries satisfying uniformly a sub-exponential tail inequality or with i.i.d. columns with log-concave density, the so-called log-concave Ensemble, have been studied in [**ALPTJ10**] and in [**ALPTJ09**] where it is shown that they also satisfy a RIP with $m \sim n/\log^2(2N/n)$.

CHAPTER 3

INTRODUCTION TO CHAINING METHODS

The restricted isometry property has been introduced in Chapter 2 in order to provide a simple way of showing that a $n \times N$ matrix A satisfies an exact reconstruction property. Indeed, if A is a $n \times N$ matrix such that for every $2m$-sparse vector $x \in \mathbb{R}^N$,

$$(1 - \delta_{2m})|x|_2^2 \leqslant |Ax|_2^2 \leqslant (1 + \delta_{2m})|x|_2^2$$

where $\delta_{2m} < \sqrt{2} - 1$ then A satisfies the exact reconstruction property of order m by ℓ_1-minimization (cf. Chapter 2). In particular, if A is a random matrix with rows vectors $n^{-1/2}Y_1, \ldots, n^{-1/2}Y_n$, this property can be translated in terms of an empirical processes property since

$$\delta_{2m} = \sup_{x \in S_2(\Sigma_{2m})} \Big| \frac{1}{n} \sum_{i=1}^{n} \langle Y_i, x \rangle^2 - 1 \Big|. \tag{3.1}$$

If we show an upper bound on the supremum (3.1) smaller than $\sqrt{2} - 1$, this will prove that the matrix A has the exact reconstruction property of order m by ℓ_1-minimization. In Chapter 2, it was shown that matrices from the subgaussian Ensemble satisfy the restricted isometry property (with high probability) thanks to a technique called the epsilon-net argument. In this chapter, we present a technique called the chaining method used to obtain upper bounds on the supremum of stochastic processes.

3.1. The chaining method

The chaining mechanism is a technique used to obtain upper bounds on the supremum $\sup_{t \in T} X_t$ of a stochastic process $(X_t)_{t \in T}$ indexed by a set T. These upper bounds are usually expressed in terms of some metric complexity measure of T.

One key idea behind the chaining method is the trade-off between the deviation or concentration estimates of the increments of the process $(X_t)_{t \in T}$ and the complexity of T endowed with a metric structure connected with the stochastic process $(X_t)_{t \in T}$.

As an introduction, we show an upper bound on the supremum $\sup_{t \in T} X_t$ in terms of an entropy integral known as the *Dudley entropy integral*. This entropy integral is based on some metric quantities of T that were introduced in Chapter 1 and that we recall now.

Definition 3.1.1. — *Let (T,d) be a semi-metric space (that is for every x,y and z in T, $d(x,y)=d(y,x)$ and $d(x,y)\leqslant d(x,z)+d(z,y)$). For every $\varepsilon>0$, the ε-covering number $N(T,d,\varepsilon)$ of (T,d) is the minimal number of open balls for the semi-metric d of radius ε needed to cover T. The metric entropy is the logarithm of the ε-covering number as a function of ε.*

We develop the chaining argument under a subgaussian assumption on the increments of the process $(X_t)_{t\in T}$ saying that for every $s,t\in T$ and $u>0$,

$$\mathbb{P}\Big[|X_s-X_t|>ud(s,t)\Big]\leqslant 2\exp(-cu^2), \tag{3.2}$$

where d is a semi-metric on T and c is an absolute positive constant. To avoid some technical complications that are less important from our point of view, we will only consider processes indexed by finite sets T. To handle more general sets one may study the random variables $\sup_{T'\subset T: T' \text{ finite}}\sup_{t\in T'}|X_t|$ or $\sup_{T'\subset T: T' \text{ finite}}\sup_{t,s\in T'}|X_t-X_s|$ which suffices for our goals.

Theorem 3.1.2. — *There exist absolute constants c_0,c_1,c_2 and c_3 for which the following holds. Let (T,d) be a semi-metric space and assume that $(X_t)_{t\in T}$ is a stochastic process with increments satisfying the subgaussian condition (3.2). Then, for every $v\geqslant c_0$, with probability greater than $1-c_1\exp(-c_2v^2)$*

$$\sup_{s,t\in T}|X_t-X_s|\leqslant c_3 v\int_0^\infty \sqrt{\log N(T,d,\varepsilon)}\ d\varepsilon.$$

In particular,

$$\mathbb{E}\sup_{s,t\in T}|X_t-X_s|\leqslant c_3\int_0^\infty \sqrt{\log N(T,d,\varepsilon)}\ d\varepsilon.$$

Proof. — Put $\eta_{-1}=\operatorname{diam}(T,d)$ and for every integer $i\geqslant 0$ set

$$\eta_i=\inf\big\{\eta>0: N(T,d,\eta)\leqslant 2^{2^i}\big\}.$$

Let $(T_i)_{i\geqslant 0}$ be a sequence of subsets of T where T_0 is a subset of T containing only one element and for every $i\geqslant 0$, by definition of η_i, we take T_{i+1} as a subset of T of cardinality smaller than $2^{2^{i+1}}$ such that

$$T\subset\bigcup_{x\in T_{i+1}}\Big(x+\eta_i B_d\Big),$$

where B_d is the unit ball associated with the semi-metric d. For every $t\in T$ and integer i, put $\pi_i(t)$ a nearest point to t in T_i. In particular, $d(t,\pi_i(t))\leqslant\eta_{i-1}$.

Since T is finite, then for every $t\in T$,

$$X_t-X_{\pi_0(t)}=\sum_{i=0}^\infty X_{\pi_{i+1}(t)}-X_{\pi_i(t)}. \tag{3.3}$$

Let $i \in \mathbb{N}$ and $t \in T$. By the subgaussian assumption (3.2), for every $u > 0$, with probability greater than $1 - 2\exp(-cu^2)$,

$$|X_{\pi_{i+1}(t)} - X_{\pi_i(t)}| \leqslant ud(\pi_{i+1}(t), \pi_i(t)) \leqslant u(\eta_{i-1} + \eta_i) \leqslant 2u\eta_{i-1}. \tag{3.4}$$

To get this result uniformly over every links $(\pi_{i+1}(t), \pi_i(t))$ for $t \in T$ at level i, we use an union bound (note that there are at most $|T_{i+1}||T_i| \leqslant 2^{3.2^i}$ such links): with probability greater than $1 - 2|T_{i+1}||T_i|\exp(-cu^2) \geqslant 1 - 2\exp\big(3.2^i \log 2 - cu^2\big)\big)$, for every $t \in T$

$$|X_{\pi_{i+1}(t)} - X_{\pi_i(t)}| \leqslant 2u\eta_{i-1}.$$

To balance the "complexity" of the set of "links" with our deviation estimate, we take $u = v2^{i/2}$, where v is larger than $\sqrt{(6\log 2)/c}$. Thus, for the level i, we obtain with probability greater than $1 - 2\exp\big(-(c/2)v^2 2^i\big)$, for all $t \in T$,

$$|X_{\pi_{i+1}(t)} - X_{\pi_i(t)}| \leqslant 2v2^{i/2}\eta_{i-1},$$

for every v larger than an absolute constant.

By (3.3) and summing over all levels $i \in \mathbb{N}$, we have with probability greater than $1 - 2\sum_{i=0}^{\infty} \exp\big(-(c/2)v^2 2^i\big) \geqslant 1 - c_1\exp(-c_2 v^2)$, for every $t \in T$,

$$|X_t - X_{\pi_0(t)}| \leqslant 2v\sum_{i=0}^{\infty} 2^{i/2}\eta_{i-1} = 2^{3/2}v\sum_{i=-1}^{\infty} 2^{i/2}\eta_i. \tag{3.5}$$

Observe that if $i \in \mathbb{N}$ and $\eta < \eta_i$ then $N(T, d, \eta) > 2^{2^i}$. Hence $N(T, d, \eta) \geqslant 2^{2^i} + 1$ and thus

$$\sqrt{\log(1 + 2^{2^i})}(\eta_i - \eta_{i+1}) \leqslant \int_{\eta_{i+1}}^{\eta_i} \sqrt{\log N(T, d, \eta)}d\eta,$$

and since $\log(1 + 2^{2^i}) \geqslant 2^i \log 2$ then summing over all $i \geqslant -1$,

$$\sqrt{\log 2}\sum_{i=-1}^{\infty} 2^{i/2}(\eta_i - \eta_{i+1}) \leqslant \int_0^{\eta_{-1}} \sqrt{\log N(T, d, \eta)}d\eta$$

and

$$\sum_{i=-1}^{\infty} 2^{i/2}(\eta_i - \eta_{i+1}) = \sum_{i=-1}^{\infty} 2^{i/2}\eta_i - \sum_{i=0}^{\infty} 2^{(i-1)/2}\eta_i \geqslant \Big(1 - \frac{1}{\sqrt{2}}\Big)\sum_{i=-1}^{\infty} 2^{i/2}\eta_i.$$

This proves that

$$\sum_{i=-1}^{\infty} 2^{i/2}\eta_i \leqslant c_3\int_0^{\infty} \sqrt{\log N(T, d, \eta)}d\eta. \tag{3.6}$$

We conclude that, for every v larger than $\sqrt{(6\log 2)/c}$, with probability greater than $1 - c_1\exp(-c_2 v^2)$, we have

$$\sup_{t\in T}|X_t - X_{\pi_0(t)}| \leqslant c_4 v\int_0^{\infty} \sqrt{\log N(T, d, \eta)}d\eta.$$

Integrating the tail estimate,

$$\begin{aligned}\mathbb{E}\sup_{t\in T}|X_t - X_{\pi_0(t)}| &= \int_0^\infty \mathbb{P}\Big[\sup_{t\in T}|X_t - X_{\pi_0(t)}| > u\Big]du \\ &\leqslant c_5\int_0^\infty \sqrt{\log N(T,d,\varepsilon)}d\varepsilon.\end{aligned}$$

Finally, since $|T_0| = 1$, it follows that, for every $t, s \in T$,

$$|X_t - X_s| \leqslant |X_t - X_{\pi_0(t)}| + |X_s - X_{\pi_0(s)}|$$

and the theorem is shown. □

In the case of a stochastic process with subgaussian increments (cf. condition (3.2)), the entropy integral

$$\int_0^\infty \sqrt{\log N(T,d,\varepsilon)}d\varepsilon$$

is called the Dudley entropy integral.

A careful look at the previous proof reveals one potential source of looseness. At each level of the chaining mechanism, we used a uniform bound (depending only on the level) to control each link. Instead, one can use "individual" bounds for every link rather than the worst at every level. This idea is the basis of what is now called the generic chaining. The natural metric complexity measure coming out of this method is the γ_2-functional which is now introduced.

Definition 3.1.3. — *Let (T,d) be a semi-metric space. A sequence $(T_s)_{s\geqslant 0}$ of subsets of T is admissible if $|T_0| \leqslant 1$ and $|T_s| \leqslant 2^{2^s}$ for every $s \geqslant 1$. The γ_2-functional of (T,d) is*

$$\gamma_2(T,d) = \inf_{(T_s)}\sup_{t\in T}\Big(\sum_{s=0}^\infty 2^{s/2}d(t,T_s)\Big)$$

where the infimum is taken over all admissible sequences $(T_s)_{s\in\mathbb{N}}$ and $d(t,T_s) = \min_{y\in T_s} d(t,y)$ for every $t \in T$ and $s \in \mathbb{N}$.

We note that the γ_2-functional is upper bounded by the Dudley entropy integral:

$$\gamma_2(T,d) \leqslant c_0\int_0^\infty \sqrt{\log N(T,d,\varepsilon)}d\varepsilon, \tag{3.7}$$

where c_0 is an absolute positive constant. Indeed, we construct an admissible sequence $(T_s)_{s\in\mathbb{N}}$ in the following way: let T_0 be a subset of T containing one element and for every $s \in \mathbb{N}$, let T_{s+1} be a subset of T of cardinality smaller than $2^{2^{s+1}}$ such that for every $t \in T$ there exists $x \in T_{i+1}$ satisfying $d(t,x) \leqslant \eta_s$, where η_s is defined by

$$\eta_s = \inf\big(\eta > 0 : N(T,d,\eta) \leqslant 2^{2^s}\big).$$

Inequality (3.7) follows from (3.6) and

$$\sup_{t\in T}\Big(\sum_{s=0}^\infty 2^{s/2}d(t,T_s)\Big) \leqslant \sum_{s=0}^\infty 2^{s/2}\sup_{t\in T}d(t,T_s) \leqslant \sum_{s=0}^\infty 2^{s/2}\eta_{s-1}.$$

Now, we apply the generic chaining mechanism to show an upper bound on the supremum of processes whose increments satisfy the subgaussian assumption (3.2).

Theorem 3.1.4. — *There exist absolute constants c_0, c_1, c_2 and c_3 such that the following holds. Let (T, d) be a semi-metric space. Let $(X_t)_{t\in T}$ be a stochastic process satisfying the subgaussian condition (3.2). For every $v \geqslant c_0$, with probability greater than $1 - c_1 \exp(-c_2 v^2)$*

$$\sup_{s,t\in T} |X_t - X_s| \leqslant c_3 v \gamma_2(T, d)$$

and

$$\mathbb{E} \sup_{s,t\in T} |X_t - X_s| \leqslant c_3 \gamma_2(T, d).$$

Proof. — Let $(T_s)_{s\in\mathbb{N}}$ be an admissible sequence. For every $t \in T$ and $s \in \mathbb{N}$ denote by $\pi_s(t)$ one point in T_s such that $d(t, T_s) = d(t, \pi_s(t))$. Since T is finite, we can write for every $t \in T$,

$$|X_t - X_{\pi_0(t)}| \leqslant \sum_{s=0}^{\infty} |X_{\pi_{s+1}(t)} - X_{\pi_s(t)}|. \tag{3.8}$$

Let $s \in \mathbb{N}$. For every $t \in T$ and $v > 0$, with probability greater than $1 - 2\exp(-c_0 2^s v^2)$,

$$|X_{\pi_{s+1}(t)} - X_{\pi_s(t)}| \leqslant v 2^{s/2} d(\pi_{s+1}(t), \pi_s(t)).$$

We extend the last inequality to every link of the chains at level s by using an union bound: for every $v \geqslant c_1$, with probability greater than $1 - 2\exp(-c_2 2^s v^2)$, for every $t \in T$,

$$|X_{\pi_{s+1}(t)} - X_{\pi_s(t)}| \leqslant v 2^{s/2} d(\pi_{s+1}(t), \pi_s(t)).$$

An union bound on every level $s \in \mathbb{N}$ yields: for every $v \geqslant c_1$, with probability greater than $1 - 2\sum_{s=0}^{\infty} \exp(-c_2 2^s v^2)$, for every $t \in T$,

$$|X_t - X_{\pi_0(t)}| \leqslant c_2 v \sum_{s=0}^{\infty} 2^{s/2} d(\pi_s(t), \pi_{s+1}(t)) \leqslant c_3 v \sum_{s=0}^{\infty} 2^{s/2} d(t, T_s).$$

The claim follows since the sum in the last probability estimate is comparable to its first term. □

For Gaussian processes, the upper bound in expectation obtained in Theorem 3.1.4 is sharp up to some absolute constants. This deep result, called the Majorizing measure theorem, makes an equivalence between two different quantities measuring the complexity of a set $T \subset \mathbb{R}^N$:

1. a metric complexity measure given by the γ_2 functional

$$\gamma_2(T, \ell_2^N) = \inf_{(T_s)} \sup_{t\in T} \sum_{s=0}^{\infty} 2^{s/2} d_{\ell_2^N}(t, T_s),$$

   where the infimum is taken over all admissible sequences $(T_s)_{s\in\mathbb{N}}$ of T;

2. a probabilistic complexity measure given by the expectation of the supremum of the canonical Gaussian process indexed by T:
$$\ell_*(T) = \mathbb{E}\sup_{t\in T}\Big|\sum_{i=1}^N g_i t_i\Big|,$$
where $g_1, \dots, g_N$ are N i.i.d. standard Gaussian variables.

Theorem 3.1.5 **(Majorizing measure Theorem)**. — *There exist two absolute positive constants c_0 and c_1 such that for every subset T of $\mathbb{R}^N$,*
$$c_0\ell_*(T) \leqslant \gamma_2(T, \ell_2^N) \leqslant c_1\ell_*(T).$$

3.2. An example of a more sophisticated chaining argument

In this section, we show upper bounds on the supremum
$$\sup_{f\in F}\left|\frac{1}{n}\sum_{i=1}^n f^2(X_i) - \mathbb{E}f^2(X)\right|, \tag{3.9}$$
where $X_1, \dots, X_n$ are n i.i.d. random variables with values in a measurable space $\mathcal{X}$ and F is a class of real-valued functions defined on $\mathcal{X}$.

In Chapter 2, this bound is used to show the restricted isometry property in Theorem 2.7.1. In this example, the class F is a class of linear functions indexed by a set of sparse vectors. In particular, for this example, the class F is not uniformly bounded.

In general, when $\|F\|_\infty = \sup_{f\in F}\|f\|_{L_\infty(\mu)} < \infty$, a bound on (3.9) follows from a symmetrization argument combined with the contraction principle. In the present study, we do not assume that F is uniformly bounded but we only assume that F has a finite diameter in $L_{\psi_2(\mu)}$ where μ is the probability distribution of $X \overset{d}{=} X_1$. This means that the norm
$$\|f\|_{\psi_2(\mu)} = \inf\Big(c > 0 : \mathbb{E}\exp\big(|f(X)|^2/c^2\big) \leqslant e\Big)$$
is uniformly bounded over every f in F. We denote this bound by α and thus we assume that
$$\alpha = \mathrm{diam}(F, \psi_2(\mu)) = \sup_{f\in F}\|f\|_{\psi_2(\mu)} < \infty. \tag{3.10}$$
In terms of random variables, Assumption (3.10) means that for all $f \in F$, $f(X)$ has a subgaussian behaviour and its ψ_2 norm is uniformly bounded over F.

Under (3.10), we can apply the classical generic chaining mechanism and obtain a bound on (3.9). Indeed, denote by $(X_f)_{f\in F}$ the empirical process where $X_f = n^{-1}\sum_{i=1}^n f^2(X_i) - \mathbb{E}f^2(X)$ for every $f \in F$. Assume that for every f and g in F, $\mathbb{E}f^2(X) = \mathbb{E}g^2(X)$. In this case, the increments of the process $(X_f)_{f\in F}$ are
$$X_f - X_g = \frac{1}{n}\sum_{i=1}^n f^2(X_i) - g^2(X_i)$$

and we have (cf. Chapter 1)

$$\left\|f^2-g^2\right\|_{\psi_1(\mu)} \leqslant \|f+g\|_{\psi_2(\mu)} \|f-g\|_{\psi_2(\mu)} \leqslant 2\alpha \|f-g\|_{\psi_2(\mu)}. \tag{3.11}$$

In particular, the increment $X_f - X_g$ is a sum of i.i.d. mean-zero ψ_1 random variables. Hence, the concentration properties of the increments of $(X_f)_{f\in F}$ follow from Theorem 1.2.7. Provided that for some $f_0 \in F$, we have $X_{f_0} = 0$ or $(X_f)_{f\in F}$ is a symmetric process then running the classical generic chaining mechanism with this increment condition yields the following: for every $u \geqslant c_0$, with probability greater than $1 - c_1 \exp(-c_2 u)$,

$$\sup_{f\in F}\left|\frac{1}{n}\sum_{i=1}^n f^2(X_i) - \mathbb{E}f^2(X)\right| \leqslant c_3 u\alpha\left(\frac{\gamma_2(F,\psi_2(\mu))}{\sqrt{n}} + \frac{\gamma_1(F,\psi_2(\mu))}{n}\right) \tag{3.12}$$

for some absolute positive constants c_0, c_1, c_2 and c_3 and with

$$\gamma_1(F,\psi_2(\mu)) = \inf_{(F_s)}\sup_{f\in F}\Big(\sum_{s=0}^\infty 2^s d_{\psi_2(\mu)}(f,F_s)\Big)$$

where the infimum is taken over all admissible sequences $(F_s)_{s\in\mathbb{N}}$ and $d_{\psi_2(\mu)}(f,F_s) = \min_{g\in F_s}\|f-g\|_{\psi_2(\mu)}$ for every $f\in F$ and $s\in\mathbb{N}$. Result (3.12) can be derived from theorem 1.2.7 of [**Tal05**].

In some cases, computing $\gamma_1(F,d)$ for some metric d can be involved and only weak estimates can be shown. Obtaining upper bounds on (3.9) which does not require the computation of $\gamma_1(F,\psi_2(\mu))$ can be of importance. In particular, upper bounds depending only on $\gamma_2(F,\psi_2(\mu))$ can be useful when the metrics $L_{\psi_2}(\mu)$ and $L_2(\mu)$ are equivalent on F because of the Majorizing measure theorem (cf. Theorem 3.1.5). In the next result, we show an upper bound on the supremum (3.9) depending only on the $\psi_2(\mu)$ diameter of F and on the complexity measure $\gamma_2(F,\psi_2(\mu))$.

Theorem 3.2.1. — *There exists absolute constants c_0, c_1, c_2 and c_3 such that the following holds. Let F be a finite class of real-valued functions in $\mathcal{S}(L_2(\mu))$, the unit sphere of $L_2(\mu)$ and denote by α the diameter $\mathrm{diam}(F,\psi_2)$. Then, with probability at least $1 - c_1\exp\Big(-(c_2/\alpha^2)\min\Big(n\alpha^2, \gamma_2(F,\psi_2)^2\Big)\Big)$,*

$$\sup_{f\in F}\left|\frac{1}{n}\sum_{i=1}^n f^2(X_i) - \mathbb{E}f^2(X)\right| \leqslant c_3\max\left(\alpha\frac{\gamma_2(F,\psi_2)}{\sqrt{n}}, \frac{\gamma_2(F,\psi_2)^2}{n}\right).$$

Moreover, if F is a symmetric subset of $\mathcal{S}(L_2(\mu))$ then,

$$\mathbb{E}\sup_{f\in F}\left|\frac{1}{n}\sum_{i=1}^n f^2(X_i) - \mathbb{E}f^2(X)\right| \leqslant c_3\max\left(\alpha\frac{\gamma_2(F,\psi_2)}{\sqrt{n}}, \frac{\gamma_2(F,\psi_2)^2}{n}\right).$$

To show Theorem 3.2.1, we introduce the following notation. For every $f\in L_2(\mu)$, we set

$$Z(f) = \frac{1}{n}\sum_{i=1}^n f^2(X_i) - \mathbb{E}f^2(X) \text{ and } W(f) = \Big(\frac{1}{n}\sum_{i=1}^n f^2(X_i)\Big)^{1/2}. \tag{3.13}$$

Moreover, for the sake of shortness, in what follows, L_2, ψ_2 and ψ_1 stand for the spaces $L_2(\mu)$, $\psi_1(\mu)$ and $\psi_2(\mu)$, for which we omit to write the probability measure μ.

To obtain upper bounds on the supremum (3.9) we study the deviation behaviour of the increments of the underlying process. Namely, we need deviation results for $Z(f) - Z(g)$ for every $f, g \in F$. Moreover, since the "end of the chains" will be analysed by different means, the deviation behaviour of the increments $W(f-g)$ will be of importance as well.

Lemma 3.2.2. — *There exists an absolute constant C_1 such that the following holds. Let $F \subset \mathcal{S}(L_2(\mu))$. Denote by α the diameter* $\operatorname{diam}(F, \psi_2)$. *For every $f, g \in F$, we have:*

1. *for every $u \geqslant 2$,*
$$\mathbb{P}\Big[W(f-g) \geqslant u \|f-g\|_{\psi_2}\Big] \leqslant 2\exp\big(-C_1 n u^2\big)$$
2. *for every $u > 0$,*
$$\mathbb{P}\Big[|Z(f) - Z(g)| \geqslant u\alpha \|f-g\|_{\psi_2}\Big] \leqslant 2\exp\big(-C_1 n \min(u, u^2)\big)$$
and for every $u > 0$,
$$\mathbb{P}\Big[|Z(f)| \geqslant u\alpha^2\Big] \leqslant 2\exp\big(-C_1 n \min(u, u^2)\big).$$

Proof. — Let $f, g \in F$. Since $f, g \in L_{\psi_2}$, we have $\left\|(f-g)^2\right\|_{\psi_1} = \|f-g\|_{\psi_2}^2$ and by Theorem 1.2.7, for every $t \geqslant 1$,
$$\mathbb{P}\Big[\frac{1}{n}\sum_{i=1}^{n}(f-g)^2(X_i) - \|f-g\|_{L_2}^2 \geqslant t\|f-g\|_{\psi_2}^2\Big] \leqslant 2\exp(-c_1 n t). \tag{3.14}$$
Using $\|f-g\|_{\psi_2} \geqslant \sqrt{e-1}\,\|f-g\|_{L_2}$ together with Equation (3.14), it is easy to get for every $u \geqslant 2$,
$$\begin{aligned}
&\mathbb{P}\Big[W(f-g) \geqslant u\|f-g\|_{\psi_2}\Big] \\
&\leqslant \mathbb{P}\Big[\frac{1}{n}\sum_{i=1}^{n}(f-g)^2(X_i) - \|f-g\|_{L_2}^2 \geqslant (u^2 - (e-1))\,\|f-g\|_{\psi_2}^2\Big] \\
&\leqslant 2\exp\big(-c_2 n u^2\big).
\end{aligned}$$

For the second statement, since $\mathbb{E}f^2 = \mathbb{E}g^2$, the increments are
$$Z(f) - Z(g) = \frac{1}{n}\sum_{i=1}^{n} f^2(X_i) - g^2(X_i).$$
Thanks to (3.11), $Z(f) - Z(g)$ is a sum of mean-zero ψ_1 random variables and the result follows from Theorem 1.2.7. The last statement is also a consequence of Theorem 1.2.7 and (3.11). □

Once obtained the deviation properties of the increments of the underlying process(es) (that is $(Z(f))_{f\in F}$ and $(W(f))_{f\in F}$), we use the generic chaining mechanism to obtain a uniform bound on (3.9). Since we work in a special framework (sum of squares of ψ_2 random variables), we will perform a particular chaining argument which will allow us to avoid the $\gamma_1(F,\psi_2)$ term coming from the classical generic chaining (cf. (3.12)).

If $\gamma_2(F,\psi_2)=\infty$ then the upper bound of Theorem 3.2.1 is trivial, otherwise consider an almost optimal admissible sequence $(F_s)_{s\in\mathbb{N}}$ of F with respect to $\psi_2(\mu)$. That is an admissible sequence $(F_s)_{s\in\mathbb{N}}$ such that

$$\gamma_2(F,\psi_2)\geqslant\frac{1}{2}\sup_{f\in F}\Big(\sum_{s=0}^{\infty}2^{s/2}d_{\psi_2}(f,F_s)\Big).$$

For every $f\in F$ and integer s, put $\pi_s(f)$ a nearest point to f in F_s.

The idea of the proof is for every $f\in F$ to analyze the links $\pi_{s+1}(f)-\pi_s(f)$ for $s\in\mathbb{N}$ of the chain $(\pi_s(f))_{s\in\mathbb{N}}$ in three different regions - values of the level s in $[0,s_1]$, $[s_1+1,s_0-1]$ or $[s_0,\infty)$ for some well chosen s_1 ans s_0 - depending on the deviation properties of the increments of the underlying process(es) at the s stage:

1. The end of the chain: we study the link $f-\pi_{s_0}(f)$. In this part of the chain, we work with the process $(W(f-\pi_{s_0}(f)))_{f\in F}$ which is subgaussian (cf. Lemma 3.2.2). Thanks to this remark, we can avoid the sub-exponential behaviour of the process $(Z(f))_{f\in F}$ and thus the term $\gamma_1(F,\psi_2)$ appearing in (3.12);
2. The middle of the chain: we work at these stages with the process $(Z(\pi_{s_0-1}(f))-Z(\pi_{s_1}(f)))_{f\in F}$ which has subgaussian increments in this range;
3. The beginning of the chain: we study the process $(Z(\pi_{s_1}(f))_{f\in F}$. For this part of the chain, the complexity of F_{s_1} is so small that a trivial comparison of the process with the ψ_2-diameter of F will be enough.

***Proposition 3.2.3* (End of the chain).** — *There exist absolute constant c_0,c_1,c_2 and c_3 for which the following holds. Let $F\subset\mathcal{S}(L_2(\mu))$ and $\alpha=\mathrm{diam}(F,\psi_2)$. For every $v\geqslant c_0$, with probability greater than $1-c_1\exp(-c_2nv)$,*

$$\sup_{f\in F}W(f-\pi_{s_0}(f))\leqslant c_3\sqrt{v}\frac{\gamma_2(F,\psi_2)}{\sqrt{n}},$$

where $s_0=\min\big(s\geqslant 0:2^s\geqslant n\big)$.

Proof. — Let f be in F. Since F is finite, we can write

$$f-\pi_{s_0}(f)=\sum_{s=s_0}^{\infty}\pi_{s+1}(f)-\pi_s(f),$$

and, since W is the empirical $L_2(P_n)$ norm (where P_n is the empirical distribution $n^{-1}\sum_{i=1}^n\delta_{X_i}$), it is sub-additive and so

$$W(f-\pi_{s_0}(f))\leqslant\sum_{s=s_0}^{\infty}W(\pi_{s+1}(f)-\pi_s(f)).$$

Now, fix a level $s \geqslant s_0$. Using a union bound on the set of links $\{(\pi_{s+1}(f), \pi_s(f)) : f \in F\}$ (note that there are at most $|F_{s+1}||F_s|$ such links) and the subgaussian property of W (i.e. Lemma 3.2.2), we get, for every $u \geqslant 2$, with probability greater than $1 - 2|F_{s+1}||F_s|\exp(-C_1 n u^2)$, for every $f \in F$,

$$W(\pi_{s+1}(f) - \pi_s(f)) \leqslant u \left\| \pi_{s+1}(f) - \pi_s(f) \right\|_{\psi_2}.$$

Then, note that for every $s \in \mathbb{N}$, $|F_{s+1}||F_s| \leqslant 2^{2^s} 2^{2^{s+1}} = 2^{3.2^s}$ so that a union bound over all the levels $s \geqslant s_0$ yields for every u such that $u^2 2^{s_0}$ is larger than some absolute constant, with probability greater than $1 - 2\sum_{s=s_0}^{\infty} |F_{s+1}||F_s| \exp(-C_1 n 2^s u^2) \geqslant 1 - c_1 \exp(-c_0 n u^2 2^{s_0})$, for every $f \in F$,

$$\begin{aligned} W(f - \pi_{s_0}(f)) &\leqslant \sum_{s=s_0}^{\infty} W(\pi_{s+1}(f) - \pi_s(f)) \leqslant \sum_{s=s_0}^{\infty} u 2^{s/2} \left\| \pi_{s+1}(f) - \pi_s(f) \right\|_{\psi_2} \\ &\leqslant 2u \sum_{s=s_0}^{\infty} 2^{s/2} d_{\psi_2}(f, F_s). \end{aligned}$$

We conclude with $v = u^2 2^{s_0}$ for v large enough, s_0 such that $2^{s_0} \sim n$ and with the quasi-optimality of the admissible sequence $(F_s)_{s \geqslant 0}$. □

***Proposition 3.2.4* (Middle of the chain)**. — *There exist absolute constants c_0, c_1, c_2 and c_3 for which the following holds. Let $s_1 \in \mathbb{N}$ be such that $s_1 \leqslant s_0$ (where s_0 has been defined in Proposition 3.2.3). Let $F \subset \mathcal{S}(L_2(\mu))$ and $\alpha = \operatorname{diam}(F, \psi_2)$. For every $u \geqslant c_0$, with probability greater than $1 - c_1 \exp(-c_2 2^{s_1} u)$,*

$$\sup_{f \in F} \left| Z(\pi_{s_0}(f)) - Z(\pi_{s_1}(f)) \right| \leqslant c_3 u \alpha \frac{\gamma_2(F, \psi_2)}{\sqrt{n}}.$$

Proof. — For every $f \in F$, we write

$$Z(\pi_{s_0 - 1}(f)) - Z(\pi_{s_1}(f)) = \sum_{s=s_1+1}^{s_0-1} Z(\pi_s(f)) - Z(\pi_{s-1}(f)).$$

Let $s_1 \leqslant s \leqslant s_2$ and $u > 0$. Thanks to the second deviation result of Lemma 3.2.2, with probability greater than $1 - 2\exp\big(-C_1 n \min\big((u 2^{s/2}/\sqrt{n}), (u^2 2^s/n)\big)\big)$,

$$|Z(\pi_s(f)) - Z(\pi_{s-1}(f))| \leqslant \frac{u 2^{s/2}}{\sqrt{n}} \alpha \left\| \pi_s(f) - \pi_{s-1}(f) \right\|_{\psi_2}. \tag{3.15}$$

Now, $s \leqslant s_0$, thus $2^s/n \leqslant 2$ and so $\min\big(u 2^{s/2}/\sqrt{n}, u^2 2^s/n\big) \geqslant \min(u, u^2)(2^s/(2n))$. In particular, (3.15) holds with probability greater than

$$1 - 2\exp\big(-C_1 2^s \min(u, u^2)\big)\big).$$

Now, a union bound on the set of links for every levels $s = s_1, \dots, s_0 - 1$ yields, for any $u > 0$, with probability greater than $1 - 2\sum_{s=s_1+1}^{s_0-1} |F_{s+1}||F_s| \exp\big(-C_1 2^s \min(u, u^2)\big)$, for every $f \in F$,

$$\left| Z(\pi_{s_0}(f)) - Z(\pi_{s_1}(f)) \right| \leqslant \sum_{s=s_1+1}^{s_0-1} \frac{u 2^{s/2}}{\sqrt{n}} \alpha \left\| \pi_s(f) - \pi_{s-1}(f) \right\|_{\psi_2}.$$

The result follows since $|F_s||F_{s+1}| \leqslant 2^{3.2^s}$ for every integer s and for u large enough. $\square$

***Proposition 3.2.5* (Beginning of the chain).** — *There exist* $c_0, c_1 > 0$ *such that the following holds. Let* $w > 0$ *and* s_1 *be such that* $2^{s_1} < (C_1/2)n\min(w, w^2)$ *(where* C_1 *is the constant appearing in Lemma 3.2.2). Let* $F \subset \mathcal{S}(L_2(\mu))$ *and* $\alpha = \mathrm{diam}(F, \psi_2)$*. With probability greater than* $1 - c_0\exp(-c_1 n \min(w, w^2))$,

$$\sup_{f\in F}\big|Z(\pi_{s_1}(f))\big| \leqslant w\alpha^2.$$

Proof. — It follows from the third deviation result of Lemma 3.2.2 and a union bound over F_{s_1}, that with probability greater than $1 - 2|F_{s_1}|\exp\big(-C_1 n\min(w, w^2)\big)$, for every $f \in F$,

$$|Z(\pi_{s_1}(f)| \leqslant w\alpha^2.$$

Since $|F_{s_1}| \leqslant 2^{2^{s_1}} < \exp\big((C_1/2)n\min(w, w^2)\big)$, the result follows. $\square$

Proof of Theorem 3.2.1. — Denote by $(F_s)_{s\in\mathbb{N}}$ an almost optimal admissible sequence of F with respect to the ψ_2-norm and, for every $s \in \mathbb{N}$ and $f \in F$, denote by $\pi_s(f)$ one of the closest point of f in F_s with respect to ψ_2. Let $s_0 \in \mathbb{N}$ be such that $s_0 = \min\big(s \geqslant 0 : 2^s \geqslant n\big)$. We have, for every $f \in F$,

$$\begin{aligned}
|Z(f)| &= \Big|\frac{1}{n}\sum_{i=1}^n f^2(X_i) - \mathbb{E}f^2(X)\Big| = \Big|\frac{1}{n}\sum_{i=1}^n (f - \pi_{s_0}(f) + \pi_{s_0}(f))^2(X_i) - \mathbb{E}f^2(X)\Big| \\
&= \Big|P_n(f - \pi_{s_0}(f))^2 + 2P_n(f - \pi_{s_0}(f))\pi_{s_0}(f) + P_n\pi_{s_0}(f)^2 - \mathbb{E}\pi_{s_0}(f)^2\Big| \\
&\leqslant W(f - \pi_{s_0}(f))^2 + 2W(f - \pi_{s_0}(f))W(\pi_{s_0}(f)) + |Z(\pi_{s_0}(f))| \\
&\leqslant W(f - \pi_{s_0}(f))^2 + 2W(f - \pi_{s_0}(f))\big(Z(\pi_{s_0}(f)) + 1\big)^{1/2} + |Z(\pi_{s_0}(f))| \\
&\leqslant 3W(f - \pi_{s_0}(f))^2 + 2W(f - \pi_{s_0}(f)) + 3|Z(\pi_{s_0}(f))| \qquad (3.16)
\end{aligned}$$

where we used $\|\pi_{s_0}(f)\|_{L_2} = 1 = \|f\|_{L_2}$ and the notation P_n stands for the empirical probability distribution $n^{-1}\sum_{i=1}^n \delta_{X_i}$.

Thanks to Proposition 3.2.3 for $v = 1$, with probability greater than $1 - c_0\exp(-c_1 n)$, for every $f \in F$,

$$W(f - \pi_{s_0}(f))^2 \leqslant c_2\frac{\gamma_2(F, \psi_2)^2}{n}. \qquad (3.17)$$

Let $w > 0$ to be chosen later and take $s_1 \in \mathbb{N}$ such that

$$s_1 = \max\Big(s \geqslant 0 : 2^s \leqslant \min\big(2^{s_0}, (C_1/2)n\min(w, w^2)\big)\Big) \qquad (3.18)$$

where C_1 is the constant defined in Lemma 3.2.2. We apply Proposition 3.2.4 with $u = 1$ and Proposition 3.2.5 to get, with probability greater than $1 - c_3\exp(-c_4 2^{s_1})$ that for every $f \in F$,

$$\begin{aligned}
|Z(\pi_{s_0}(f))| &\leqslant |Z(\pi_{s_0}(f)) - Z(\pi_{s_1}(f))| + |Z(\pi_{s_1}(f))| \\
&\leqslant c_5\alpha\frac{\gamma_2(F, \psi_2)}{\sqrt{n}} + \alpha^2 w. \qquad (3.19)
\end{aligned}$$

We combine Equation (3.16), (3.17) and (3.19) to get, with probability greater than $1 - c_6 \exp(-c_7 2^{s_1})$ that for every $f \in F$,

$$|Z(f)| \leqslant c_8 \frac{\gamma_2(F,\psi_2)^2}{n} + c_9 \frac{\gamma_2(F,\psi_2)}{\sqrt{n}} + c_{10}\alpha \frac{\gamma_2(F,\psi_2)}{\sqrt{n}} + 3\alpha^2 w.$$

First statement of Theorem 3.2.1 follows for

$$w = \max\Big(\frac{\gamma_2(F,\psi_2)}{\alpha\sqrt{n}}, \frac{\gamma_2(F,\psi_2)^2}{\alpha^2 n}\Big). \tag{3.20}$$

For the last statement, we use Proposition 3.2.3 to get

$$\mathbb{E}\sup_{f\in F} W(f - \pi_{s_0}(f))^2 = \int_0^\infty \mathbb{P}\big[\sup_{f\in F} W(f-\pi_{s_0}(f))^2 \geqslant t\big]dt \leqslant c_{11}\frac{\gamma_2(F,\psi_2)^2}{n} \tag{3.21}$$

and

$$\mathbb{E}\sup_{f\in F} W(f-\pi_{s_0}(f)) \leqslant c_{12}\frac{\gamma_2(F,\psi_2)}{\sqrt{n}}. \tag{3.22}$$

It follows from Propositions 3.2.4 and 3.2.5 for s_1 and w defined in (3.18) and (3.20) that

$$\begin{aligned}
\mathbb{E}\sup_{f\in F}|Z(\pi_{s_0}(f))| &\leqslant \mathbb{E}\sup_{f\in F}|Z(\pi_{s_0}(f)) - Z(\pi_{s_1}(f))| + \mathbb{E}\sup_{f\in F}|Z(\pi_{s_1}(f))| \\
&\leqslant \int_0^\infty \mathbb{P}\Big[\sup_{f\in F}|Z(\pi_{s_0}(f)) - Z(\pi_{s_1}(f))| \geqslant t\Big]dt + \int_0^\infty \mathbb{P}\Big[\sup_{f\in F}|Z(\pi_{s_0}(f))| \geqslant t\Big]dt \\
&\leqslant c\alpha\frac{\gamma_2(F,\psi_2)}{\sqrt{n}}.
\end{aligned} \tag{3.23}$$

The claim follows by combining equations (3.21), (3.22) and (3.23) in Equation (3.16). □

3.3. Application to Compressed Sensing

In this section, we apply Theorem 3.2.1 to prove that a $n \times N$ random matrix with i.i.d. isotropic rows vectors which are ψ_2 with constant α satisfies $\mathrm{RIP}_{2m}(\delta)$ with overwhelming probability under suitable assumptions on n, N, m and δ. Let A be such a matrix and denote by $n^{-1/2}Y_1, \dots, n^{-1/2}Y_n$ its rows vectors distributed according to a probability measure μ.

For a functions class F in $\mathcal{S}(L_2(\mu))$, it follows from Theorem 3.2.1 that with probability greater than $1 - c_1\exp\Big(-(c_2/\alpha^2)\min\Big(n\alpha^2, \gamma_2(F,\psi_2)^2\Big)\Big)$,

$$\sup_{f\in F}\left|\frac{1}{n}\sum_{i=1}^n f^2(Y_i) - \mathbb{E}f^2(Y)\right| \leqslant c_3 \max\left(\alpha\frac{\gamma_2(F,\psi_2)}{\sqrt{n}}, \frac{\gamma_2(F,\psi_2)^2}{n}\right).$$

where $\alpha = \mathrm{diam}(F, \psi_2(\mu))$. In particular, for a class F of linear functions indexed by a subset T of $\mathcal{S}^{N-1}$, the $\psi_2(\mu)$ norm and the $L_2(\mu)$ norm are equivalent on F and so with probability greater than $1 - c_1 \exp\big(-c_2 \min\big(n, \gamma_2(T, \ell_2^N)^2\big)\big)$,

$$\sup_{x\in T}\left|\frac{1}{n}\sum_{i=1}^{n}\langle Y_i, x\rangle^2 - 1\right| \leqslant c_3\alpha^2 \max\left(\frac{\gamma_2(T,\ell_2^N)}{\sqrt{n}}, \frac{\gamma_2(T,\ell_2^N)^2}{n}\right). \tag{3.24}$$

A bound on the restricted isometry constant δ_{2m} follows from (3.24). Indeed let $T = S_2(\Sigma_{2m})$ then with probability greater than

$$1 - c_1 \exp\big(-c_2 \min\big(n, \gamma_2(S_2(\Sigma_{2m}), \ell_2^N)^2\big)\big),$$

$$\delta_{2m} \leqslant c_3\alpha^2 \max\left(\frac{\gamma_2(S_2(\Sigma_{2m}),\ell_2^N)}{\sqrt{n}}, \frac{\gamma_2(S_2(\Sigma_{2m}),\ell_2^N)^2}{n}\right).$$

Now, it remains to bound $\gamma_2(S_2(\Sigma_{2m}), \ell_2^N)$. Such a bound may follow from the Majorizing measure theorem (cf. Theorem 3.1.5):

$$\gamma_2(S_2(\Sigma_{2m}), \ell_2^N) \sim \ell_*(S_2(\Sigma_{2m})).$$

Since $S_2(\Sigma_{2m})$ can be written as a union of sphere with short support, it is easy to obtain

$$\ell_*(S_2(\Sigma_{2m})) = \mathbb{E}\Big(\sum_{i=1}^{2m}(g_i^*)^2\Big)^{1/2} \tag{3.25}$$

where $g_1, \dots, g_N$ are N i.i.d. standard Gaussian variables and $(g_i^*)_{i=1}^N$ is a non-decreasing rearrangement of $(|g_i|)_{i=1}^N$. A bound on (3.25) follows from the following result.

Lemma 3.3.1. — *There exist absolute positive constants c_1, c_2 and c_3 such that the following holds. Let $(g_i)_{i=1}^N$ be a family of N i.i.d. standard Gaussian variables. Denote by $(g_i^*)_{i=1}^N$ a non-increasing rearrangement of $(|g_i|)_{i=1}^N$. For any $k = 1, \dots, N$, we have*

$$\sqrt{c_1 \log\Big(\frac{c_2 N}{k}\Big)} \leqslant \mathbb{E}\Big(\frac{1}{k}\sum_{i=1}^{k}(g_i^*)^2\Big)^{1/2} \leqslant \sqrt{\log\Big(\frac{c_3 N}{k}\Big)}.$$

Proof. — Let g be a standard real-valued Gaussian variable and take $c_0 > 0$ such that $\mathbb{E}\exp(g^2) \leqslant c_0$. By convexity, it follows that

$$\exp\Big(\mathbb{E}\Big(\frac{1}{k}\sum_{i=1}^{k}(g_i^*)^2\Big)\Big) \leqslant \frac{1}{k}\sum_{i=1}^{k}\mathbb{E}\exp\big((g_i^*)^2\big) \leqslant \frac{1}{k}\sum_{i=1}^{N}\mathbb{E}\exp(g_i^2) \leqslant \frac{c_0 N}{k}.$$

Finally,

$$\mathbb{E}\Big(\frac{1}{k}\sum_{i=1}^{k}(g_i^*)^2\Big)^{1/2} \leqslant \Big(\mathbb{E}\frac{1}{k}\sum_{i=1}^{k}(g_i^*)^2\Big)^{1/2} \leqslant \sqrt{\log\big(c_0 N/k\big)}.$$

For the lower bound, we note that for any $x > 0$,

$$\sqrt{\frac{2}{\pi}}\int_x^\infty \exp(-s^2/2)ds \geqslant \sqrt{\frac{2}{\pi}}\int_x^{2x} \exp(-s^2/2)ds \geqslant \sqrt{\frac{2}{\pi}}x\exp(-2x^2).$$

In particular, for any $c_0 > 0, c_1 \geqslant e$ and $1 \leqslant k \leqslant N$,

$$\mathbb{P}\Big[|g| \geqslant \sqrt{c_0 \log\big(c_1 N/k\big)}\Big] \geqslant \sqrt{\frac{2c_0}{\pi} \log\Big(\frac{c_1 N}{k}\Big)} \Big(\frac{k}{c_1 N}\Big)^{2c_0}. \tag{3.26}$$

It follows from Markov inequality that

$$\begin{aligned}
\mathbb{E}\Big(\frac{1}{k}\sum_{i=1}^{k}(g_i^*)^2\Big)^{1/2} &\geqslant \mathbb{E}g_k^* \geqslant \sqrt{c_0 \log\big(c_1 N/k\big)}\,\mathbb{P}\Big[g_k^* \geqslant \sqrt{c_0 \log\big(c_1 N/k\big)}\Big] \\
&= \sqrt{c_0 \log\big(c_1 N/k\big)}\,\mathbb{P}\Big[\sum_{i=1}^{N} I\Big(|g_i| \geqslant \sqrt{c_0 \log\big(c_1 N/k\big)}\Big) \geqslant k\Big] \\
&= \sqrt{c_0 \log\big(c_1 N/k\big)}\,\mathbb{P}\Big[\sum_{i=1}^{N} \delta_i \geqslant k\Big]
\end{aligned}$$

where $I(\cdot)$ denotes the indicator function and $\delta_i = I\Big(|g_i| \geqslant \sqrt{2\log\big(c_1 N/k\big)}\Big)$ for every $i = 1, \dots, N$. Since $(\delta_i)_{i=1}^N$ is a family of i.i.d. Bernoulli variables with mean $\delta = \mathbb{P}\Big[|g| \geqslant \sqrt{2\log\big(c_1 N/k\big)}\Big]$, it follows from Bernstein inequality (cf. Theorem 1.2.6) that, as long as $k \leqslant \delta N/2$ and $N\delta \geqslant 10\log 4$,

$$\mathbb{P}\Big[\sum_{i=1}^{N} \delta_i \geqslant k\Big] \geqslant \mathbb{P}\Big[\frac{1}{N}\sum_{i=1}^{N} \delta_i - \delta \geqslant \frac{-\delta}{2}\Big] \geqslant 1/2.$$

Thanks to (3.26), it is easy to check that for $c_0 = 1/4$ and $c_1 = 4$, we have $k \leqslant \delta N/2$. □

It is now possible to obtain the result announced at the beginning of the section.

Theorem 3.3.2. — *There exist absolute positive constants c_0, c_1, c_2 and c_3 such that the following holds. Let A be a $n \times N$ random matrix with rows vectors $n^{-1/2}Y_1, \dots, n^{-1/2}Y_n$. Assume that $Y_1, \dots, Y_n$ are i.i.d. isotropic vectors of $\mathbb{R}^N$, which are ψ_2 with constant α. Let m be an integer and $\delta \in (0, 1)$ such that*

$$m \log\big(c_0 N/m\big) = c_1 n \delta^2,$$

then, with probability greater than $1 - c_2 \exp(-c_3 n\delta^2/\alpha^4)$, the restricted isometry constant of order $2m$ of A is such that $\delta_{2m} \leqslant \delta$.

3.4. Notes and comments

Dudley entropy bound (cf. Theorem 3.1.2) can be found in [**Dud67**]. Other Dudley type entropy bounds for processes $(X_t)_{t\in T}$ with Orlicz norm of the increments satisfying, for every $s, t \in T$,

$$\|X_t - X_s\|_\psi \leqslant d(s,t) \tag{3.27}$$

may be obtained (see [**Pis80**] and [**Kôn80**]). Under the increment condition (3.27), the Dudley entropy integral

$$\int_0^\infty \psi^{-1}\big(N(T,d,\epsilon)\big)d\epsilon,$$

where ψ^{-1} is the inverse function of the Orlicz function ψ, is an upper bound on $\mathbb{E}\sup_{t\in T}$ coming out of the chaining argument.

For the partition scheme method used in the generic chaining mechanism of Theorem 3.1.4, we refer to [**Tal05**] and [**Tal01**]. The generic chaining mechanism was first introduced using majorizing measures. This tool was introduced in [**Fer74, Fer75**] and is implicit in earlier work by Preston based on an important result of Garcia, Rodemich and Rumsey. In [**Tal87**], the author proves that majorizing measures are the key quantities to analyze the supremum of Gaussian processes. In particular, the majorizing measure theorem (cf. Theorem 3.1.5) is shown in [**Tal87**]. More about majorizing measures and majorizing measure theorems for other processes than Gaussian processes can be found in [**Tal96a**] and [**Tal95**]. Connections between the majorizing measures and partition schemes have been showed in [**Tal05**] and [**Tal01**].

The upper bounds on the process

$$\sup_{f\in F}\Big|\frac{1}{n}\sum_{i=1}^n f^2(X_i)-\mathbb{E}f^2(X)\Big| \tag{3.28}$$

developed in Section 3.2 follow the line of [**MPTJ07**]. Other bounds on (3.28) can be found in the next chapter (cf. Theorem 5.3.14).

CHAPTER 4

SINGULAR VALUES

The singular values of a matrix are very natural geometrical quantities which play an important role in pure and applied mathematics. The first part of this chapter is a compendium on the singular values of matrices. The second part of the chapter concerns random matrices, and constitutes a quick tour in this vast subject. It starts with important facts on Gaussian random matrices, gives a proof of the universal Marchenko-Pastur theorem regarding the counting probability measure of the singular values, and ends with the Bai-Yin theorem statement on the extremal singular values.

For every square matrix $A \in \mathcal{M}_{n,n}(\mathbb{C})$, we denote by $\lambda_1(A), \ldots, \lambda_n(A)$ the eigenvalues of A which are the roots in $\mathbb{C}$ of the characteristic polynomial $\det(A - ZI) \in \mathbb{C}[Z]$. We label the eigenvalues of A so that $|\lambda_1(A)| \geqslant \cdots \geqslant |\lambda_n(A)|$. In all this chapter, $\mathbb{K}$ stands for $\mathbb{R}$ or $\mathbb{C}$, and we say that $U \in \mathcal{M}_{n,n}(\mathbb{K})$ is $\mathbb{K}$-unitary when $UU^* = I$.

4.1. The notion of singular values

This section gathers a selection of classical results from linear algebra. We begin with the Singular Value Decomposition (SVD), a fundamental tool in matrix analysis, which expresses a diagonalization up to unitary transformations of the space.

***Theorem 4.1.1* (Singular Value Decomposition).** — *For every $A \in \mathcal{M}_{m,n}(\mathbb{K})$, there exists a couple of $\mathbb{K}$–unitary matrices U $(m \times m)$ and V $(n \times n)$ and a sequence of real numbers $s_1 \geqslant \cdots \geqslant s_{m \wedge n} \geqslant 0$ such that*

$$U^* A V = \operatorname{diag}(s_1, \ldots, s_{m \wedge n}) \in \mathcal{M}_{m,n}(\mathbb{K}).$$

This sequence of real numbers does not depend on the particular choice of U, V.

Proof. — Let $v \in \mathbb{K}^n$ be such that $|Av|_2 = \max_{|x|_2=1} |Ax|_2 = \|A\|_{2\to 2} = s$. If $|v|_2 = 0$ then $A = 0$ and the desired result is trivial. If $s > 0$ then let us define $u = Av/s$. One can find a $\mathbb{K}$-unitary $m \times m$ matrix U with first column vector equal to u, and a $\mathbb{K}$-unitary $n \times n$ matrix V with first column vector equal to v. It follows that

$$U^* A V = \begin{pmatrix} s & w^* \\ 0 & B \end{pmatrix} = A_1$$

for some $w \in \mathcal{M}_{n-1,1}(\mathbb{K})$ and $B \in \mathcal{M}_{m-1,n-1}(\mathbb{K})$. If t is the first row of A_1 then $|A_1 t^*|_2^2 \geqslant (s^2 + |w|_2^2)^2$ and therefore $\|A_1\|_{2\to 2}^2 \geqslant s^2 + |w|_2^2 \geqslant \|A\|_{2\to 2}^2$. On the other hand, since A and A_1 are unitary equivalent, we have $\|A_1\|_{2\to 2} = \|A\|_{2\to 2}$. Therefore $w = 0$, and the desired decomposition follows by a simple induction.

If one sees the diagonal matrix $D := \operatorname{diag}(s_1(A)^2, \ldots, s_{m\wedge n}(A)^2)$ as an element of $\mathcal{M}_{m,m}(\mathbb{K})$ or $\mathcal{M}_{n,n}(\mathbb{K})$ by appending as much zeros as needed, we have

$$U^*AA^*U = D \quad \text{and} \quad V^*A^*AV = D.$$

The positive semidefinite Hermitian matrices $AA^* \in \mathcal{M}_{m,m}(\mathbb{K})$ and $A^*A \in \mathcal{M}_{n,n}(\mathbb{K})$ share the same sequence of eigenvalues, up to the multiplicity of the eigenvalue 0, and for every $k \in \{1, \ldots, m \wedge n\}$,

$$s_k(A) = \lambda_k(\sqrt{AA^*}) = \sqrt{\lambda_k(AA^*)} = \sqrt{\lambda_k(A^*A)} = \lambda_k(\sqrt{A^*A}) = s_k(A^*).$$

This shows the uniqueness of $s_1, \ldots, s_{m\wedge n}$. The columns of U and V are the eigenvectors of the positive semidefinite Hermitian matrices AA^* and A^*A. □

The numbers $s_k(A) := s_k$ for $k \in \{1, \ldots, m \wedge n\}$ are called the *singular values* of A. It is sometimes convenient to use the convention $s_k(A) = 0$ if $k > m \wedge n$. For any $A \in \mathcal{M}_{m,n}(\mathbb{K})$, the matrices A, $\bar{A}$, $A^\top$, A^*, UA, AV share the same sequences of singular values, for any $\mathbb{K}$–unitary matrices U, V.

Remark 4.1.2 **(Normal matrices)**. — *When A is normal (i.e. $AA^* = A^*A$) then $m = n$ and $s_k(A) = |\lambda_k(A)|$ for every $k \in \{1, \ldots, n\}$. Hermitian matrices are normal. Note also that $s_k(A^r) = s_k(A)^r$ for any $r \geqslant 1$ (not true in general if A is not normal).*

Remark 4.1.3 **(Hermitization)**. — *For any $A \in \mathcal{M}_{m,n}(\mathbb{K})$, the eigenvalues of the $(m+n) \times (m+n)$ Hermitian matrix*

$$H = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix}$$

are given by $+s_1(A), -s_1(A), \ldots, +s_{m\wedge n}(A), -s_{m\wedge n}(A), 0, \ldots, 0$ where the notation $0, \ldots, 0$ stands for a sequence of 0's of length $m+n-2(m\wedge n) = (m\vee n)-(m\wedge n)$. One may deduce the singular values of A from the eigenvalues of H, and $H^2 = A^*A \oplus AA^*$. If $m = n$ and $A_{i,j} \in \{0, 1\}$ for all i, j, then A is the adjacency matrix of an oriented graph, and H is the adjacency matrix of a compagnon nonoriented bipartite graph.

Remark 4.1.4 **(Left and right eigenvectors)**. — *If $u_1 \perp \cdots \perp u_m \in \mathbb{K}^m$ and $v_1 \perp \cdots \perp v_n \in \mathbb{K}^n$ are the columns of U, V then for every $k \in \{1, \ldots, m \wedge n\}$,*

$$Av_k = s_k(A)u_k \quad \text{and} \quad A^*u_k = s_k(A)v_k \tag{4.1}$$

while $Av_k = 0$ and $A^*u_k = 0$ for $k > m \wedge n$. The SVD gives an intuitive geometrical interpretation of A and A^* as a dual correspondence/dilation between two orthonormal bases known as the left and right eigenvectors of A and A^*. Additionally, A has exactly $r = \operatorname{rank}(A)$ nonzero singular values $s_1(A), \ldots, s_r(A)$ and

$$A = \sum_{k=1}^r s_k(A)u_k v_k^* \quad \text{and} \quad \begin{cases} \operatorname{kernel}(A) & = \operatorname{span}\{v_{r+1}, \ldots, v_n\}, \\ \operatorname{range}(A) & = \operatorname{span}\{u_1, \ldots, u_r\}. \end{cases}$$

We have also $s_k(A) = |Av_k|_2 = |A^*u_k|_2$ *for every* $k \in \{1,\dots,m\wedge n\}$.

Condition number. — The condition number of $A \in \mathcal{M}_{n,n}(\mathbb{K})$ is given by

$$\kappa(A) = \|A\|_{2\to 2}\left\|A^{-1}\right\|_{2\to 2} = \frac{s_1(A)}{s_n(A)}.$$

The condition number quantifies the numerical sensitivity of linear systems involving A. For instance, if $x \in \mathbb{K}^n$ is the solution of the linear equation $Ax = b$ then $x = A^{-1}b$. If b is known up to precision $\delta \in \mathbb{K}^n$ then x is known up to precision $A^{-1}\delta$. Therefore, the ratio of relative errors for the determination of x is given by

$$R(b,\delta) = \frac{\left|A^{-1}\delta\right|_2/\left|A^{-1}b\right|_2}{|\delta|_2/|b|_2} = \frac{\left|A^{-1}\delta\right|_2}{|\delta|_2}\frac{|b|_2}{|A^{-1}b|_2}.$$

Consequently, we obtain

$$\max_{b\neq 0,\delta\neq 0} R(b,\delta) = \left\|A^{-1}\right\|_{2\to 2}\|A\|_{2\to 2} = \kappa(A).$$

Geometrically, $\kappa(A)$ measures the "spherical defect" of the ellipsoid in figure (1).

Basic properties. — The eigenvalues of a Hermitian matrix can be expressed in terms of the entries of the matrix via minimax variational formulas. The following theorem is the counterpart for the singular values. It can be deduced from its Hermitian cousin.

Theorem 4.1.5 **(Courant–Fischer variational formulas for singular values)** *For every* $A \in \mathcal{M}_{m,n}(\mathbb{K})$ *and every* $k \in \{1,\dots,m\wedge n\}$,

$$s_k(A) = \max_{V\in\mathcal{G}_{n,k}}\min_{\substack{x\in V\\ |x|_2=1}}|Ax|_2 = \min_{V\in\mathcal{G}_{n,n-k+1}}\max_{\substack{x\in V\\ |x|_2=1}}|Ax|_2$$

where $\mathcal{G}_{n,k}$ *is the set of all subspaces of* $\mathbb{K}^n$ *of dimension* k. *In particular, we have*

$$s_1(A) = \max_{\substack{x\in\mathbb{K}^n\\ |x|_2=1}}|Ax|_2 \quad \text{and} \quad s_{m\wedge n}(A) = \min_{\substack{x\in\mathbb{K}^n\\ |x|_2=1}}|Ax|_2.$$

We have also the following alternative formulas, for every $k \in \{1,\dots,m\wedge n\}$,

$$s_k(A) = \max_{\substack{V\in\mathcal{G}_{n,k}\\ W\in\mathcal{G}_{m,k}}}\min_{\substack{(x,y)\in V\times W\\ |x|_2=|y|_2=1}}\langle Ax, y\rangle.$$

As an exercise, one can check that if $A \in \mathcal{M}_{m,n}(\mathbb{R})$ then the variational formulas for $\mathbb{K} = \mathbb{C}$, if one sees A as an element of $\mathcal{M}_{m,n}(\mathbb{C})$, coincide actually with the formulas for $\mathbb{K} = \mathbb{R}$. Geometrically, the matrix A maps the Euclidean unit ball to an ellipsoid, and the singular values of A are exactly the half lengths of the $m\wedge n$ largest principal axes of this ellipsoid, see figure 1. The remaining axes have a zero length. In particular, for $A \in \mathcal{M}_{n,n}(\mathbb{K})$, the variational formulas for the extremal singular values $s_1(A)$ and $s_n(A)$ correspond to the half length of the longest and shortest axes.

From the Courant–Fischer variational formulas, the largest singular value is the operator norm of A for the Euclidean norm $|\cdot|_2$, namely

$$s_1(A) = \|A\|_{2\to 2}.$$

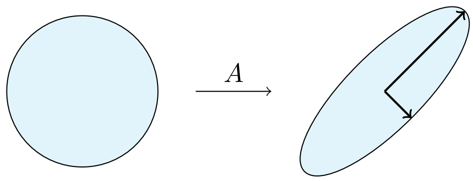

FIGURE 1. Largest and smallest singular values of $A \in \mathcal{M}_{2,2}(\mathbb{R})$.

The map $A \mapsto s_1(A)$ is Lipschitz and convex. In the same spirit, if U, V are the couple of $\mathbb{K}$–unitary matrices from an SVD of A, then for any $k \in \{1, \ldots, \mathrm{rank}(A)\}$,

$$s_k(A) = \min_{\substack{B \in \mathcal{M}_{m,n}(\mathbb{K}) \\ \mathrm{rank}(B)=k-1}} \|A - B\|_{2\to 2} = \|A - A_k\|_{2\to 2} \quad \text{where} \quad A_k = \sum_{i=1}^{k-1} s_i(A) u_i v_i^*$$

with u_i, v_i as in (4.1). Let $A \in \mathcal{M}_{n,n}(\mathbb{K})$ be a square matrix. If A is invertible then the singular values of A^{-1} are the inverses of the singular values of A, in other words

$$\forall k \in \{1, \ldots, n\}, \quad s_k(A^{-1}) = s_{n-k+1}(A)^{-1}.$$

Moreover, a square matrix $A \in \mathcal{M}_{n,n}(\mathbb{K})$ is invertible iff $s_n(A) > 0$, and in this case

$$s_n(A) = s_1(A^{-1})^{-1} = \left\|A^{-1}\right\|_{2\to 2}^{-1}.$$

Contrary to the map $A \mapsto s_1(A)$, the map $A \mapsto s_n(A)$ is Lipschitz but is not convex when $n \geqslant 2$. Regarding the Lipschitz nature of the singular values, the Courant–Fischer variational formulas provide the following more general result, which has a Hermitian counterpart.

***Theorem 4.1.6* (Additive perturbations).** — *If $A, B \in \mathcal{M}_{m,n}(\mathbb{K})$ then for every $i, j \in \{1, \ldots, m \wedge n\}$ with $i + j \leqslant 1 + (m \wedge n)$,*

$$s_{i+j-1}(A) \leqslant s_i(B) + s_j(A - B).$$

Theorem 4.1.6 implies that $A \mapsto s(A) := (s_1(A), \ldots, s_n(A))$ is 1-Lipschitz from $(\mathcal{M}_{m,n}(\mathbb{K}), \|\cdot\|_{2\to 2})$ to $(\mathbb{R}_+^{m\wedge n}, \|\cdot\|_\infty)$ since

$$\max_{1 \leqslant k \leqslant m\wedge n} |s_k(A) - s_k(B)| \leqslant \|A - B\|_{2\to 2}.$$

From the Courant–Fischer variational formulas we obtain also the following result.

***Theorem 4.1.7* (Interlacing by rows deletion).** — *Let $A \in \mathcal{M}_{m,n}(\mathbb{K})$ and $k \in \{1, 2, \ldots\}$ with $1 \leqslant k \leqslant m \leqslant n$ and let $B \in \mathcal{M}_{m-k,n}(\mathbb{K})$ be a matrix obtained from A by deleting k rows. Then for every $i \in \{1, \ldots, m-k\}$,*

$$s_i(A) \geqslant s_i(B) \geqslant s_{i+k}(A).$$

Theorem 4.1.7 implies $[s_{m-k}(B), s_1(B)] \subset [s_m(A), s_1(A)]$. Row deletions produce a compression of the singular values interval. Another way to express this phenomenon consists in saying that if we add a row to B then the largest singular value increases while the smallest singular value is diminished.

The following result is an immediate consequence of theorem 4.1.6. It is used in the proof of the Marchenko-Pastur theorem 4.3.1 for a rank one additive perturbation. We recall that the *cumulative distribution function* of a probability measure μ on $\mathbb{R}$ is the function $F_\mu : \mathbb{R} \to [0,1]$ defined by $F_\mu(t) := \mu((-\infty, t])$ for every $t \in \mathbb{R}$.

***Theorem 4.1.8* (Rank inequality).** — *If $A, B \in \mathcal{M}_{m,n}(\mathbb{K})$ and if F_{μ_A} and F_{μ_B} denote the cumulative distribution functions of the counting probability measures $\mu_A := \frac{1}{m}\sum_{k=1}^m \delta_{\lambda_k(AA^*)}$ and $\mu_B := \frac{1}{m}\sum_{k=1}^m \delta_{\lambda_k(BB^*)}$ then*

$$\|F_{\mu_A} - F_{\mu_B}\|_\infty \leqslant \frac{\operatorname{rank}(A-B)}{m}.$$

Hilbert-Schmidt norm. — For every $A \in \mathcal{M}_{m,n}(\mathbb{K})$ we set

$$\|A\|_{\mathrm{HS}}^2 := \operatorname{Tr}(AA^*) = \operatorname{Tr}(A^*A) = \sum_{i,j=1}^n |A_{i,j}|^2 = s_1(A)^2 + \cdots + s_{m\wedge n}(A)^2.$$

This defines the so called Hilbert–Schmidt or Frobenius norm $\|\cdot\|_{\mathrm{HS}}$. We have always

$$\|A\|_{2\to 2} \leqslant \|A\|_{\mathrm{HS}} \leqslant \sqrt{\operatorname{rank}(A)}\ \|A\|_{2\to 2}$$

where equalities are achieved when $\operatorname{rank}(A) = 1$ and $A = \lambda I \in \mathcal{M}_{m,n}(\mathbb{K})$ with $\lambda \in \mathbb{K}$ respectively. The advantage of $\|\cdot\|_{\mathrm{HS}}$ over $\|\cdot\|_{2\to 2}$ lies in its convenient expression in terms of the matrix entries. Actually, the Frobenius norm is Hilbertian for the Hermitian product

$$\langle A, B\rangle = \operatorname{Tr}(AB^*).$$

We have seen that a matrix $A \in \mathcal{M}_{m,n}(\mathbb{K})$ has exactly $r = \operatorname{rank}(A)$ non zero singular values. If $k \in \{0, 1, \ldots, r\}$ and if A_k is obtained from the SVD of A by forcing $s_i = 0$ for all $i > k$ then we have the Eckart and Young observation:

$$\min_{\substack{B \in \mathcal{M}_{m,n}(\mathbb{K}) \\ \operatorname{rank}(B) = k}} \|A - B\|_{\mathrm{HS}}^2 = \|A - A_k\|_{\mathrm{HS}}^2 = s_{k+1}(A)^2 + \cdots + s_r(A)^2.$$

The following result shows that $A \mapsto s(A)$ is 1-Lipschitz for $\|\cdot\|_{\mathrm{HS}}$ and $\|\cdot\|_2$.

***Theorem 4.1.9* (Hoffman-Wielandt).** — *If $A, B \in \mathcal{M}_{m,n}(\mathbb{K})$ then*

$$\sum_{k=1}^{m\wedge n} (s_k(A) - s_k(B))^2 \leqslant \|A - B\|_{\mathrm{HS}}^2.$$

The following result is used for the proof of the Marchenko-Pastur theorem 4.3.1.

***Theorem 4.1.10* (Lévy distance inequality).** — *If $A, B \in \mathcal{M}_{m,n}(\mathbb{K})$ and if F_{μ_A} and F_{μ_B} are the cumulative distribution functions of the counting probability measures $\mu_A := \frac{1}{m}\sum_{k=1}^m \delta_{\lambda_k(AA^*)}$ and $\mu_B := \frac{1}{m}\sum_{k=1}^m \delta_{\lambda_k(BB^*)}$ then*

$$L(\mu_A, \mu_B) \leqslant \left(\frac{2}{m^2}(\|A\|_{\mathrm{HS}}^2 + \|B\|_{\mathrm{HS}}^2)\,\|A - B\|_{\mathrm{HS}}^2\right)^{1/4}$$

where $L(\mu_A, \mu_B)$ is the Lévy distance between μ_A and μ_B defined by

$$L(\mu_A, \mu_B) = \inf\{\varepsilon > 0 : F_{\mu_A}(\cdot - \varepsilon) - \varepsilon \leqslant F_{\mu_B}(\cdot) \leqslant F_{\mu_A}(\cdot + \varepsilon) + \varepsilon\}.$$

We end up with an inequality due to von Neumann.

***Theorem 4.1.11* (von Neumann inequality).** — *If $A, B \in \mathcal{M}_{m,n}(\mathbb{K})$ then*

$$|\mathrm{Tr}(AB^*)| \leqslant \sum_{k=1}^{m\wedge n} s_k(A)s_k(B).$$

***Remark 4.1.12* (Unitary invariance).** — *For every $k \in \{1, \ldots, m \wedge n\}$ and any real number $p \geqslant 1$, the map $A \in \mathcal{M}_{m,n}(\mathbb{K}) \mapsto (s_1(A)^p + \cdots + s_k(A)^p)^{1/p}$ is a unitary invariant norm on $\mathcal{M}_{m,n}(\mathbb{K})$. We recover the operator norm $\|A\|_{2\to 2}$ for $k = 1$ and the Frobenius norm $\|A\|_{\mathrm{HS}}$ for $(k, p) = (m \wedge n, 2)$. The special case $(k, p) = (m \wedge n, 1)$ gives the Ky Fan norms, while the special case $k = m \wedge n$ gives the Schatten norms, a concept already considered in the first chapter.*

Basic relationships between eigenvalues and singular values. — We know that if $A \in \mathcal{M}_{n,n}(\mathbb{K})$ is normal (i.e. $AA^* = A^*A$) then $s_k(A) = |\lambda_k(A)|$ for every $k \in \{1, \ldots, n\}$. Beyond normal matrices, for every $A \in \mathcal{M}_{n,n}(\mathbb{K})$ with row vectors $R_1, \ldots, R_n$, we have, by viewing $|\det(A)|$ as the volume of a parallelepiped,

$$|\det(A)| = \prod_{k=1}^{n} |\lambda_k(A)| = \prod_{k=1}^{n} s_k(A) = \prod_{k=1}^{n} \mathrm{dist}(R_k, \mathrm{span}\{R_1, \ldots, R_{k-1}\}) \qquad (4.2)$$

(basis $\times$ height etc). The following result, due to Weyl, is less global and more subtle.

***Theorem 4.1.13* (Weyl inequalities).** — *If $A \in \mathcal{M}_{n,n}(\mathbb{K})$, then*

$$\forall k \in \{1, \ldots, n\}, \quad \prod_{i=1}^{k} |\lambda_i(A)| \leqslant \prod_{i=1}^{k} s_i(A) \quad \text{and} \quad \prod_{i=k}^{n} s_i(A) \leqslant \prod_{i=k}^{n} |\lambda_i(A)| \qquad (4.3)$$

Moreover, for every increasing function φ from $(0, \infty)$ to $(0, \infty)$ such that $t \mapsto \varphi(e^t)$ is convex on $(0, \infty)$ and $\varphi(0) := \lim_{t\to 0^+} \varphi(t) = 0$, we have

$$\forall k \in \{1, \ldots, n\}, \quad \sum_{i=1}^{k} \varphi(|\lambda_i(A)|^2) \leqslant \sum_{i=1}^{k} \varphi(s_i(A)^2). \qquad (4.4)$$

Observe that from (4.4) with $\varphi(t) = t$ for every $t > 0$ and $k = n$, we obtain

$$\sum_{k=1}^{n} |\lambda_k(A)|^2 \leqslant \sum_{k=1}^{n} s_k(A)^2 = \mathrm{Tr}(AA^*) = \sum_{i,j=1}^{n} |A_{i,j}|^2 = \mathrm{Tr}(AA^*) = \|A\|_{\mathrm{HS}}^2. \qquad (4.5)$$

The following result, due to Horn, constitutes a converse to Weyl inequalities 4.2. It explains why so many generic relationships between eigenvalues and singular values are consequences of (4.2), for instance via majorization inequalities and techniques.

***Theorem 4.1.14* (Horn inverse problem).** — *If $\lambda \in \mathbb{C}^n$ and $s \in [0, \infty)^n$ satisfy $|\lambda_1| \geqslant \cdots \geqslant |\lambda_n|$ and $s_1 \geqslant \cdots \geqslant s_n$ and the Weyl relationships (4.3) then there exists $A \in \mathcal{M}_{n,n}(\mathbb{C})$ such that $\lambda_i(A) = \lambda_i$ and $s_i(A) = s_i$ for every $i \in \{1, \ldots, n\}$.*

From (4.2) we get $s_n(A) \leqslant |\lambda_n(A)| \leqslant |\lambda_1(A)| \leqslant s_1(A)$ for any $A \in \mathcal{M}_{n,n}(\mathbb{K})$. In particular, we have the following spectral radius / operator norm comparison:

$$\rho(A) = |\lambda_1(A)| \leqslant s_1(A) = \|A\|_{2\to 2}.$$

In this spirit, the following result, due to Gelfand, allows to estimate the spectral radius $\rho(A)$ with the singular values of the powers of A.

***Theorem 4.1.15* (Gelfand spectral radius formula).** — *Let $\|\cdot\|$ be a submultiplicative matrix norm on $\mathcal{M}_{n,n}(\mathbb{K})$ such as the operator norm $\|\cdot\|_{2\to 2}$ or the Frobenius norm $\|\cdot\|_{\mathrm{HS}}$. Then for every matrix $A \in \mathcal{M}_{n,n}(\mathbb{K})$ we have*

$$\rho(A) := |\lambda_1(A)| = \lim_{k\to\infty} \sqrt[k]{\|A^k\|}.$$

The eigenvalues of non normal matrices are far more sensitive to perturbations than the singular values, and this is captured by the notion of pseudo spectrum:

$$\mathrm{pseudospec}_\varepsilon(A) := \bigcup_{\|B-A\|_{2\to 2}\leqslant\varepsilon} \{\lambda_1(B), \ldots, \lambda_n(B)\}.$$

If A is normal then $\mathrm{pseudospec}_\varepsilon(A)$ is the ε-neighborhood of the spectrum of A.

Relation with rows distances. — The following couple of lemmas relate the singular values of matrices to distances between rows (or columns). For square random matrices, they provide a convenient control on the operator norm and Frobenius norm of the inverse respectively.

***Lemma 4.1.16* (Rows and operator norm).** — *If $A \in \mathcal{M}_{m,n}(\mathbb{K})$ has row vectors $R_1, \ldots, R_m$, then, denoting $R_{-i} = \mathrm{span}\{R_j : j \neq i\}$, we have*

$$m^{-1/2} \min_{1\leqslant i\leqslant m} \mathrm{dist}_2(R_i, R_{-i}) \leqslant s_{m\wedge n}(A) \leqslant \min_{1\leqslant i\leqslant m} \mathrm{dist}_2(R_i, R_{-i}).$$

Proof. — Since A and $A^\top$ have same singular values, we can prove the statement for the column vectors $C_1, \ldots, C_n$ of A (swap m and n). For every $x \in \mathbb{K}^n$ and every $i \in \{1, \ldots, n\}$, the triangle inequality and the identity $Ax = x_1C_1 + \cdots + x_nC_n$ give

$$|Ax|_2 \geqslant \mathrm{dist}_2(Ax, C_{-i}) = \min_{y\in C_{-i}} |Ax - y|_2 = \min_{y\in C_{-i}} |x_iC_i - y|_2 = |x_i|\mathrm{dist}_2(C_i, C_{-i}).$$

If $|x|_2 = 1$ then necessarily $|x_i| \geqslant n^{-1/2}$ for some $i \in \{1, \ldots, n\}$, and therefore

$$s_{m\wedge n}(A) = \min_{|x|_2=1} |Ax|_2 \geqslant n^{-1/2} \min_{1\leqslant i\leqslant n} \mathrm{dist}_2(C_i, C_{-i}).$$

Conversely, for any $i \in \{1, \ldots, n\}$, there exists a vector $y \in \mathbb{K}^n$ with $y_i = 1$ such that

$$\mathrm{dist}_2(C_i, C_{-i}) = |y_1C_1 + \cdots + y_nC_n|_2 = |Ay|_2 \geqslant |y|_2 \min_{|x|_2=1} |Ax|_2 \geqslant s_{m\wedge n}(A)$$

where we used the fact that $|y|_2^2 = |y_1|^2 + \cdots + |y_n|^2 \geqslant |y_i|^2 = 1$. $\square$

***Lemma 4.1.17* (Rows and trace norm).** — *If $A \in \mathcal{M}_{m,n}(\mathbb{K})$ with $m \leqslant n$ has rows $R_1, \dots, R_m$ and if* $\mathrm{rank}(A) = m$ *then, denoting* $R_{-i} = \mathrm{span}\{R_j : j \neq i\}$,

$$\sum_{i=1}^{m} s_i^{-2}(A) = \sum_{i=1}^{m} \mathrm{dist}_2(R_i, R_{-i})^{-2}.$$

Proof. — The orthogonal projection of R_i^* on R_{-i}^* is $B^*(BB^*)^{-1}BR_i^*$ where B is the $(m-1)\times n$ matrix obtained from A by removing the row R_i. In particular, we have

$$|R_i|_2^2 - \mathrm{dist}_2(R_i, R_{-i})^2 = \left|B^*(BB^*)^{-1}BR_i^*\right|_2^2 = (BR_i^*)^*(BB^*)^{-1}BR_i^*$$

by the Pythagoras theorem. On the other hand, the Schur bloc inversion formula states that if M is an $m \times m$ matrix then for every partition $\{1, \dots, m\} = I \cup I^c$,

$$(M^{-1})_{I,I} = (M_{I,I} - M_{I,I^c}(M_{I^c,I^c})^{-1}M_{I^c,I})^{-1}.$$

Now we take $M = AA^*$ and $I = \{i\}$, and we note that $(AA^*)_{i,j} = R_iR_j^*$, which gives

$$((AA^*)^{-1})_{i,i} = (R_iR_i^* - (BR_i^*)^*(BB^*)^{-1}BR_i^*)^{-1} = \mathrm{dist}_2(R_i, R_{-i})^{-2}.$$

The desired formula follows by taking the sum over $i \in \{1, \dots, m\}$. □

Unitary bidiagonalization and computation of the SVD. — To compute the SVD of $A \in \mathcal{M}_{m,n}(\mathbb{K})$ one can diagonalize AA^* or diagonalize the Hermitian matrix H defined in remark 4.1.3. Unfortunately, this approach can lead to a loss of precision numerically. In practice, and up to machine precision, the SVD is better computed by using for instance a variant of the QR algorithm after unitary bidiagonalization.

Let us explain how works the unitary bidiagonalization of a matrix $A \in \mathcal{M}_{m,n}(\mathbb{K})$ with $m \leqslant n$. If r_1 is the first row of A, the Gram–Schmidt (or Householder) algorithm provides an $m \times m$ $\mathbb{K}$–unitary matrix V_1 which maps r_1^* to a multiple of e_1. Since V_1 is unitary the matrix AV_1^* has first row equal to $|r_1|_2^{-1}e_1$. Now one can construct similarly a $n \times n$ $\mathbb{K}$–unitary matrix U_1 with first row and column equal to e_1 which maps the first row of AV_1^* to an element of $\mathrm{span}(e_1, e_2)$. This gives to $U_1AV_1^*$ a nice structure and suggests a recursion on the dimension m. Indeed by induction one may construct $m \times m$ bloc diagonal $\mathbb{K}$–unitary matrices $U_1, \dots, U_{m-2}$ and bloc diagonal $n \times n$ $\mathbb{K}$–unitary matrices $V_1, \dots, V_{m-1}$ such that if $U := U_{m-2} \cdots U_1$ and $V := V_1^* \cdots V_{m-1}^*$ then the matrix

$$B = UAV \tag{4.6}$$

is real $m \times n$ lower triangular bidiagonal i.e. $B_{i,j} = 0$ for every i and every $j \notin \{i, i+1\}$. If A is symmetric or Hermitian then taking $U = V$ provides a symmetric or Hermitian tridiagonal matrix $B = UAU^*$ having the same spectrum as A.

4.2. Gaussian random matrices

This section gathers some facts concerning random matrices with i.i.d. Gaussian entries. The standard Gaussian law on $\mathbb{K}$ is $\mathcal{N}(0,1)$ if $\mathbb{K} = \mathbb{R}$ and $\mathcal{N}(0, \frac{1}{2}I_2)$ if $\mathbb{K} = \mathbb{C} = \mathbb{R}^2$. If Z is a standard Gaussian random variable on $\mathbb{K}$ then

$$\mathrm{Var}(Z) := \mathbb{E}(|Z - \mathbb{E}Z|^2) = \mathbb{E}(|Z|^2) = 1.$$

Let $(G_{i,j})_{i,j\geqslant 1}$ be i.i.d. standard Gaussian random variables on $\mathbb{K}$. For any $m,n\geqslant 1$,

$$G := (G_{i,j})_{1\leqslant i\leqslant m,\,1\leqslant j\leqslant n}$$

is a random $m\times n$ matrix with density in $\mathcal{M}_{m,n}(\mathbb{K})\equiv\mathbb{K}^{nm}$ proportional to

$$G\mapsto \exp\left(-\frac{\beta}{2}\sum_{i=1}^m\sum_{j=1}^n |G_{i,j}|^2\right)=\exp\left(-\frac{\beta}{2}\mathrm{Tr}(GG^*)\right)=\exp\left(-\frac{\beta}{2}\|G\|_{\mathrm{HS}}^2\right)$$

where

$$\beta := \begin{cases}1 & \text{if } \mathbb{K}=\mathbb{R},\\ 2 & \text{if } \mathbb{K}=\mathbb{C}.\end{cases}$$

The law of G is unitary invariant in the sense that $UGV\stackrel{d}{=}G$ for every deterministic $\mathbb{K}$–unitary matrices U $(m\times m)$ and V $(n\times n)$. We say that the random $m\times n$ matrix G belongs to the Ginibre Ensemble, real if $\beta=1$ and complex if $\beta=2$.

Remark 4.2.1 **(Complex Ginibre and the GUE)**. — *If $m=n$ and $\beta=2$ then $H_1=(G+G^*)/2$ and $H_2=(G-G^*)/2$ are independent and in the Gaussian Unitary Ensemble (GUE). Conversely, if H_1 and H_2 are two $m\times m$ independent random matrices in the GUE then $H_1+\sqrt{-1}\,H_2$ has the law of G with $m=n$ and $\beta=2$.*

Theorem 4.2.2 **(Wishart)**. — *Let $\mathcal{S}_m^+$ be the cone of $m\times m$ Hermitian positive definite matrices. If $m\leqslant n$ then the law of the random Hermitian matrix $W=GG^*$ is a Wishart distribution with Lebesgue density on $\mathcal{S}_m^+$ proportional to*

$$W\mapsto \det(W)^{\beta(n-m+1)/2-1}\exp\left(-\frac{\beta}{2}\mathrm{Tr}(W)\right).$$

Idea of the proof. — The Gram–Schmidt algorithm for the rows of G furnishes a $n\times m$ matrix V such that $T:=GV$ is $m\times m$ lower triangular with a real positive diagonal. Note that V can be completed into an $n\times n$ $\mathbb{K}$–unitary matrix. We have

$$W=GVV^*G^*=TT^*,\quad \det(W)=\det(T)^2=\prod_{k=1}^m T_{k,k}^2,\quad \mathrm{Tr}(W)=\sum_{i,j=1}^m |T_{i,j}|^2.$$

The desired result follows from the formulas for the Jacobian of the change of variables $G\mapsto(T,V)$ and $T\mapsto TT^*$ and the integration of the independent variable V. □

The Wishart distribution can be understood as a sort of multivariate χ^2 distribution. The correlation between the entries of the random matrix $W=GG^*$ is captured by the determinent $\det(W^{\beta(n-m+1)/2-1})$, which disappears when $n=m+(2-\beta)/\beta$.

Theorem 4.2.3 **(Bidiagonalization)**. — *If $m\leqslant n$ then there exists two random $\mathbb{K}$–unitary matrices U $(m\times m)$ and V $(n\times n)$ such that $B:=\sqrt{\beta}UGV\in\mathcal{M}_{m,n}(\mathbb{K})$*

is lower triangular and bidiagonal with independent real entries of law

$$\begin{pmatrix} \chi_{\beta n} & 0 & 0 & 0 & & \cdots & & & 0 \\ \chi_{\beta(m-1)} & \chi_{\beta(n-1)} & 0 & 0 & & \cdots & & & 0 \\ 0 & \chi_{\beta(m-2)} & \chi_{\beta(n-2)} & 0 & & \cdots & & & 0 \\ 0 & 0 & \ddots & \ddots & & \cdots & & & 0 \\ \vdots & \vdots & & & & \cdots & & & 0 \\ 0 & 0 & 0 & \cdots\ 0 & \chi_{\beta} & \chi_{\beta(n-(m-1))} & 0 & \cdots & 0 \end{pmatrix}.$$

Recall that if $X_1, \ldots, X_\ell$ are independent and identically distributed with law $\mathcal{N}(0,1)$ then $\|X\|_2^2 = X_1^2 + \cdots + X_\ell^2 \sim \chi_\ell^2$ and $\|X\|_2 = \sqrt{X_1^2 + \cdots + X_\ell^2} \sim \chi_\ell$. The densities of χ_ℓ^2 and χ_ℓ are proportional to $t \mapsto t^{\ell/2-1}e^{-t/2}$ and $t \mapsto t^{\ell-1}e^{-t^2/2}$.

Proof. — The desired result follows from 4.6 and basic properties of Gaussian laws (Cochran's theorem on the orthogonal Gaussian projections). ☐

Here is an application of theorem 4.2.3 : since B and G have same singular values, one may use B for their simulation, reducing the dimension from nm to $2m-1$.

***Theorem 4.2.4* (Laguerre Ensembles).** — *If $m \leqslant n$ then the random vector*

$$(s_1^2(G), \ldots, s_m^2(G)) = (\lambda_1(GG^*), \ldots, \lambda_m(GG^*))$$

admits a density on $\{\lambda \in [0,\infty)^m : \lambda_1 \geqslant \cdots \geqslant \lambda_n\}$ proportional to

$$\lambda \mapsto \exp\left(-\frac{\beta}{2}\sum_{i=1}^m \lambda_i\right)\prod_{i=1}^m \lambda_i^{\beta(n-m+1)/2-1}\prod_{1\leqslant i<j\leqslant m}|\lambda_i - \lambda_j|^\beta.$$

The correlation is captured by the Vandermonde determinant and expresses an electrostatic logarithmic repulsive potential. We recognize the Laguerre weight $t \mapsto t^\alpha e^{-t}$. Also we say that GG^* belongs to the β–Laguerre ensemble or Laguerre Orthogonal Ensemble (LOE) for $\beta = 1$ and Laguerre Unitary Ensemble (LUE) for $\beta = 2$.

Proof. — Let us consider the $m \times m$ tridiagonal real symmetric matrix

$$T = \begin{pmatrix} a_m & b_{m-1} & & & \\ b_{m-1} & a_{m-1} & b_{m-2} & & \\ & \ddots & \ddots & \ddots & \\ & & b_2 & a_2 & b_1 \\ & & & b_1 & a_1 \end{pmatrix}.$$

We denote by $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ its eigenvalues. Let $v_1, \ldots, v_m$ be an orthonormal system of eigenvectors. If V is the $m \times m$ orthogonal matrix with columns $v_1, \ldots, v_m$ then $T = V\mathrm{Diag}(\lambda_1, \ldots, \lambda_m)V^\top$. For every $k \in \{1, \ldots, m\}$, the equation $Tv_k = \lambda_k v_k$ writes, for every $i \in \{1, \ldots, m\}$, with the convention $b_0 = b_m = v_{k,0} = v_{k,m+1} = 0$,

$$b_{m-i+1}v_{k,i-1} + a_{m-i+1}v_{k,i} + b_{m-i+1}v_{k,i+1} = \lambda_k v_{k,i}.$$

It follows from these recursive equations that the matrix V is entirely determined by its first row $r = (r_1, \ldots, r_m) = (v_{1,1}, \ldots, v_{m,1})$ and $\lambda_1, \ldots, \lambda_m$. From now on, we assume that $\lambda_i \neq \lambda_j$ for every $i \neq j$ and that $r_1 > 0, \ldots, r_m > 0$, which makes V unique. Our first goal is to compute the Jacobian of the change of variable

$$(a, b) \mapsto (\lambda, r).$$

Note that $r_1^2 + \cdots + r_m^2 = 1$. For every $\lambda \notin \{\lambda_1, \ldots, \lambda_m\}$ we have

$$((T - \lambda I)^{-1})_{1,1} = \sum_{i=1}^m \frac{r_i^2}{\lambda_i - \lambda}.$$

On the other hand, for every $m \times m$ matrix A with $\det(A) \neq 0$, we have

$$(A^{-1})_{1,1} = \frac{\det(A_{m-1})}{\det(A)}$$

where A_k stands for the $k \times k$ right bottom sub matrix $A_k := (A_{i,j})_{m-k+1 \leqslant i,j \leqslant m}$. If $\lambda_{k,1}, \ldots, \lambda_{k,k}$ are the eigenvalues of T_k, then we obtain, with $A = T - \lambda I$,

$$\frac{\prod_{i=1}^{m-1}(\lambda_{m-1,i} - \lambda)}{\prod_{i=1}^m(\lambda_i - \lambda)} = \sum_{i=1}^m \frac{r_i^2}{\lambda_i - \lambda}.$$

Recall that $\lambda_1, \ldots, \lambda_m$ are all distinct. By denoting $P_k(\lambda) := \prod_{i=1}^k(\lambda - \lambda_{k,i})$ the characteristic polynomial of T_k, we get, for every $i \in \{1, \ldots, m\}$,

$$\frac{P_{m-1}(\lambda_i)}{P_m'(\lambda_i)} = r_i^2.$$

Since $P_m'(\lambda_i) = \prod_{1 \leqslant j \neq i \leqslant m}(\lambda_i - \lambda_j)$ we obtain

$$\prod_{i=1}^m r_i^2 = \frac{\prod_{i=1}^m |P_{m-1}(\lambda_i)|}{\prod_{1 \leqslant i < j \leqslant m}(\lambda_i - \lambda_j)^2}.$$

Let us rewrite the numerator of the right hand side. By expanding the first row in the determinant $\det(\lambda I - T) = P_m(\lambda)$, we get, with $P_{-1} := 0$ and $P_0 := 1$,

$$P_m(\lambda) = (\lambda - a_m)P_{m-1}(\lambda) - b_{m-1}^2 P_{m-2}(\lambda).$$

This shows that the spectrum of $T_m = T$ does not depend on the signs of $b_1, \ldots, b_{m-1}$. We can then safely assume that $b_1 > 0, \ldots, b_{m-1} > 0$. Additionally, we obtain

$$\prod_{i=1}^{m-1} |P_m(\lambda_{m-1,i})| = b_{m-1}^{2(m-1)} \prod_{i=1}^{m-1} |P_{m-2}(\lambda_{m-1,i})|.$$

Now the observation

$$\prod_{i=1}^{m-1} |P_{m-2}(\lambda_{m-1,i})| = \prod_{i=1}^{m-1} \prod_{j=1}^{m-2} |\lambda_{m-2,j} - \lambda_{m-1,i}| = \prod_{j=1}^{m-2} |P_{m-1}(\lambda_{m-2,j})|$$

leads by induction to the identity

$$\prod_{i=1}^{m-1} |P_m(\lambda_{m-1,i})| = \prod_{i=1}^{m-1} b_i^{2i}.$$

Finally, we have shown that

$$\prod_{1\leqslant i<j\leqslant m}(\lambda_i-\lambda_j)^2=\frac{\prod_{i=1}^{m-1}b_i^{2i}}{\prod_{i=1}^{m}r_i^2}. \tag{4.7}$$

To compute the Jacobian of the change of variable $(a,b)\mapsto(\lambda,r)$, we start from

$$((I-\lambda T)^{-1})_{1,1}=\sum_{i=1}^{m}\frac{r_i^2}{1-\lambda\lambda_i}$$

with $|\lambda|<1/\max(\lambda_1,\ldots,\lambda_m)$ (this gives $\|\lambda T\|_{2\to 2}<1$). By expanding both sides in power series of λ and identifying the coefficients, we get the system of equations

$$(T^k)_{1,1}=\sum_{i=1}^{m}r_i^2\lambda_i^k \quad \text{where} \quad k\in\{0,1,\ldots,2m-1\}.$$

Since $(T^k)_{1,1}=\langle T^k e_1,e_1\rangle$ and since T is tridiagonal, we see that this system of equations is triangular with respect to the variables $a_m,b_{m-1},a_{m-1},b_{m-2},\ldots$. The first equation is $1=r_1^2+\cdots+r_m^2$ and gives $-r_m dr_m=r_1dr_1+\cdots+r_{m-1}dr_{m-1}$. This identity and the remaining triangular equations give, after some tedious calculus,

$$dadb=\pm\frac{1}{r_m}\frac{\prod_{i=1}^{m-1}b_i}{\prod_{i=1}^{m}r_i}\left(\frac{\prod_{i=1}^{m}r_i^2}{\prod_{i=1}^{m-1}b_i^{2i}}\right)^2\prod_{1\leqslant i<j\leqslant m}(\lambda_i-\lambda_j)^4\,d\lambda dr.$$

which gives, using (4.7),

$$dadb=\pm\frac{1}{r_m}\frac{\prod_{i=1}^{m-1}b_i}{\prod_{i=1}^{m}r_i}\,d\lambda dr. \tag{4.8}$$

Let us consider now the $m\times n$ lower triangular bidiagonal real matrix ($m\leqslant n$)

$$B=\begin{pmatrix} x_n & & & \\ y_{m-1} & x_{n-1} & & \\ & \ddots & \ddots & \\ & & y_1 & x_{n-(m-1)} \end{pmatrix}$$

The matrix $T=BB^\top$ is $m\times m$ symmetric tridiagonal and for $i\in\{1,\ldots,m-1\}$,

$$a_m=x_n^2,\quad a_i=y_i^2+x_{n-(m-i)}^2,\quad b_i=y_ix_{n-(m-i)+1}. \tag{4.9}$$

Let us assume that B has real non negative entries. We get, after some calculus,

$$dxdy=\left(2^n x_{n-(m-1)}\prod_{i=0}^{m-2}x_{n-i}^2\right)^{-1}dadb.$$

From theorem 4.2.3 we have, with a normalizing constant $c_{m,n,\beta}$,

$$dB=c_{m,n,\beta}\prod_{i=0}^{m-1}x_{n-i}^{\beta(n-i)-1}\prod_{i=1}^{m-1}y_i^{\beta i-1}\exp\left(-\frac{\beta}{2}\sum_{i=0}^{m-1}x_{n-i}^2-\frac{\beta}{2}\sum_{i=1}^{m-1}y_i^2\right)dxdy.$$

Let us consider T as a function of λ and r. We first note that

$$\sum_{i=0}^{m-1} x_{n-i}^2 + \sum_{i=1}^{m-1} y_i^2 = \mathrm{Tr}(BB^\top) = \mathrm{Tr}(T) = \sum_{i=1}^{m} \lambda_i.$$

The unitary invariance of the law of B implies that r is uniform and that with probability one the component of λ are all distinct. Using equations (4.8-4.9), we obtain

$$dB = c_{m,n,\beta} \frac{\prod_{i=0}^{m-1} x_{n-i}^{\beta(n-i)-2} \prod_{i=1}^{m-1} y_i^{\beta i}}{r_m \prod_{i=1}^{m} r_i} \exp\left(-\frac{\beta}{2}\sum_{i=1}^{m}\lambda_i\right) d\lambda dr.$$

But using equation (4.7-4.9) we have

$$\prod_{1\leqslant i<j\leqslant m} |\lambda_i - \lambda_j| = \frac{\prod_{i=1}^{m-1} b_i^i}{\prod_{i=1}^{m} r_i} = \frac{\prod_{i=1}^{m-1} y_i^i x_{n-(m-i)+1}^i}{\prod_{i=1}^{m} r_i} = \frac{\prod_{i=0}^{m-1} x_{n-i}^{m-i-1} \prod_{i=1}^{m-1} y_i^i}{\prod_{i=1}^{m} r_i}$$

and therefore

$$dB = c_{m,n,\beta} \frac{\left(\prod_{i=0}^{m-1} x_{n-i}^2\right)^{\frac{1}{2}\beta(n-m+1)-1}}{r_m \prod_{i=1}^{m} r_i} \prod_{1\leqslant i<j\leqslant m} |\lambda_i - \lambda_j|^\beta \exp\left(-\frac{\beta}{2}\sum_{i=1}^{m}\lambda_i\right) d\lambda dr.$$

Now it remains to use the identity $\prod_{i=0}^{m-1} x_{n-i}^2 = \det(B)^2 = \det(T) = \prod_{i=1}^{m} \lambda_i$ to get only (λ, r) variables, and to eliminate the r variable by separation and integration. □

***Remark 4.2.5* (Universality of Gaussian models).** — *Gaussian models of random matrices have the advantage to allow explicit computations. However, in some applications such as in compressed sensing, Gaussian models can be less relevant than discrete models such as Bernoulli/Rademacher models. It turns out that most large dimensional properties are the same, such as in the Marchenko-Pastur theorem.*

4.3. The Marchenko-Pastur theorem

The Marchenko-Pastur theorem concerns the asymptotics of the counting probability measure of the singular values of large random rectangular matrices, with i.i.d. entries, when the aspect ratio (number of rows over number of columns) of the matrix converges to a finite positive real number.

***Theorem 4.3.1* (Marchenko-Pastur).** — *Let $(M_{i,j})_{i,j\geqslant 1}$ be an infinite table of i.i.d. random variables on $\mathbb{K}$ with unit variance and arbitrary mean. Let*

$$\nu_{m,n} = \frac{1}{m}\sum_{k=1}^{m} \delta_{s_k(\frac{1}{\sqrt{n}}M)} = \frac{1}{m}\sum_{k=1}^{m} \delta_{\lambda_k(\sqrt{\frac{1}{n}MM^*})}$$

be the counting probability measure of the singular values of the $m \times n$ random matrix

$$\frac{1}{\sqrt{n}} M = \left(\frac{1}{\sqrt{n}} M_{i,j}\right)_{1\leqslant i\leqslant m, 1\leqslant j\leqslant n}.$$

Suppose that $m = m_n$ depends on n in such a way that

$$\lim_{n\to\infty} \frac{m_n}{n} = \rho \in (0,\infty).$$

Then with probability one, for any bounded continuous function $f : \mathbb{R}_+ \to \mathbb{R}$,

$$\int f\, d\nu_{m,n} \underset{n\to+\infty}{\longrightarrow} \int f\, d\nu_\rho$$

where ν_ρ is the Marchenko-Pastur law with shape parameter ρ given by

$$\left(1-\frac{1}{\rho}\right)_+ \delta_0 + \frac{1}{\rho\pi x}\sqrt{(b-x^2)(x^2-a)}\,\mathbf{1}_{[\sqrt{a},\sqrt{b}]}(x)dx. \tag{4.10}$$

where $a = (1-\sqrt{\rho})^2$ and $b = (1+\sqrt{\rho})^2$ (atom at point 0 if and only if $\rho > 1$).

Theorem 4.3.1 is a sort of strong law of large numbers: it states the almost sure convergence of the sequence $(\nu_{m,n})_{n\geqslant 1}$ to a deterministic probability measure ν_ρ.

***Remark 4.3.2* (Weak convergence).** — *Recall that for probability measures, the weak convergence with respect to bounded continuous functions is equivalent to the pointwise convergence of cumulative distribution functions at every continuity point of the limit. This convergence, known as the narrow convergence, corresponds also to the convergence in law of random variables. Consequently, the Marchenko-Pastur theorem 4.3.1 states that if m depends on n with $\lim_{n\to\infty} m/n = \rho \in (0,\infty)$ then with probability one, for every $x \in \mathbb{R}$ ($x \neq 0$ if $\rho > 1$) denoting $I = (-\infty, x]$,*

$$\lim_{n\to\infty} \nu_{m,n}(I) = \nu_\rho(I).$$

***Remark 4.3.3* (Atom at 0).** — *The atom at 0 in ν_ρ when $\rho > 1$ can be understood by the fact that $s_k(M) = 0$ for any $k > m \wedge n$. If $m \geqslant n$ then $\nu_\rho(\{0\}) \geqslant (m-n)/m$.*

***Remark 4.3.4* (Quarter circle law).** — *When $\rho = 1$ then M is asymptotically square, $a = 0$, $b = 4$, and ν_1 is the so called* quarter circle law

$$\frac{1}{\pi}\sqrt{4-x^2}\,\mathbf{1}_{[0,2]}(x)dx.$$

Actually, the normalization factor makes it an ellipse instead of a circle.

Alternate formulation. — Recall that $s_k^2(\frac{1}{\sqrt{n}}M) = \lambda_k(\frac{1}{n}MM^*)$ for every $k \in \{1,\dots,m\}$. The image of $\nu_{m,n}$ by the map $x \mapsto x^2$ is the probability measure

$$\mu_{m,n} = \frac{1}{m}\sum_{k=1}^m \delta_{\lambda_k(\frac{1}{n}MM^*)}.$$

Similarly, the image μ_ρ of ν_ρ by the map $x \mapsto x^2$ is given by

$$\left(1-\frac{1}{\rho}\right)_+ \delta_0 + \frac{1}{\rho 2\pi x}\sqrt{(b-x)(x-a)}\,\mathbf{1}_{[a,b]}(x)dx \tag{4.11}$$

where $a = (1-\sqrt{\rho})^2$ and $b = (1+\sqrt{\rho})^2$ as in theorem 4.3.1. As an immediate consequence, the Marchenko-Pastur theorem 4.3.1 can be usefully rephrased as follows:

***Theorem 4.3.5* (Marchenko-Pastur).** — *Let $(M_{i,j})_{i,j\geqslant 1}$ be an infinite table of i.i.d. random variables on $\mathbb{K}$ with unit variance and arbitrary mean. Let*

$$\mu_{m,n} = \frac{1}{m}\sum_{k=1}^{m}\delta_{\lambda_k(\frac{1}{n}MM^*)}$$

be the counting probability measure of the eigenvalues of the $m\times m$ random matrix $\frac{1}{n}MM^*$ where $M=(M_{i,j})_{1\leqslant i\leqslant m, 1\leqslant j\leqslant n}$. Suppose that $m=m_n$ depends on n with

$$\lim_{n\to\infty}\frac{m_n}{n} = \rho \in (0,\infty)$$

then with probability one, for any bounded continuous function $f:\mathbb{R}_+\to\mathbb{R}$,

$$\int f\,d\mu_{m,n} \underset{n\to+\infty}{\longrightarrow} \int f\,d\mu_\rho$$

where μ_ρ is the Marchenko-Pastur law defined by (4.11).

***Remark 4.3.6* (First moment and tightness).** — *By the strong law of large numbers, we have, with probability one,*

$$\begin{aligned}\int x\,d\mu_{m,n}(x) &= \frac{1}{m}\sum_{k=1}^{m}s_k^2(\frac{1}{\sqrt{n}}M)\\ &= \frac{1}{m}\mathrm{Tr}\Big(\frac{1}{n}MM^*\Big)\\ &= \frac{1}{nm}\sum_{\substack{1\leqslant i\leqslant m\\ 1\leqslant j\leqslant n}}|M_{ij}|^2 \underset{n,m\to+\infty}{\longrightarrow} 1.\end{aligned}$$

This shows the almost sure convergence of the first moment in the Marchenko-Pastur theorem. Moreover, by Markov's inequality, for any $r>0$, we have

$$\mu_{m,n}([0,r]^c) \leqslant \frac{1}{r}\int x\,d\mu_{m,n}(x).$$

This shows that almost surely the sequence $(\mu_{m_n,n})_{n\geqslant 1}$ is tight.

***Remark 4.3.7* (Covariance matrices).** — *Suppose that M has centered entries. The column vectors $C_1,\ldots,C_n$ of M are independent and identically distributed random vectors of $\mathbb{R}^m$ with mean 0 and covariance I_m, and $\frac{1}{n}MM^*$ is the empirical covariance matrix of this sequence of vectors seen as a sample of $\mathcal{N}(0,I_m)$. We have*

$$\frac{1}{n}MM^* = \frac{1}{n}\sum_{k=1}^{n}C_k{C_k}^*.$$

Also, if m is fixed then by the strong law of large numbers, with probability one, $\lim_{n\to\infty}\frac{1}{n}MM^* = \mathbb{E}(C_1{C_1}^*) = I_m$. This is outside the regime of the Marchenko-Pastur theorem, for which m depends on n in such a way that $\lim_{n\to\infty}m/n\in(0,\infty)$.

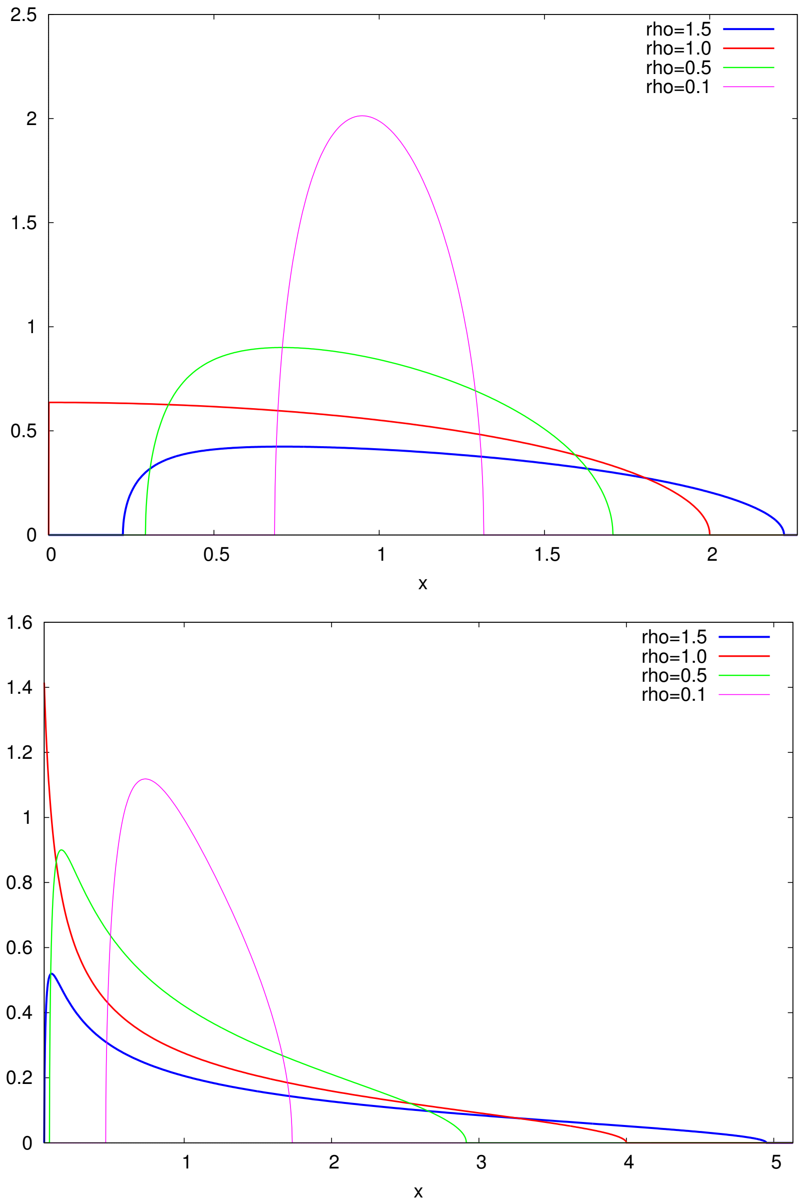

FIGURE 2. Absolutely continuous parts of the Marchenko-Pastur laws ν_ρ (4.10) and μ_ρ (4.11) for different values of the shape parameter ρ. These graphics were produced with the wxMaxima free software package.

4.4. Proof of the Marchenko-Pastur theorem

This section is devoted to a proof of theorem 4.3.1. We will actually provide a proof of the equivalent version formulated in theorem 4.3.5, by using the method of moments. Let us define the truncated matrix $\widetilde{M} = (\widetilde{M}_{i,j})_{1\leqslant i\leqslant m, 1\leqslant j\leqslant n}$ where

$$\widetilde{M}_{i,j} = M_{i,j}\mathbf{1}_{\{|M_{i,j}|\leqslant C\}}$$

with $C > 0$. By theorem 4.1.10 we have

$$L^4(\mu_{\frac{1}{\sqrt{n}}M}, \mu_{\frac{1}{\sqrt{n}}\widetilde{M}}) \leqslant \frac{2}{(nm)^2}\sum_{\substack{1\leqslant i\leqslant m\\1\leqslant j\leqslant n}}\left(|M_{i,j}|^2 + |\widetilde{M}_{i,j}|^2\right)\sum_{\substack{1\leqslant i\leqslant m\\1\leqslant j\leqslant n}}|M_{i,j}|^2\mathbf{1}_{\{|M_{i,j}|>C\}}.$$

Now, by the strong law of large numbers, with probability one,

$$\limsup_{m,n\to\infty} L^4(\mu_{\frac{1}{\sqrt{n}}M}, \mu_{\frac{1}{\sqrt{n}}\widetilde{M}}) \leqslant 4\mathbb{E}(|M_{1,1}|^2\mathbf{1}_{\{|M_{1,1}|>C\}}).$$

The right hand side can be made arbitrarily close to zero by choosing C large enough, and since the convergence for the Lévy distance implies the weak convergence with respect to bounded continuous functions, one may assume that the entries of M have bounded support (remark that by scaling, one may take entries of arbitrary variance). Let us define the $m \times n$ centered matrix $\overline{M} = M - \mathbb{E}(M)$. By theorem 4.1.8 we have

$$\|F_{\mu_{\frac{1}{\sqrt{n}}M}} - F_{\mu_{\frac{1}{\sqrt{n}}\overline{M}}}\|_\infty \leqslant \frac{\mathrm{rank}(\mathbb{E}(M))}{m} \leqslant \frac{1}{m}.$$

Consequently, one may assume that M has mean 0. Additionally, lemma 4.4.1 below reduces the problem to the convergence of $\mathbb{E}\mu_{m,n}$ to μ_ρ (via the first Borel-Cantelli lemma and the countable test functions $f = \mathbf{1}_{(-\infty,x]}$ with x rational). Next, lemmas 4.4.4 and 4.4.5 below reduce in turn the problem to the convergence of the moments of $\mathbb{E}\mu_{m,n}$ to the ones of μ_ρ computed in lemma 4.4.6 below.

Summarizing, it remains to show that if M has i.i.d. entries of mean 0, variance 1, and support $[-C, C]$, and if $\lim_{n\to\infty} m/n = \rho \in (0,\infty)$, then, for every $r \geqslant 1$,

$$\lim_{n\to\infty}\mathbb{E}\int x^r\,d\mu_{m,n} = \sum_{k=0}^{r-1}\frac{\rho^k}{k+1}\binom{r}{k}\binom{r-1}{k}. \tag{4.12}$$

The result is immediate for the first moment ($r = 1$) since

$$\begin{aligned}\mathbb{E}\int x\,d\mu_{m,n} &= \frac{1}{mn}\mathbb{E}\sum_{k=1}^{m}\lambda_k(MM^*)\\ &= \frac{1}{nm}\mathbb{E}\mathrm{Tr}(MM^*)\\ &= \frac{1}{nm}\sum_{\substack{1\leqslant i\leqslant m\\1\leqslant j\leqslant n}}\mathbb{E}(|M_{i,j}|^2) = 1.\end{aligned}$$

This show actually that $\mathbb{E}\mu_{m,n}$ and μ_ρ have even the same first moment for all values of m and n. The convergence of the second moment ($r = 2$) is far more subtle:

$$\begin{aligned}\mathbb{E}\int x^2\,d\mu_{m,n} &= \frac{1}{mn^2}\mathbb{E}\sum_{k=1}^m \lambda_k^2(MM^*)\\ &= \frac{1}{mn^2}\mathbb{E}\mathrm{Tr}(MM^*MM^*)\\ &= \frac{1}{mn^2}\sum_{\substack{1\leqslant i,k\leqslant m\\ 1\leqslant j,l\leqslant n}} \mathbb{E}(M_{ij}\overline{M}_{kj}M_{kl}\overline{M}_{il}).\end{aligned}$$

If an element of $\{(ij),(kj),(kl),(il)\}$ appears one time and exactly one in the product $M_{ij}\overline{M}_{kj}M_{kl}\overline{M}_{il}$ then by the assumptions of independence and mean 0 we get $\mathbb{E}(M_{ij}\overline{M}_{kj}M_{kl}\overline{M}_{il}) = 0$. The case when the four elements are the same appears with mn possibilities and is thus asymptotically negligible. It remains only to consider the cases where two different elements appear two times. The case $(ij) = (kj)$ and $(kl) = (il)$ gives $i = k$ and contributes $\mathbb{E}(|M_{ij}|^2|M_{il}|^2) = 1$ with $m(n^2-n)$ possibilities (here $j \neq l$ since the case $j = l$ was already considered). The case $(ij) = (kl)$ and $(kj) = (il)$ gives $i = j = k = l$ which was already considered. The case $(ij) = (il)$ and $(kj) = (kl)$ gives $j = l$ and contributes $\mathbb{E}(|M_{ij}|^2|M_{kj}|^2) = 1$ with $n^2(m^2 - m)$ possibilities (here $i \neq k$ since the case $i = k$ was already considered). We used here the assumptions of independence, mean 0, and variance 1. At the end, the second moment of $\mathbb{E}\mu_{m,n}$ tends to $\lim_{n\to\infty}(m(n^2 - n) + n(m^2 - m))/(mn^2) = 1 + \rho$ which is the second moment of μ_ρ. We have actually in hand a method reducing the proof of (4.12) to combinatorial arguments. Namely, for all $r \geqslant 1$, we write

$$\int x^r\,d\mu_{m,n}(x) = \frac{1}{mn^r}\sum_{k=1}^m \lambda_k(MM^*)^r = \frac{1}{mn^r}\mathrm{Tr}((MM^*)^r)$$

which gives

$$\mathbb{E}\int x^r\,d\mu_{m,n}(x) = \frac{1}{mn^r}\sum_{\substack{1\leqslant i_1,\dots,i_r\leqslant m\\ 1\leqslant j_1,\dots,j_r\leqslant n}} \mathbb{E}(M_{i_1j_1}\overline{M}_{i_2j_1}M_{i_2j_2}\overline{M}_{i_3j_2}\cdots M_{i_rj_r}\overline{M}_{i_1j_r}).$$

Draw $i_1,\dots,i_r$ on a horizontal line representing $\mathbb{N}$ and $j_1,\dots,j_r$ on another parallel horizontal line below the previous one representing another copy of $\mathbb{N}$. Draw r down edges from i_s to j_s and r up edges from j_s to i_{s+1}, with the convention $i_{r+1} = i_1$, for all $s = 1,\dots,r$. This produces an oriented "MP" graph with possibly multiple edges between two nodes (certain vertices or edges of this graph may have a degree larger that one due to the possible coincidence of certain values of i_s or of j_s). We have

$$\mathbb{E}\int x^r\,d\mu_{m,n}(x) = \frac{1}{n^r m}\sum_G \mathbb{E}M_G$$

where the sum $\sum_G$ runs over the set of MP graphs and where M_G is the product of M_{ab} or $\overline{M}_{ab}$ over the edges ab of G. We say that two MP graphs are equivalent when they are identical up to a permutation of $\{1,\dots,m\}$ and $\{1,\dots,n\}$. Each equivalent class contains a unique canonical graph such that $i_1 = j_1 = 1$ and $i_s \leqslant \max\{i_1, \dots, i_{s-1}\}+1$ and $j_s \leqslant \max\{j_1, \dots, j_{s-1}\} + 1$ for all s. A canonical graph possesses $\alpha + 1$ distinct i-vertices and β distinct j-vertices with $0 \leqslant \alpha \leqslant r - 1$ and $1 \leqslant \beta \leqslant r$. We say that such a canonical graph is $T(\alpha, \beta)$, and we distinguish three types :

- $T_1(\alpha, \beta)$: $T(\alpha, \beta)$ graphs for which each down edge coincides with one and only one up edge. We have necessarily $\alpha + \beta = r$ and we abridge $T_1(\alpha, \beta)$ into $T_1(\alpha)$
- $T_2(\alpha, \beta)$: $T(\alpha, \beta)$ graphs with at least one edge of multiplicity exactly 1
- $T_3(\alpha, \beta)$: $T(\alpha, \beta)$ graphs which are not $T_1(\alpha, \beta)$ nor $T_2(\alpha, \beta)$

We admit the following combinatorial facts :

(C1) the cardinal of the equivalent class of each $T(\alpha, \beta)$ canonical graph is
$$m(m-1)\cdots(m-\alpha)n(n-1)\cdots(n-\beta+1).$$
(C2) each $T_3(\alpha, \beta)$ canonical graph has at most r distinct vertices (i.e. $\alpha + \beta < r$).
(C3) the number of $T_1(\alpha, \beta)$ canonical graphs is
$$\frac{1}{\alpha+1}\binom{r}{\alpha}\binom{r-1}{\alpha}.$$

The quantity $\mathbb{E}(M_G)$ depends only on the equivalent class of G. We denote by $\mathbb{E}(M_{T(\alpha,\beta)})$ the common value to all $T(\alpha, \beta)$ canonical graphs. We get, using (C1),
$$\frac{1}{n^r m}\sum_G M_G = \frac{1}{n^r m}\sum_{T(\alpha,\beta)} m(m-1)\cdots(m-\alpha)n(n-1)\cdots(n-\beta+1)\mathbb{E}(M_{T(\alpha,\beta)})$$
where the sum runs over the set of all canonical graphs. The contribution of T_2 graphs is zero thanks to the assumption of independence and mean 0. The contribution of T_3 graphs is asymptotically negligible since there are few of them. Namely, by the bounded support assumption we have $|M_{T_3(\alpha,\beta)}| \leqslant C^{2r}$, and thanks to (C2) we obtain
$$\frac{1}{n^r m}\sum_{T_3(\alpha,\beta)} m(m-1)\cdots(m-\alpha)n(n-1)\cdots(n-\beta+1)\mathbb{E}(M_{T(\alpha,\beta)})$$
$$\leqslant \frac{r^2}{n^r m} C^{2r} m^{\alpha+1} n^{\beta} = O(n^{-1}).$$
Therefore we know now that only T_1 graphs contributes asymptotically. Let us consider a $T_1(\alpha, \beta) = T_1(\alpha)$ canonical graph ($\beta = r - \alpha$). Since $M_{T(\alpha,\beta)} = M_{T(\alpha)}$ is a product of squared modules of distinct entries of M, which are independent, of mean 0, and variance 1, we have $\mathbb{E}(M_{T(\alpha)}) = 1$. Consequently, using (C3) we obtain
$$\frac{1}{n^r m}\sum_{T_1(\alpha)} m(m-1)\cdots(m-\alpha)n(n-1)\cdots(n-r+\alpha+1)\mathbb{E}(M_{T(\alpha,r-\alpha)})$$
$$= \sum_{\alpha=0}^{r-1} \frac{1}{1+\alpha}\binom{r}{\alpha}\binom{r-1}{\alpha}\frac{1}{n^r m} m(m-1)\cdots(m-\alpha)n(n-1)\cdots(n-r+\alpha+1)$$
$$= \sum_{\alpha=0}^{r-1} \frac{1}{1+\alpha}\binom{r}{\alpha}\binom{r-1}{\alpha}\prod_{i=1}^{\alpha}\left(\frac{m}{n}-\frac{i}{n}\right)\prod_{i=1}^{r-\alpha+1}\left(1-\frac{i-1}{n}\right).$$

Therefore, denoting $\rho_n = m/n$, we have

$$\mathbb{E}\int x^r \, d\mu_n(x) = \sum_{\alpha=0}^{r-1} \frac{\rho_n^\alpha}{\alpha+1}\binom{r}{\alpha}\binom{r-1}{\alpha} + O(n^{-1}).$$

This achieves the proof of (4.12), and thus of the Marchenko-Pastur theorem 4.3.5.

Concentration for empirical spectral distributions. — This section is devoted to the proof of lemma 4.4.1 below. The total variation of $f : \mathbb{R} \to \mathbb{R}$ is

$$\|f\|_{\mathrm{TV}} := \sup_{(x_k)_{k\in\mathbb{Z}}} \sum_{k\in\mathbb{Z}} |f(x_{k+1}) - f(x_k)|,$$

where the supremum runs over all non decreasing sequences $(x_k)_{k\in\mathbb{Z}}$. If f is differentiable with $f' \in \mathrm{L}^1(\mathbb{R})$ then $\|f\|_{\mathrm{TV}} = \|f'\|_1$. If $f = \mathbf{1}_{(-\infty,s]}$ for a real s then $\|f\|_{\mathrm{TV}} = 1$, and consequently, for probability measures, the weak convergence with respect to bounded continuous functions can be checked on measurable test functions with $\|\cdot\|_{\mathrm{TV}} \leqslant 1$ (or even $= 1$).

***Lemma 4.4.1* (Concentration).** — *Let M be an $m \times n$ complex random matrix with independent rows. Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function such that $\|f\|_{\mathrm{TV}} \leqslant 1$. Assume that $\mathbb{E}\left|\int f \, d\mu_M\right| < \infty$ where $\mu_M := \frac{1}{m}\sum_{k=1}^m \delta_{\lambda_k(MM^*)}$. This always holds true for instance when f is bounded. Then for any $r \geqslant 0$,*

$$\mathbb{P}\left(\left|\int f \, d\mu_M - \mathbb{E}\int f \, d\mu_M\right| \geqslant r\right) \leqslant 2\exp\left(-2mr^2\right).$$

Proof. — Let A and B be two $m \times n$ complex matrices and let F_{μ_A} and F_{μ_B} be the cumulative distributions functions of $\mu_A = \frac{1}{m}\sum_{k=1}^m \delta_{s_k^2(A)}$ and $\mu_B = \frac{1}{m}\sum_{k=1}^m \delta_{s_k^2(B)}$. For any differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that $f' \in \mathrm{L}^1(\mathbb{R})$, an integration by parts and theorem 4.1.8 give

$$\left|\int f \, d\mu_A - \int f \, d\mu_B\right| = \left|\int_{\mathbb{R}} f'(t)(F_{\mu_A}(t) - F_{\mu_B}(t)) \, dt\right| \leqslant \frac{\mathrm{rank}(A-B)}{m}\int_{\mathbb{R}} |f'(t)| \, dt.$$

Since the left hand side depends on at most $2m$ points, we get, by approximation, for every measurable function $f : \mathbb{R} \to \mathbb{R}$ with $\|f\|_{\mathrm{TV}} \leqslant 1$,

$$\left|\int f \, d\mu_A - \int f \, d\mu_B\right| \leqslant \frac{\mathrm{rank}(A-B)}{m}.$$

From now on, $f : \mathbb{R} \to \mathbb{R}$ is a fixed measurable function with $\|f\|_{\mathrm{TV}} \leqslant 1$. For every row vectors $x_1, \dots, x_m$ in $\mathbb{C}^n$, we denote by $A(x_1, \dots, x_m)$ the $m \times n$ matrix with row vectors $x_1, \dots, x_m$ and we define $F : (\mathbb{C}^n)^m \to \mathbb{R}$ by

$$F(x_1, \dots, x_m) := \int f \, d\mu_{A(x_1,\dots,x_m)}.$$

For any $i \in \{1, \dots, m\}$ and any row vectors $x_1, \dots, x_m, x_i'$ of $\mathbb{C}^n$, we have

$$\mathrm{rank}(A(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_m) - A(x_1, \dots, x_{i-1}, x_i', x_{i+1}, \dots, x_m)) \leqslant 1$$

and thus

$$|F(x_1,\dots,x_{i-1},x_i,x_{i+1},\dots,x_m) - F(x_1,\dots,x_{i-1},x_i',x_{i+1},\dots,x_m)| \leqslant \frac{1}{m}.$$

Finally, the desired concentration result follows from McDiarmid's lemma 4.4.2 used for the random variable $X = F(R_1,\dots,R_m)$ where $R_1,\dots,R_m$ are the rows of M. $\square$

***Lemma 4.4.2* (McDiarmid).** — *Let $R_1,\dots,R_n$ be independent random variables taking values in $E_1,\dots,E_n$. Let $F : E_1\times\cdots\times E_n \to \mathbb{R}$ be a measurable function such that the random variable $X = F(R_1,\dots,R_n)$ is integrable. Then for any $r \geqslant 0$,*

$$\mathbb{P}(|X-\mathbb{E}(X)|\geqslant r) \leqslant 2\exp\left(-\frac{2r^2}{c_1^2+\cdots+c_n^2}\right)$$

where $c_k = \sup_{(x,x')\in\mathcal{D}_k}|F(x)-F(x')|$ and $\mathcal{D}_k = \{(x,x') \;:\; x_i = x_i' \text{ for all } i\neq k\}$.

Proof. — Let $R_1',\dots,R_n'$ be an independent copy of $R_1,\dots,R_n$. If $\mathcal{F}_k$ stands for the σ-field generated by $R_1,\dots,R_k$ then for every $1\leqslant k\leqslant n$ we have, with $\mathcal{F}_0 = \{\varnothing,\Omega\}$,

$$\mathbb{E}(X\,|\,\mathcal{F}_{k-1}) = \mathbb{E}(F(R_1,\dots,R_k,\dots,R_n)\,|\,\mathcal{F}_{k-1}) = \mathbb{E}(F(R_1,\dots,R_k',\dots,R_n)\,|\,\mathcal{F}_k).$$

Now the desired result follows from the Azuma-Hoeffding lemma 4.4.3 since

$$\begin{aligned} d_k &= \mathbb{E}(X\,|\,\mathcal{F}_k) - \mathbb{E}(X\,|\,\mathcal{F}_{k-1}) \\ &= \mathbb{E}(F(R_1,\dots,R_k,\dots,R_n) - F(R_1,\dots,R_k',\dots,R_n)\,|\,\mathcal{F}_k) \end{aligned}$$

gives $\|d_k\|_\infty \leqslant c_k$ for every $1\leqslant k\leqslant n$. $\square$

The following lemma on concentration of measure for sums of bounded differences is close in spirit to theorem 1.2.1. The condition on the oscillation (support diameter) rather than on the variance (second moment) is typical of Hoeffding type statements.

***Lemma 4.4.3* (Azuma-Hoeffding).** — *If $X\in \mathrm{L}^1(\Omega,\mathcal{F},\mathbb{P})$ then for every $r\geqslant 0$*

$$\mathbb{P}(|X-\mathbb{E}(X)|\geqslant r) \leqslant 2\exp\left(-\frac{2r^2}{\|d_1\|_\infty^2+\cdots+\|d_n\|_\infty^2}\right)$$

where $d_k = \mathbb{E}(X\,|\,\mathcal{F}_k) - \mathbb{E}(X\,|\,\mathcal{F}_{k-1})$ for an arbitrary filtration

$$\{\varnothing,\Omega\} = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n = \mathcal{F}.$$

Proof. — If U is a random variable with $\mathbb{E}(U)=0$ and $a\leqslant U\leqslant b$, then, by convexity, $e^{tx} \leqslant \frac{x-a}{b-a}e^{tb} + \frac{b-x}{b-a}e^{ta}$ for all $t\geqslant 0$ and $a\leqslant x\leqslant b$, which gives after some analysis

$$\mathbb{E}(e^{tU}) \leqslant \frac{b}{b-a}e^{ta} - \frac{a}{b-a}e^{tb} \leqslant e^{\frac{t^2}{8}(b-a)^2}.$$

Used with $U = d_k = \mathbb{E}(X\,|\,\mathcal{F}_k) - \mathbb{E}(X\,|\,\mathcal{F}_{k-1})$ conditional on $\mathcal{F}_{k-1}$, this gives

$$\mathbb{E}(e^{td_k}\,|\,\mathcal{F}_{k-1}) \leqslant e^{\frac{t^2}{8}\|d_k\|_\infty^2}.$$

By writing the Doob martingale telescopic sum $X-\mathbb{E}(X) = d_n+\cdots+d_1$, we get

$$\mathbb{E}(e^{t(X-\mathbb{E}(X))}) = \mathbb{E}(e^{t(d_{n-1}+\cdots+d_1)}\mathbb{E}(e^{td_n}\,|\,\mathcal{F}_{n-1})) \leqslant \cdots \leqslant e^{\frac{t^2}{8}(\|d_1\|_\infty^2+\cdots+\|d_n\|_\infty^2)}.$$

Now the desired result follows from Markov's inequality and an optimization of t. $\square$

Moments and weak convergence. — This section is devoted to the proof of lemmas 4.4.5 and 4.4.6 below. Let $\mathcal{P}$ be the set of probability measures μ on $\mathbb{R}$ such that $\mathbb{R}[X] \subset \mathrm{L}^1(\mu)$. We say that $\mu_1, \mu_2 \in \mathcal{P}$ are equivalent when

$$\int P\,d\mu_1 = \int P\,d\mu_2$$

for all $P \in \mathbb{R}[X]$, in other words μ_1 and μ_2 have the same moments. We say that $\mu \in \mathcal{P}$ is *characterized by its moments* when its equivalent class is a singleton. The celebrated Carleman theorem states that $\mu \in \mathcal{P}$ is characterized by its moments $(\kappa_n)_{n\geqslant 1}$ iff $\sum_n \kappa_{2n}^{-1/(2n)} = \infty$. Lemma 4.4.4 below provides a simpler sufficient condition, which is strong enough to imply that every compactly supported probability measure, such as the Marchenko-Pastur law μ_ρ, is characterized by its moments. Note that by the Weierstrass theorem on the density of polynomials, we already know that every compactly supported probability measure is characterized by its moments among the class of compactly supported probability measures.

***Lemma 4.4.4* (Moments and analyticity).** — *Let $\mu \in \mathcal{P}$, $\varphi(t) = \int e^{itx}\,d\mu(x)$ and $\kappa_n = \int x^n\,d\mu(x)$. The following three statements are equivalent :*

1. *φ is analytic in a neighborhood of the origin*
2. *φ is analytic on $\mathbb{R}$*
3. *$\overline{\lim}_n \left(\frac{1}{n!}|\kappa_n|\right)^{\frac{1}{n}} < \infty$.*

If these statement hold true then μ is characterized by its moments.
This is the case for instance if μ is compactly supported.

Proof. — For all n we have $\int |x|^n\,d\mu < \infty$ and thus φ is n times differentiable on $\mathbb{R}$. Moreover, $\varphi^{(n)}$ is continuous on $\mathbb{R}$ and for all $t \in \mathbb{R}$,

$$\varphi^{(n)}(t) = \int_{\mathbb{R}} (ix)^n e^{itx}\,d\mu(x).$$

In particular, $\varphi^{(n)}(0) = i^n \kappa_n$, and the Taylor series of φ at the origin is determined by $(\kappa_n)_{n\geqslant 1}$. The convergence radius r of the power series $\sum_n a_n z^n$ associated to a sequence of complex numbers $(a_n)_{n\geqslant 0}$ is given by Hadamard's formla $r^{-1} = \overline{\lim}_n |a_n|^{\frac{1}{n}}$. This shows the equivalence of properties 1 and 3 by taking $a_n = i^n\kappa_n/n!$. Since for all $n \in \mathbb{N}$, $s, t \in \mathbb{R}$,

$$\left| e^{isx}\left(e^{itx} - 1 - \frac{itx}{1!} - \cdots - \frac{(itx)^{n-1}}{(n-1)!} \right) \right| \leqslant \frac{|tx|^n}{n!},$$

we get for all even $n \in \mathbb{N}$ and all $s, t \in \mathbb{R}$,

$$\left| \varphi(s+t) - \varphi(s) - \frac{t}{1!}\varphi'(s) - \cdots - \frac{t^{n-1}}{(n-1)!}\varphi^{(n-1)}(s) \right| \leqslant \kappa_n \frac{|t|^n}{n!},$$

and thus property 3 implies property 2. Since property 2 implies property 1 we get that properties 1-2-3 are equivalent. If these properties hold then by the preceding arguments, there exists $r > 0$ such that the series expansion of φ at any $x \in \mathbb{R}$ has radius $> r$, and thus, φ is characterized by its sequence of derivatives at point 0. If μ is compactly supported then $\sup_n |\kappa_n|^{\frac{1}{n}} < \infty$ and thus property 3 holds. $\square$

***Lemma 4.4.5* (Moments convergence).** — *Let $\mu \in \mathcal{P}$ be characterized by its moments. If $(\mu_n)_{n\geqslant 1}$ is a sequence in $\mathcal{P}$ such that for every polynomial $P \in \mathbb{R}[X]$,*

$$\lim_{n\to\infty} \int P\,d\mu_n = \int P\,d\mu$$

then for every bounded continuous function $f : \mathbb{R} \to \mathbb{R}$,

$$\lim_{n\to\infty} \int f\,d\mu_n = \int f\,d\mu.$$

Proof. — By assumption, for any $P \in \mathbb{R}[X]$, we have $C_P := \sup_{n\geqslant 1} \int P\,d\mu_n < \infty$, and therefore, by Markov's inequality, for any real $R > 0$,

$$\mu_n([-R,R]^c) \leqslant \frac{C_{X^2}}{R^2}.$$

This shows that $(\mu_n)_{n\geqslant 1}$ is tight. Thanks to Prohorov's theorem, it suffices then to show that if a subsequence $(\mu_{n_k})_{k\geqslant 1}$ converges with respect to bounded continuous functions toward a probability measure ν as $k \to \infty$ then $\nu = \mu$. Let us fix $P \in \mathbb{R}[X]$ and a real number $R > 0$. Let $\varphi_R : \mathbb{R} \to [0,1]$ be continuous and such that

$$\mathbf{1}_{[-R,R]} \leqslant \varphi_R \leqslant \mathbf{1}_{[-R-1,R+1]}.$$

We have the decomposition

$$\int P\,d\mu_{n_k} = \int \varphi_R P\,d\mu_{n_k} + \int (1-\varphi_R) P\,d\mu_{n_k}.$$

Since $(\mu_{n_k})_{k\geqslant 1}$ converges weakly to ν we have

$$\lim_{k\to\infty} \int \varphi_R P\,d\mu_{n_k} = \int \varphi_R P\,d\nu.$$

Moreover, by the Cauchy-Schwarz and Markov inequalities we have

$$\left|\int (1-\varphi_R) P\,d\mu_{n_k}\right|^2 \leqslant \mu_{n_k}([-R,R]^c) \int P^2\,d\mu_{n_k} \leqslant \frac{C_{X^2} C_{P^2}}{R^2}.$$

On the other hand, we know that $\displaystyle\lim_{k\to\infty} \int P\,d\mu_{n_k} = \int P\,d\mu$ and thus

$$\lim_{R\to\infty} \int \varphi_R P\,d\nu = \int P\,d\mu.$$

Using this for P^2 provides via monotone convergence that $P \in \mathrm{L}^2(\nu) \subset \mathrm{L}^1(\nu)$ and by dominated convergence that $\int P\,d\nu = \int P\,d\mu$. Since P is arbitrary and μ is characterized by its moments, we obtain $\mu = \nu$. □

***Lemma 4.4.6* (Moments of the M.-P. law μ_ρ).** — *The sequence of moments of the Marchenko-Pastur distribution μ_ρ defined by (4.11) is given for all $r \geqslant 1$ by*

$$\int x^r\,d\mu_\rho(x) = \sum_{k=0}^{r-1} \frac{\rho^k}{k+1} \binom{r}{k} \binom{r-1}{k}.$$

In particular, μ_ρ has mean 1 and variance ρ.

Proof. — Since $a+b=2(1+\rho)$ and $ab=(1-\rho)^2$ we have

$$\sqrt{(b-x)(x-a)}=\sqrt{\frac{(a+b)^2}{4}-ab-\left(x-\frac{a+b}{2}\right)^2}=\sqrt{4\rho-(x-(1+\rho))^2}$$

The change of variable $y=(x-(1+\rho))/\sqrt{\rho}$ gives

$$\int x^r\,d\mu_\rho(x)=\int x^r\,d\mu_\rho(x)=\frac{1}{2\pi}\int_{-2}^{2}(\sqrt{\rho}y+1+\rho)^{r-1}\sqrt{4-y^2}\,dy.$$

Recall that the even moments of the semicircle law are the Catalan numbers :

$$\frac{1}{2\pi}\int_{-2}^{2}y^{2k+1}\sqrt{4-y^2}\,dy=0\quad\text{and}\quad\frac{1}{2\pi}\int_{-2}^{2}y^{2k}\sqrt{4-y^2}\,dy=\frac{1}{1+k}\binom{2k}{k}.$$

By using binomial expansions and the Vandermonde convolution identity,

$$\begin{aligned}\int x^r\,d\mu_\rho(x)&=\sum_{k=0}^{\lfloor(r-1)/2\rfloor}\rho^k(1+\rho)^{r-1-2k}\binom{r-1}{2k}\binom{2k}{k}\frac{1}{1+k}\\&=\sum_{k=0}^{\lfloor(r-1)/2\rfloor}\rho^k(1+\rho)^{r-1-2k}\frac{(r-1)!}{(r-1-2k)!k!(k+1)!}\\&=\sum_{k=0}^{\lfloor(r-1)/2\rfloor}\sum_{s=0}^{r-1-2k}\rho^{k+s}\frac{(r-1)!}{k!(k+1)!(r-1-2k-s)!s!}\\&=\sum_{t=0}^{r-1}\rho^t\sum_{k=0}^{\min(t,r-1-t)}\frac{(r-1)!}{k!(k+1)!(r-1-t-k)!(t-k)!}\\&=\frac{1}{r}\sum_{t=0}^{r-1}\rho^t\binom{r}{t}\sum_{k=0}^{\min(t,r-1-t)}\binom{t}{k}\binom{r-t}{k+1}\\&=\frac{1}{r}\sum_{t=0}^{r-1}\rho^t\binom{r}{t}\binom{r}{t+1}\\&=\sum_{t=0}^{r-1}\frac{\rho^t}{t+1}\binom{r}{t}\binom{r-1}{t}.\end{aligned}$$

□

Other proof of the Marchenko-Pastur theorem. — An alternate proof of the Marchenko-Pastur theorem 4.3.1 is based on the Cauchy-Stieltjes transform. Recall that the Cauchy-Stieltjes transform of a probability measure μ on $\mathbb{R}$ is

$$z\in\mathbb{C}_+=\{z\in\mathbb{C}:\mathrm{Im}(z)>0\}\mapsto S_\mu(z)=\int\frac{1}{x-z}\,d\mu(x).$$

For instance, the Cauchy-Stieltjes transform of the Marchenko-Pastur law μ_ρ is

$$S_{\mu_\rho}(z)=\frac{1-\rho-z+\sqrt{(z-1-\rho)^2-4\rho}}{2\rho z}.$$

The knowledge of S_μ fully characterizes μ, and the pointwise convergence along a sequence of probability measures implies the weak convergence of the sequence. For any $m \times m$ Hermitian matrix H with spectral distribution $\mu_H = \frac{1}{m}\sum_{k=1}^m \delta_{\lambda_k(H)}$, the Cauchy-Stieltjes transform S_{μ_H} is the normalized trace of the resolvent of H since

$$S_{\mu_H}(z) = \frac{1}{m}\mathrm{Tr}((H - zI)^{-1}).$$

This makes the Cauchy-Stieltjes transform an analogue of the Fourier transform, well suited for spectral distributions of matrices. To prove the Marchenko-Pastur theorem one takes $H = \frac{1}{n}MM^*$ and one first shows that $S_{\mu_H} - \mathbb{E}S_{\mu_H}$ tends to 0 with probability one as $n \to \infty$. Beware that $\mathbb{E}S_{\mu_A} \neq S_{\mathbb{E}\mu_A}$. Next the Schur bloc inversion allows to deduce a recursive equation for $\mathbb{E}S_{\mu_H}$ leading to the fixed point equation $S = 1/(1 - z - \rho - \rho z S)$ at the limit. This quadratic equation in S admits two solutions including the Cauchy-Stieltjes transform S_{μ_ρ} of the M.-P. law μ_ρ.

The behavior of μ_H when H is random can by captured by looking at $\mathbb{E}\int f\,d\mu_H$ with a test function f running over a sufficiently large family $\mathcal{F}$. The method of moments corresponds to the family $\mathcal{F} = \{x \mapsto x^r : r \in \mathbb{N}\}$ whereas the Cauchy-Stieltjes transform method corresponds to the family $\mathcal{F} = \{z \mapsto 1/(x-z) : z \in \mathbb{C}_+\}$. Each of these allows to prove theorem 4.3.5, with advantages and drawbacks.

4.5. The Bai-Yin theorem

The convergence stated by the Marchenko-Pastur theorem 4.3.1 is too weak to provide the convergence of the smallest and largest singular values. More precisely, one can only deduce from theorem 4.3.1 that with probability one,

$$\liminf_{n\to\infty} s_{n\wedge m}\left(\frac{1}{\sqrt{n}}M\right) \leqslant \sqrt{a} \quad \text{and} \quad \limsup_{n\to\infty} s_1\left(\frac{1}{\sqrt{n}}M\right) \geqslant \sqrt{b}$$

where $a = (1-\sqrt{\rho})^2$ and $b = (1+\sqrt{\rho})^2$. Of course if $\rho = 1$ then $a = 0$ and we obtain $\lim_{n\to\infty} s_{n\wedge m}\left(\frac{1}{\sqrt{n}}M\right) = 0$. The Bai and Yin theorem below provides a complete answer for any value of ρ when the entries have mean zero and finite fourth moment.

***Theorem 4.5.1* (Bai-Yin).** — *Let $(M_{i,j})_{i,j\geqslant 1}$ be an infinite table of i.i.d. random variables on $\mathbb{K}$ with mean 0, variance 1 and finite fourth moment : $\mathbb{E}(|M_{1,1}|^4) < \infty$. As in the Marchenko-Pastur theorem 4.3.1, let M be the $m \times n$ random matrix*

$$M = (M_{i,j})_{1\leqslant i\leqslant m, 1\leqslant j\leqslant n}.$$

Suppose that $m = m_n$ depends on n in such a way that

$$\lim_{n\to\infty} \frac{m_n}{n} = \rho \in (0,\infty).$$

Then with probability one

$$\lim_{n\to\infty} s_{m\wedge n}\left(\frac{1}{\sqrt{n}}M\right) = \sqrt{a} \quad \text{and} \quad \lim_{n\to\infty} s_1\left(\frac{1}{\sqrt{n}}M\right) = \sqrt{b}.$$

Regarding the assumptions, it can be shown that if M is not centered or does not have finite fourth moment then the largest singular value tends to infinity.

When $m < n$ the Bai-Yin theorem can be roughly rephrased as follows

$$\sqrt{n} - \sqrt{m} + \sqrt{n}o_{n\to\infty}(1) \leqslant s_{m\wedge n}(M) \leqslant s_1(M) \leqslant \sqrt{n} + \sqrt{m} + \sqrt{n}o_{n\to\infty}(1).$$

The proof of the Bai-Yin theorem is tedious and is outside the scope of this book. In the Gaussian case, the result may be deduced from theorem 4.2.3. It is worthwhile to mention that in the Gaussian case, we have the following result due to Gordon:

$$\sqrt{n} - \sqrt{m} \leqslant \mathbb{E}(s_{m\wedge n}(M)) \leqslant \mathbb{E}(s_1(M)) \leqslant \sqrt{n} + \sqrt{m}.$$

***Remark 4.5.2* (Jargon).** — *The Marchenko-Pastur theorem 4.3.1 concerns the global behavior of the spectrum using the counting probability measure: we say* bulk of the spectrum. *The Bai-Yin theorem 4.5.1 concerns the boundary of the spectrum: we say* edge of the spectrum. *When $\rho = 1$ then the left limit $\sqrt{a} = 0$ acts like a hard wall forcing single sided fluctuations, and we speak about a* hard edge. *In contrast, we have a* soft edge *at $\sqrt{b}$ for any ρ and at $\sqrt{a}$ for $\rho \neq 1$ in the sense that the spectrum can fluctuate around the limit at both sides. The asymptotic fluctuation at the edge depends on the nature of the edge: soft edges give rise to Tracy-Widom laws, while hard edges give rise to (deformed) exponential laws (depending on $\mathbb{K}$).*

4.6. Notes and comments

The singular values of deterministic matrices are studied in the reference books [**HJ90, HJ94, Bha97, Zha02, BS10**] for the static aspects, and [**GVL96, CG05**] for the algorithmic aspects. The notion of pseudo-spectrum is studied in [**TE05**]. The SVD is typically used for dimension reduction and for regularization. For instance, the SVD allows to construct the so called Moore–Penrose pseudoinverse [**Moo20, Pen56**] of a matrix by replacing the non null singular values by their inverse while leaving in place the null singular values. Generalized inverses of integral operators were introduced earlier by Fredholm in [**Fre03**]. Such generalized inverse of matrices provide for instance least squares solutions to degenerate systems of linear equations. A diagonal shift in the SVD is used in the so called Tikhonov regularization [**Tik43, Tar05**] or ridge regression for solving over determined systems of linear equations. The SVD is at the heart of the so called principal component analysis (PCA) technique in applied statistics for multivariate data analysis [**Jol02**]. The partial least squares (PLS) regression technique is also connected to PCA/SVD. In the last decade, the PCA was used together with the so called kernel methods in learning theory. Generalizations of the SVD are used for the regularization of ill posed inverse problems [**BB98**]. The couple of lemmas connecting the rows distances of a matrix with the norm of its inverse are taken from [**RV08a**] (operator norm) and [**TV10**] (trace norm).

The study of the singular values of random matrices takes its roots in the works of Wishart [**Wis28**] on the empirical covariance matrices of Gaussian samples, and in the works of von Neumann and Goldstine in numerical analysis [**vNG47**]. The singular values of Gaussian random matrices were extensively studied and we refer to [**Jam60, DS01, ER05, HT03, For10**]. Our presentation is inspired by [**For10**].

For simplicity, we have skipped the link with Laguerre orthogonal polynomials, which plays an important role in the asymptotic analysis of the extremal singular values. The bidiagonalization of Gaussian random matrices is due to Silverstein [**Sil85**]. The Marchenko-Pastur theorem goes back to Marchenko and Pastur [**MP67**]. The modern universal version with minimal moments assumptions was obtained after a sequence of works including [**Gir75**] and can be found in the books [**PS11, BS10**]. Most of the proof given in this chapter is taken from [**BS10**]. The argument for the almost sure convergence based on concentration for finite variation test functions is due to Bordenave but can be found in [**GL09**]. The McDiarmid-Azuma-Hoeffding inequalities are taken from [**McD89**]. An extension to random matrices with independent row vectors or column vectors is given in [**MP06**] and [**PP09**]. In the Gaussian case, the Marchenko-Pastur theorem can be proved using Laguerre orthogonal polynomials with an approach due to Haagerup and Thorbjørnsen [**HT03**] developed in [**Led04**].

The Bai and Yin theorem was obtained after a series of works by Bai and Yin [**BY93**], and is proved in great generality in [**BS10**]. The non-asymptotic analysis of the singular values of random matrices is the subject of the recent survey [**Ver11**].

CHAPTER 5

EMPIRICAL METHODS AND SELECTION OF CHARACTERS

The purpose of this chapter is to present the connections between two different topics. The first one is the recent subject about reconstruction of signals with small supports from a small amount of linear measurements, called also compressed sensing. It has been presented in Chapter 2. A big amount of work was recently made to develop some strategy to construct an encoder (to compress a signal) and an associate decoder (to reconstruct exactly or approximately the original signal). Several deterministic methods are known but recently, some random methods allow the reconstruction of signal with much larger size of support. A lot of ideas are common with a subject of harmonic analysis, going back to the construction of $\Lambda(p)$ sets which are not $\Lambda(q)$ for $q > p$. The most powerful method was to select a random choice of characters via the method of selectors. We will discuss about the problem of selecting a large part of a bounded orthonormal system such that on the vectorial span of this family, the L_2 and the L_1 norms are as close as possible. Solving this type of problems leads to questions about the Euclidean radius of the intersection of the kernel of a matrix with the unit ball of a normed space. That is exactly the subject of study of Gelfand numbers and Kashin splitting theorem. In all this theory, empirical processes are essential tools. Numerous results of this theory are at the heart of the proofs and we will present some of them.

Notations. — We briefly indicate some notations that will be used in this section. For any $p \geq 1$ and $t \in \mathbb{R}^N$, we define its ℓ_p-norm by

$$|t|_p = \left(\sum_{i=1}^{N} |t_i|^p \right)^{1/p}$$

and its L_p-norm by

$$\|t\|_p = \left(\frac{1}{N} \sum_{i=1}^{N} |t_i|^p \right)^{1/p}.$$

For $p \in (0,1)$, the definition is still valid but it is not a norm. For $p = \infty$, $|t|_\infty = \|t\|_\infty = \max\{|t_i| : i = 1, \ldots, n\}$. We denote by B_p^N the unit ball of the ℓ_p-norm in

$\mathbb{R}^N$. The radius of a set $T \subset \mathbb{R}^N$ is

$$\operatorname{rad} T = \sup_{t \in T} |t|_2.$$

More generally, if μ is a probability measure on a measurable space Ω, for any $p > 0$ and any measurable function f, we denote its L_p-norm and its L_∞-norm by

$$\|f\|_p = \left(\int |f|^p d\mu \right)^{1/p} \quad \text{and} \quad \|f\|_\infty = \sup |f|.$$

The unit ball of $L_p(\mu)$ is denoted by B_p and the unit sphere by S_p. If $T \subset L_2(\mu)$ then its radius with respect to $L_2(\mu)$ is defined by

$$\operatorname{Rad} T = \sup_{t \in T} \|t\|_2.$$

Observe that if μ is the probability counting measure on $\mathbb{R}^N$, $B_p = N^{1/p} B_p^N$ and for a subset $T \subset L_2(\mu)$, $\sqrt{N} \operatorname{Rad} T = \operatorname{rad} T$.

The letters c, C are used for numerical constants which do not depend on any parameter (dimension, size of sparsity, ...). Since the dependence of these parameters is important in this study, it will be always indicated (as precisely as we can). Sometimes, the value of these numerical constants can change from line to line.

5.1. Selection of characters and the reconstruction property.

Exact and approximate reconstruction.— We start by recalling briefly from Chapter 2 the ℓ_1-minimization method to reconstruct any unknown sparse signal from a small number of linear measurements. Let $U \in \mathbb{R}^N$ (or $\mathbb{C}^N$) be an unknown signal. We receive ΦU where Φ is an $n \times N$ matrix with row vectors $Y_1, \ldots, Y_n \in \mathbb{R}^N$ (or $\mathbb{C}^N$) which means that

$$\Phi = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} \qquad \text{and} \qquad \Phi U = (\langle Y_i, U \rangle)_{1 \le i \le n}$$

and we assume that $n \le N - 1$. In this case the linear system to reconstruct U is ill-posed. However, the main information is that U has a small support in the canonical basis chosen at the beginning, that is $|\operatorname{supp} U| \le m$. We also say that U is m-sparse and we denote by Σ_m the set of m-sparse vectors. Our aim is to find conditions on Φ, m, n and N such that for every $U \in \Sigma_m$, the solution of the problem

$$\min_{t \in \mathbb{R}^N} \{ |t|_1 : \Phi U = \Phi t \} \tag{5.1}$$

is unique and equal to U. From Proposition 2.2.11, we know that the property "for every signal $U \in \Sigma_m$, the solution of (5.1) is unique and equal to U" is equivalent to the following

$$\forall h \in \ker \Phi, h \neq 0, \forall I \subset [N], |I| \le m, \sum_{i \in I} |h_i| < \sum_{i \notin I} |h_i|.$$

This property is also called the null space property. Let $\mathcal{C}_m$ be the cone

$$\mathcal{C}_m = \{h \in \mathbb{R}^N, \exists I \subset [N] \text{ with } |I| \le m, |h_{I^c}|_1 \le |h_I|_1\}.$$

The null space property is therefore equivalent to $\ker \Phi \cap \mathcal{C}_m = \{0\}$. Taking the intersection with the Euclidean sphere S^{N-1}, we can say that

"for every signal $U \in \Sigma_m$, the solution of (5.1) is unique and equal to U"
if and only if
$\ker \Phi \cap \mathcal{C}_m \cap S^{N-1} = \emptyset$.

We observe the following simple fact: if $t \in \mathcal{C}_m \cap S^{N-1}$ then

$$|t|_1 = \sum_{i=1}^{N} |t_i| = \sum_{i \in I} |t_i| + \sum_{i \notin I} |t_i| \le 2 \sum_{i \in I} |t_i| \le 2\sqrt{m}$$

since $|I| \le m$ and $|t|_2 = 1$. This implies that

$$\mathcal{C}_m \cap S^{N-1} \subset 2\sqrt{m} B_1^N \cap S^{N-1}$$

from which we conclude that if

$$\ker \Phi \cap 2\sqrt{m} B_1^N \cap S^{N-1} = \emptyset$$

then "for every $U \in \Sigma_m$, the solution of (5.1) is unique and equal to U". We can now state the conclusion of this introduction, which is Proposition 2.4.4.

Proposition 5.1.1. — *Denote by* $\operatorname{rad} T$ *the radius of a set* T *with respect to the Euclidean distance:* $\operatorname{rad} T = \sup_{t \in T} |t|_2$. *If*

$$\operatorname{rad}(\ker \Phi \cap B_1^N) < \rho \text{ with } \rho \le \frac{1}{2\sqrt{m}} \tag{5.2}$$

then "for every $U \in \Sigma_m$, *the solution of the basis pursuit algorithm* (5.1) *is unique and equal to* U*".*

It has also been noticed in Chapter 2 that it is very stable and allows approximate reconstructions. Indeed by Proposition 2.7.3, if $U^\sharp$ is a solution of the minimization problem (5.1)

$$\min_{t \in \mathbb{R}^N} \{|t|_1 : \Phi U = \Phi t\}.$$

and if for some integer m such that $1 \le m \le N$, we have

$$\operatorname{rad}(\ker \Phi \cap B_1^N) \le \rho < \frac{1}{2\sqrt{m}}$$

then for any set $I \subset \{1, \dots, N\}$ of cardinality less than m

$$|U^\sharp - U|_2 \le \rho\, |U^\sharp - U|_1 \le \frac{2\rho}{1 - 2\rho\sqrt{m}}\, |U_{I^c}|_1.$$

In particular: if $\operatorname{rad}(\ker \Phi \cap B_1^N) \le 1/4\sqrt{m}$ then for any subset I of cardinality less than m,

$$|U^\sharp - U|_2 \le \frac{|U^\sharp - U|_1}{4\sqrt{m}} \le \frac{|U_{I^c}|_1}{\sqrt{m}}.$$

Moreover if $U \in B_{p,\infty}^N$ i.e. if for all $s > 0$, $|\{i, |U_i| \geq s\}| \leq s^{-p}$ then

$$|U^\sharp - U|_2 \leq \frac{|U^\sharp - U|_1}{4\sqrt{m}} \leq \frac{1}{(1-1/p)\, m^{\frac{1}{p}-\frac{1}{2}}}.$$

Another problem coming from Harmonic Analysis. — Let μ be a probability measure and let $(\psi_1, \ldots, \psi_N)$ be an orthonormal system of $L_2(\mu)$ bounded in $L_\infty(\mu)$ i.e. such that for every $i \leq N$, $\|\psi_i\|_\infty \leq 1$. Typically, we consider a system of characters in $L_2(\mu)$. By the assumptions on $\{\psi_1, \ldots, \psi_N\}$, it is clear that for any subset $I \subset [N]$

$$\forall\, (a_i)_{i \in I}\,, \quad \left\| \sum_{i \in I} a_i \psi_i \right\|_1 \leq \left\| \sum_{i \in I} a_i \psi_i \right\|_2 \leq \sqrt{|I|} \left\| \sum_{i \in I} a_i \psi_i \right\|_1 .$$

The Dvoretky's theorem proved by Milman asserts that for any $\varepsilon \in (0,1)$, there exists a subspace $E \subset \operatorname{span}\{\psi_1, \ldots, \psi_N\}$ of dimension $\dim E = n = c\left(\varepsilon^2/\log(1+2/\varepsilon)\right) N$ on which the L_1 and L_2 norms are comparable, that is such that

$$\forall\, (a_i)_{i=1}^N\,, \text{ if } x = \sum_{i=1}^N a_i \psi_i \in E, \text{ then } (1-\varepsilon)\, r\, \|x\|_1 \leq \|x\|_2 \leq (1+\varepsilon)\, r\, \|x\|_1$$

where r is a number depending on the dimension N which can be bounded from above and below by some numerical constants (independent of the dimension N). Observe that E is a general subspace and the fact that $x \in E$ does not say anything about the number of non zero coordinates. Moreover the constant c which appears in the dependance of $\dim E$ is very small hence this formulation of Dvoretzy's theorem does not provide a subspace of say half dimension such that the L_1 norm and the L_2 norm are comparable up to constant factors. This question was solved by Kashin. He proved in fact a very strong result which is called now a Kashin decomposition: there exists a subspace E of dimension $[N/2]$ such that $\forall\, (a_i)_{i=1}^N$,

$$\text{if } x = \sum_{i=1}^N a_i \psi_i \in E \text{ then } \|x\|_1 \leq \|x\|_2 \leq C\, \|x\|_1\,,$$

$$\text{and if } y = \sum_{i=1}^N a_i \psi_i \in E^\perp \text{ then } \|y\|_1 \leq \|y\|_2 \leq C\, \|y\|_1$$

where C is a numerical constant. Again the subspaces E and $E^\perp$ have not any particular structure.

In the setting of Harmonic Analysis, the questions are more related with coordinate subspaces because it request to find a subset $I \subset \{1, \ldots, N\}$ such that the L_1 and L_2 norms are well comparable on $\operatorname{span}\{\psi_i\}_{i \in I}$. Talagrand, improving a result of Bourgain, showed that there exists a small constant δ_0 such that for any bounded orthonormal system $\{\psi_1, \ldots, \psi_N\}$, there exists a subset I of cardinality greater than

$\delta_0 N$ such that

$$\forall (a_i)_{i\in I}, \quad \left\| \sum_{i\in I} a_i\psi_i \right\|_2 \le C\sqrt{\log N\,(\log\log N)}\,\left\| \sum_{i\in I} a_i\psi_i \right\|_1 . \tag{5.3}$$

This is a Dvoretzky type theorem. We will present in section 5.5 an extension of this result to a Kashin type setting. An important observation that relates this study with Proposition 5.1.1 is the following. Let Ψ be the operator defined on $\mathrm{span}\,\{\psi_1,\dots,\psi_N\} \subset L_2(\mu)$ by $\Psi(f) = (\langle f,\psi_i\rangle)_{i\in I}$. Because of the orthogonality condition between the ψ_i's, the linear span of $\{\psi_i, i\in I\}$ is nothing else than the kernel of Ψ and the inequality (5.3) is equivalent to $\mathrm{Rad}\,(\ker\Psi\cap B_1) \le C\sqrt{\log N\,(\log\log N)}$ where Rad is the Euclidean radius with respect to the norm on $L_2(\mu)$ and B_1 is the unit ball of $L_1(\mu)$. The question is reduced to finding the conditions between the size of I, the dimension N and ρ_1 such that $\mathrm{Rad}\,(\ker\Psi\cap B_1)\le\rho_1$. This is analogous to the condition (5.2) in Proposition 5.1.1. Just notice that in this situation, we have a change of normalization because we work in a probability space $L_2(\mu)$ instead of ℓ_2^N.

The strategy. — In conclusion of these two paragraphs, we say that we will focus on the condition about the radius of the section of the unit ball of ℓ_1^N (or B_1) with the kernel of some matrices. As it has been noticed in Chapter 2, the RIP condition implies a control of this radius. Moreover, the condition (5.2) was deeply studied in the so called Local Theory of Banach Spaces during the seventies and the eighties and is connected with the study of Gelfand widths. These notions are presented in Chapter 2 and we recall that the strategy consists in studying the width of a truncated set $T_\rho = T\cap\rho S^{N-1}$. Indeed by Proposition 2.7.7, Φ satisfies the condition (5.2) if ρ is such that $\ker\Phi\cap T_\rho=\emptyset$. This is the purpose of the following proposition.

Proposition 5.1.2. — *Let T be a star body with respect to the origin that is a compact subset T of $\mathbb{R}^N$ such that for any $x\in T$, the segment $[0,x]$ is contained in T. Let Φ be an $n\times N$ matrix with row vectors denoted by $Y_1,\dots,Y_n$.*

$$\textit{If}\quad \inf_{y\in T\cap\rho S^{N-1}} \sum_{i=1}^n \langle Y_i, y\rangle^2 > 0 \quad \textit{then}\quad \mathrm{rad}\,(\ker\Phi\cap T) < \rho.$$

Remark 5.1.3. — *By a simple compacity argument, the reciprocal of this statement holds true. We can also replace the Euclidean norm $|\Phi z|_2$ by any other norm $\|\Phi z\|$ since the hypothesis is just made to ensure that* $\ker\Phi\cap T\cap\rho S^{N-1}=\emptyset$.

Proof. — The argument is geometric. Indeed, if $z\in T\cap\rho S^{N-1}$ then $|\Phi z|_2^2>0$ so $z\notin\ker\Phi$. Since T is star shaped, if $y\in T$ and $|y|_2\ge\rho$ then $z=\rho y/|y|_2\in T\cap\rho S^{N-1}$ so z and y do not belong to $\ker\Phi$. □

The vectors $Y_1,\dots,Y_n$ will be chosen at random and we will find the good conditions such that, in average, the key inequality of Proposition 5.1.2 holds true. An important case is when the Y_i's are independent copies of a standard random Gaussian vector in $\mathbb{R}^N$. It is exactly a way to prove Theorem 2.5.2 with Φ being this standard random Gaussian matrix. However, in the context of Compressed Sensing or Harmonic Analysis, we are looking for more structured matrices, like Fourier or Walsh matrices.

5.2. A way to construct a random data compression matrix

The setting of the study is the following. We start with a square $N \times N$ orthogonal matrix and we would like to select n rows of this matrix such that the $n \times N$ matrix Φ is a good encoder for every m-sparse vectors. In view of Proposition 5.1.1, we want to find the conditions on n, N and m such that

$$\mathrm{rad}\left(\ker \Phi \cap B_1^N\right) < \frac{1}{2\sqrt{m}}.$$

The main examples are the discrete Fourier matrix with

$$\phi_{k\ell} = \frac{1}{\sqrt{N}}\,\omega^{k\ell} \quad 1 \le k, \ell \le N \quad \text{where } \omega = \exp\left(-2i\pi/N\right),$$

and the Walsh matrix defined by induction: $W_1 = 1$ and for any $p \ge 2$,

$$W_p = \frac{1}{\sqrt{2}}\begin{pmatrix} W_{p-1} & W_{p-1} \\ -W_{p-1} & W_{p-1} \end{pmatrix}.$$

The matrix W_p is an orthogonal matrix of size $N = 2^p$ with entries $\frac{\pm 1}{\sqrt{N}}$. In each case, the column vectors form an orthonormal basis of ℓ_2^N, with ℓ_∞^N-norm bounded by $1/\sqrt{N}$. We will consider more generally a system of vectors $\phi_1, \dots, \phi_N$ such that

$$(H) \quad \begin{cases} \text{it is an orthogonal system of } \ell_2^N, \\ \forall i \le N, |\phi_i|_\infty \le 1/\sqrt{N} \text{ and } |\phi_i|_2 = K \text{ where } K \text{ is a fixed number.} \end{cases}$$

The empirical method. — The first definition of randomness is an empirical one. Let Y be the random vector defined by $Y = \phi_i$ with probability $1/N$ and let $Y_1, \dots, Y_n$ be independent copies of Y. We define the random matrix Φ by

$$\Phi = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix}.$$

We have the following properties:

$$\mathbb{E}\langle Y, y\rangle^2 = \frac{1}{N}\sum_{i=1}^N \langle \phi_i, y\rangle^2 = \frac{K^2}{N}|y|_2^2 \quad \textit{and} \quad \mathbb{E}|\Phi y|_2^2 = \frac{K^2\, n}{N}|y|_2^2. \tag{5.4}$$

In view of Proposition 5.1.2, we would like to find ρ such that

$$\mathbb{E} \inf_{y \in T \cap \rho S^{N-1}} \sum_{i=1}^n \langle Y_i, y\rangle^2 > 0.$$

However it is difficult to study the infimum of an empirical process. We shall prefer to study

$$\mathbb{E} \sup_{y \in T \cap \rho S^{N-1}} \left| \sum_{i=1}^n \langle Y_i, y\rangle^2 - \frac{K^2\, n\rho^2}{N} \right|$$

that is the supremum of the deviation of the empirical process to its mean (because of (5.4)). We will focus our attention on the following problem.

Problem 5.2.1. — *What are the conditions on ρ such that we have*

$$\mathbb{E} \sup_{y \in T \cap \rho S^{N-1}} \left| \sum_{i=1}^{n} \langle Y_i, y \rangle^2 - \frac{K^2\, n\rho^2}{N} \right| \leq \frac{2}{3} \frac{K^2\, n\rho^2}{N} \ ?$$

Indeed if this inequality is satisfied then there exists a choice of vectors $(Y_i)_{1 \leq i \leq n}$ such that

$$\forall y \in T \cap \rho S^{N-1}, \left| \sum_{i=1}^{n} \langle Y_i, y \rangle^2 - \frac{K^2\, n\rho^2}{N} \right| \leq \frac{2}{3} \frac{K^2\, n\rho^2}{N},$$

from which we deduce that

$$\forall y \in T \cap \rho S^{N-1}, \sum_{i=1}^{n} \langle Y_i, y \rangle^2 \geq \frac{1}{3} \frac{K^2\, n\rho^2}{N} > 0.$$

Therefore, by Proposition 5.1.2, we conclude that $\operatorname{rad}(\ker \Phi \cap T) < \rho$. Doing this with $T = B_1^N$, we will conclude by Proposition 5.1.1 that if

$$m \leq \frac{1}{4\rho^2}$$

then the matrix Φ is a good encoder, that is for every $U \in \Sigma_m$, the solution of the basis pursuit algorithm (5.1) is unique and equal to U.

Remark 5.2.2. — *The number $2/3$ can be replaced by any number strictly less than 1.*

The method of selectors. — The second definition of randomness uses the notion of selectors. Let $\delta \in (0,1)$ and let δ_i be i.i.d. random variables taking the value 1 with probability δ and 0 with probability $1-\delta$.

We start from the orthogonal matrix with rows $\phi_1, \ldots, \phi_N$ and we select randomly some rows to construct a matrix Φ with row vectors ϕ_i if $\delta_i = 1$. The random variables $\delta_1, \ldots, \delta_N$ are called selectors and the number of rows of Φ, equal to $|\{i : \delta_i = 1\}$, will be highly concentrated around δN. The problem 5.2.1 can be stated in the following way:

Problem 5.2.3. — *What are the conditions on ρ such that we have*

$$\mathbb{E} \sup_{y \in T \cap \rho S^{N-1}} \left| \sum_{i=1}^{N} \delta_i \langle \phi_i, y \rangle^2 - \delta K^2 \rho^2 \right| \leq \frac{2}{3} \delta K^2 \rho^2 \ ?$$

The same argument as before shows that if this inequality is satisfied for $T = B_1^N$, then there exists a choice of selectors such that $\operatorname{rad}(\ker \Phi \cap B_1^N) < \rho$ and we will conclude as before that the matrix Φ is a good encoder.

Before we state and explain the main results, we will need some tools from the theory of empirical processes to solve Problems 5.2.1 and 5.2.3. Another question is to prove that the random matrix Φ will be a good decoder with high probability.

We will also present some concentration inequalities of the supremum of empirical processes around their mean that will enable us to get better deviation inequality than the trivial Markov bound.

5.3. Empirical processes

Classical tools. — A lot is known about the supremum of empirical processes and the connection with Rademacher averages. We refer to chapter 4 of [**LT91**] for a detailed description. We recall the important comparison theorem for Rademacher average.

Theorem 5.3.1. — *Let $F : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing convex function, let $h_i : \mathbb{R} \to \mathbb{R}$ be functions such that $|h_i(s) - h_i(t)| \leq |s-t|$ and $h_i(0) = 0$. Then for any separable bounded set $T \subset \mathbb{R}^n$,*

$$\mathbb{E}F\left(\frac{1}{2}\sup_{t\in T}\left|\sum_{i=1}^{n}\varepsilon_i h_i(t_i)\right|\right) \leq \mathbb{E}F\left(\sup_{t\in T}\left|\sum_{i=1}^{n}\varepsilon_i t_i\right|\right).$$

The proof of this theorem is however beyond the scope of this chapter. We will concentrate on the study of the average of the supremum of some empirical processes. Consider n independent random vectors $Y_1, \ldots, Y_n$ taking values in a measurable space Ω and $\mathcal{F}$ be a class of measurable functions, and define

$$Z = \sup_{f\in\mathcal{F}}\left|\sum_{i=1}^{n}(f(Y_i) - \mathbb{E}f(Y_i))\right|.$$

The situation will be different from Chapter 1 because the control on the ψ_α norm of $f(Y_i)$ will not be relevant in our situation. In this case, a classical strategy consists to "symmetrize" the variable and to introduce Rademacher averages.

Theorem 5.3.2. — *Consider n independent random vectors $Y_1, \ldots, Y_n$ taking values in a measurable space Ω, $\mathcal{F}$ be a class of measurable functions and $\varepsilon_1, \ldots, \varepsilon_n$ be independent Rademacher random variables, independent of the Y_i's. Denote by $\mathbb{E}_\varepsilon$ the expectation with respect to these Rademacher random variables. Then the following inequalities hold true:*

$$\mathbb{E}\sup_{f\in\mathcal{F}}\left|\sum_{i=1}^{n}(f(Y_i) - \mathbb{E}f(Y_i))\right| \leq 2\mathbb{E}\mathbb{E}_\varepsilon\sup_{f\in\mathcal{F}}\left|\sum_{i=1}^{n}\varepsilon_i f(Y_i)\right|, \tag{5.5}$$

$$\mathbb{E}\sup_{f\in\mathcal{F}}\sum_{i=1}^{n}|f(Y_i)| \leq \sup_{f\in\mathcal{F}}\sum_{i=1}^{n}\mathbb{E}|f(Y_i)| + 4\mathbb{E}\mathbb{E}_\varepsilon\sup_{f\in\mathcal{F}}\left|\sum_{i=1}^{n}\varepsilon_i f(Y_i)\right|. \tag{5.6}$$

Moreover

$$\mathbb{E}\mathbb{E}_\varepsilon\sup_{f\in\mathcal{F}}\left|\sum_{i=1}^{n}\varepsilon_i(f(Y_i) - \mathbb{E}f(Y_i))\right| \leq 2\mathbb{E}\sup_{f\in\mathcal{F}}\left|\sum_{i=1}^{n}(f(Y_i) - \mathbb{E}f(Y_i))\right|. \tag{5.7}$$

Proof. — Let $Y_1', \dots, Y_n'$ be independent copies of $Y_1, \dots, Y_n$. We replace $\mathbb{E}f(Y_i)$ by $\mathbb{E}' f(Y_i')$ where $\mathbb{E}'$ denotes the expectation with respect to the random vectors $Y_1', \dots, Y_n'$ then by Jensen inequality,

$$\mathbb{E} \sup_{f \in \mathcal{F}} \left| \sum_{i=1}^n (f(Y_i) - \mathbb{E}f(Y_i)) \right| \le \mathbb{E}\mathbb{E}' \sup_{f \in \mathcal{F}} \left| \sum_{i=1}^n (f(Y_i) - f(Y_i')) \right|.$$

The variables $(f(Y_i) - f(Y_i'))_{1 \le i \le n}$ are independent symmetric hence $(f(Y_i) - f(Y_i'))_{1 \le i \le n}$ has the same law as $(\varepsilon_i(f(Y_i) - f(Y_i')))_{1 \le i \le n}$ where $\varepsilon_1, \dots, \varepsilon_n$ are independent Rademacher random variables. We deduce that

$$\mathbb{E} \sup_{f \in \mathcal{F}} \left| \sum_{i=1}^n (f(Y_i) - \mathbb{E}f(Y_i)) \right| \le \mathbb{E}\mathbb{E}'\mathbb{E}_\varepsilon \sup_{f \in \mathcal{F}} \left| \sum_{i=1}^n \varepsilon_i (f(Y_i) - f(Y_i')) \right|.$$

We conclude the proof of (5.5) by using the triangle inequality.
Inequality (5.6) is a consequence of (5.5) when applying it to $|f|$ instead of f, using the triangle inequality and Theorem 5.3.1 (in the case $F(x) = x$ and $h_i(x) = |x|$) to deduce that

$$\mathbb{E}_\varepsilon \sup_{f \in \mathcal{F}} \left| \sum_{i=1}^n \varepsilon_i |f(Y_i)| \right| \le 2\mathbb{E}_\varepsilon \sup_{f \in \mathcal{F}} \left| \sum_{i=1}^n \varepsilon_i f(Y_i) \right|.$$

For the proof of (5.7), we can assume that $\mathbb{E}f(Y_i) = 0$. We compute the expectation conditionally with respect to the Bernoulli random variables. Let $I = I(\varepsilon) = \{i, \varepsilon_i = 1\}$ then

$$\begin{aligned}
\mathbb{E}\mathbb{E}_\varepsilon \sup_{f \in \mathcal{F}} \left| \sum_{i=1}^n \varepsilon_i f(Y_i) \right| &\le \mathbb{E}_\varepsilon \mathbb{E} \sup_{f \in \mathcal{F}} \left| \sum_{i \in I} f(Y_i) - \sum_{i \notin I} f(Y_i) \right| \\
&\le \mathbb{E}_\varepsilon \mathbb{E} \sup_{f \in \mathcal{F}} \left| \sum_{i \in I} f(Y_i) \right| + \mathbb{E}_\varepsilon \mathbb{E} \sup_{f \in \mathcal{F}} \left| \sum_{i \notin I} f(Y_i) \right|.
\end{aligned}$$

However, since for every $i \le n$, $\mathbb{E}f(Y_i) = 0$ we deduce from Jensen inequality that for any $I \subset \{1, \dots, n\}$

$$\mathbb{E} \sup_{f \in \mathcal{F}} \left| \sum_{i \in I} f(Y_i) \right| = \mathbb{E} \sup_{f \in \mathcal{F}} \left| \sum_{i \in I} f(Y_i) + \sum_{i \notin I} \mathbb{E}f(Y_i) \right| \le \mathbb{E} \sup_{f \in \mathcal{F}} \left| \sum_{i=1}^n f(Y_i) \right|$$

which ends the proof of (5.7). □

Another simple fact about Rademacher averages is the following comparison between the supremum of Rademacher processes and the supremum of the same Gaussian processes.

Proposition 5.3.3. — *Let $\varepsilon_1, \dots, \varepsilon_n$ be independent Bernoulli random variables and $g_1, \dots, g_n$ be independent Gaussian $\mathcal{N}(0,1)$ random variables, then for any set $T \subset \mathbb{R}^n$*

$$\mathbb{E} \sup_{t \in T} \left| \sum_{i=1}^n \varepsilon_i t_i \right| \le \sqrt{\frac{2}{\pi}} \mathbb{E} \sup_{t \in T} \left| \sum_{i=1}^n g_i t_i \right|.$$

Proof. — Indeed, $(g_1, \ldots, g_n)$ has the same law as $(\varepsilon_1|g_1|, \ldots, \varepsilon_n|g_n|)$ and by Jensen inequality,

$$\mathbb{E}_\varepsilon \mathbb{E}_g \sup_{t \in T} \left| \sum_{i=1}^n \varepsilon_i |g_i| t_i \right| \geq \mathbb{E}_\varepsilon \sup_{t \in T} \left| \mathbb{E}_g \sum_{i=1}^n \varepsilon_i |g_i| t_i \right| = \sqrt{\frac{\pi}{2}} \mathbb{E} \sup_{t \in T} \left| \sum_{i=1}^n \varepsilon_i t_i \right|.$$

□

To conclude this part, we state an important result about the concentration of the supremum of empirical processes around its mean. This motivates the fact that we will focus on the estimation of the expectation of the supremum of such empirical process.

Theorem 5.3.4. — *Consider n independent random vectors $Y_1, \ldots, Y_n$ and $\mathcal{G}$ a class of measurable functions. Let*

$$Z = \sup_{g \in \mathcal{G}} \left| \sum_{i=1}^n g(Y_i) \right|, \ M = \sup_{g \in \mathcal{G}} \|g\|_\infty, \ V = \mathbb{E} \sup_{g \in \mathcal{G}} \sum_{i=1}^n g(Y_i)^2.$$

Then for any $t > 0$, we have

$$\mathbb{P}\left(|Z - \mathbb{E}Z| > t\right) \leq C \exp\left(-c \frac{t}{M} \log\left(1 + \frac{tM}{V}\right)\right).$$

Sometimes, we need a more simple quantity than V in this concentration inequality. Let $\mathcal{F}$ be a class of measurable functions, and define the function g by $g(Y) = f(Y) - \mathbb{E}f(Y)$ for any $f \in \mathcal{F}$. In this situation, we have a very useful estimate for V.

Proposition 5.3.5. — *Consider n independent random vectors $Y_1, \ldots, Y_n$ and $\mathcal{F}$ a class of measurable functions. Let*

$$Z = \sup_{f \in \mathcal{F}} \left| \sum_{i=1}^n f(Y_i) - \mathbb{E}f(Y_i) \right|, \ u = \sup_{f \in \mathcal{F}} \|f\|_\infty, \ \text{and}$$

$$v = \sup_{f \in \mathcal{F}} \sum_{i=1}^n \mathrm{Var} f(Y_i) + 32\, u\, \mathbb{E} \sup_{f \in \mathcal{F}} \left| \sum_{i=1}^n f(Y_i) - \mathbb{E}f(Y_i) \right|.$$

Then for any $t > 0$, we have

$$\mathbb{P}\left(|Z - \mathbb{E}Z| > t\right) \leq C \exp\left(-c \frac{t}{u} \log\left(1 + \frac{tu}{v}\right)\right).$$

Proof. — It is a typical use of the symmetrization principle. Let $\mathcal{G}$ be the set of functions defined by $g(Y) = f(Y) - \mathbb{E}f(Y)$ where $f \in \mathcal{F}$. Using Theorem 5.3.4, the conclusion will follow when estimating

$$M = \sup_{g \in \mathcal{G}} \|g\|_\infty \text{ and } V = \mathbb{E} \sup_{g \in \mathcal{G}} \sum_{i=1}^n g(Y_i)^2.$$

It is clear that $M \leq 2u$ and by the triangle inequality we get

$$\mathbb{E} \sup_{g \in \mathcal{G}} \sum_{i=1}^{n} g(Y_i)^2 \ \leq \ \mathbb{E} \sup_{g \in \mathcal{G}} \left| \sum_{i=1}^{n} g(Y_i)^2 - \mathbb{E} g(Y_i)^2 \right| + \sup_{g \in \mathcal{G}} \sum_{i=1}^{n} \mathbb{E} g(Y_i)^2.$$

Using inequality (5.5), we deduce that

$$\mathbb{E} \sup_{g \in \mathcal{G}} \left| \sum_{i=1}^{n} g(Y_i)^2 - \mathbb{E} g(Y_i)^2 \right| \leq 2\, \mathbb{E}\mathbb{E}_\varepsilon \sup_{g \in \mathcal{G}} \left| \sum_{i=1}^{n} \varepsilon_i g(Y_i)^2 \right| = 2\, \mathbb{E}\mathbb{E}_\varepsilon \sup_{t \in T} \left| \sum_{i=1}^{n} \varepsilon_i t_i^2 \right|$$

where T is the random set $\{t = (t_1, \dots, t_n) = (g(Y_1), \dots, g(Y_n)) : g \in \mathcal{G}\}$. Since $T \subset [-2u, 2u]^n$, we deduce that the function $h(x) = x^2$ is $4u$-Lipschitz on T. By the comparison Theorem 5.3.1, we get that

$$\mathbb{E}_\varepsilon \sup_{t \in T} \left| \sum_{i=1}^{n} \varepsilon_i t_i^2 \right| \leq 8\, u\, \mathbb{E}_\varepsilon \sup_{t \in T} \left| \sum_{i=1}^{n} \varepsilon_i t_i \right|.$$

Since for every $i \leq n$, $\mathbb{E} g(Y_i) = 0$, we deduce from (5.7) that

$$\mathbb{E}\mathbb{E}_\varepsilon \sup_{g \in \mathcal{G}} \left| \sum_{i=1}^{n} \varepsilon_i g(Y_i)^2 \right| \leq 16\, u\, \mathbb{E} \sup_{g \in \mathcal{G}} \left| \sum_{i=1}^{n} g(Y_i) \right|.$$

This allows to conclude that

$$V \leq 32\, u\, \mathbb{E} \sup_{f \in \mathcal{F}} \left| \sum_{i=1}^{n} f(Y_i) - \mathbb{E} f(Y_i) \right| + \sup_{f \in \mathcal{F}} \sum_{i=1}^{n} \mathrm{Var} f(Y_i).$$

This ends the proof of the proposition. $\square$

The study of the expectation of the supremum of some empirical processes. — We go back to the study of Problem 5.2.1 with a definition of randomness given by the empirical method. The situation is similar if we worked with the method of selectors. For any star body $T \subset \mathbb{R}^N$, we define the class $\mathcal{F}$ of functions in the following way:

$$\mathcal{F} = \left\{ \begin{array}{rcl} f_y : \mathbb{R}^N & \to & \mathbb{R} \\ Y & \mapsto & \langle Y, y \rangle \end{array} \quad : \quad y \in T \cap \rho S^{N-1} \right\}.$$

Therefore

$$Z = \sup_{f \in \mathcal{F}} \left| \sum_{i=1}^{n} (f^2(Y_i) - \mathbb{E} f^2(Y_i)) \right| = \sup_{y \in T \cap \rho S^{N-1}} \left| \sum_{i=1}^{n} \langle Y_i, y \rangle^2 - \frac{n\rho^2}{N} \right|.$$

Applying the symmetrization procedure to Z (cf (5.5)) and comparing Rademacher and Gaussian processes, we conclude that

$$\begin{aligned} \mathbb{E} \sup_{y \in T \cap \rho S^{N-1}} \left| \sum_{i=1}^{n} \langle Y_i, y \rangle^2 - \frac{n\rho^2}{N} \right| &\leq 2\mathbb{E}\mathbb{E}_\varepsilon \sup_{y \in T \cap \rho S^{N-1}} \left| \sum_{i=1}^{n} \varepsilon_i \langle Y_i, y \rangle^2 \right| \\ &\leq \sqrt{2\pi}\mathbb{E}\mathbb{E}_g \sup_{y \in T \cap \rho S^{N-1}} \left| \sum_{i=1}^{n} g_i \langle Y_i, y \rangle^2 \right|. \end{aligned}$$

We will first get a bound for the Rademacher average (or the Gaussian one) and then we will take the expectation with respect to the Y_i's. Before working with these difficult processes, we present a result of Rudelson where the supremum is taken on the unit sphere S^{N-1}.

Theorem 5.3.6. — *For any fixed vectors $Y_1, \ldots, Y_n$ in $\mathbb{R}^N$,*

$$\mathbb{E}_\varepsilon \sup_{y \in S^{N-1}} \left| \sum_{i=1}^n \varepsilon_i \langle Y_i, y\rangle^2 \right| \le C \sqrt{\log n} \max_{1\le i\le n} |Y_i|_2 \sup_{y\in S^{N-1}} \left(\sum_{i=1}^n \langle Y_i, y\rangle^2 \right)^{1/2}.$$

Proof. — For every $i = 1, \ldots, n$, we define the self-adjoint rank 1 operators

$$T_i = Y_i \otimes Y_i : \begin{cases} \mathbb{R}^N & \to & \mathbb{R}^N \\ y & \mapsto & \langle Y_i, y\rangle Y_i \end{cases}$$

in such a way that

$$\sup_{y\in S^{N-1}} \left| \sum_{i=1}^n \varepsilon_i \langle Y_i, y\rangle^2 \right| = \sup_{y\in S^{N-1}} \left| \langle \sum_{i=1}^n \varepsilon_i T_i y, y\rangle \right| = \left\| \sum_{i=1}^n \varepsilon_i T_i \right\|_{2\to 2}.$$

Let $(\lambda_i)_{1\le i\le N}$ be the eigenvalues of a self-adjoint operator S. By definition of the S_q^N norms for any $q > 0$,

$$\|S\|_{2\to 2} = \|S\|_{S_\infty^N} = \max_{1\le i\le n} |\lambda_i| \quad \text{and} \quad \|S\|_{S_q^N} = \left(\sum_{i=1}^N |\lambda_i|^q \right)^{1/q}.$$

Assume that the operator S has rank less than n then for $i \ge n+1$, $\lambda_i = 0$ and we deduce by Hölder inequality that

$$\|S\|_{S_\infty^N} \le \|S\|_{S_q^N} \le n^{1/q} \|S\|_{S_\infty^N} \le e\|S\|_{S_\infty^N} \quad \text{for} \quad q \ge \log n.$$

The non-commutative Khinchine inequality of Lust-Piquard and Pisier states that for any operator $T_1, \ldots, T_n$,

$$\mathbb{E}_\varepsilon \left\| \sum_{i=1}^n \varepsilon_i T_i \right\|_{S_q^N} \le C\sqrt{q} \max \left\{ \left\| \left(\sum_{i=1}^n T_i^* T_i \right)^{1/2} \right\|_{S_q^N}, \left\| \left(\sum_{i=1}^n T_i T_i^* \right)^{1/2} \right\|_{S_q^N} \right\}.$$

In our situation, $T_i^* T_i = T_i T_i^* = |Y_i|_2^2 T_i$ and $S = \left(\sum_{i=1}^n T_i^* T_i\right)^{1/2}$ has rank less than n, hence for $q = \log n$,

$$\left\| \left(\sum_{i=1}^n T_i^* T_i \right)^{1/2} \right\|_{S_q^N} \le e \left\| \left(\sum_{i=1}^n |Y_i|_2^2 T_i \right)^{1/2} \right\|_{S_\infty^N} \le e \max_{1\le i\le n} |Y_i|_2 \left\| \sum_{i=1}^n T_i \right\|_{S_\infty^N}^{1/2}.$$

Combining all these inequalities, we conclude that for $q = \log n$

$$\mathbb{E}_\varepsilon \left\| \sum_{i=1}^n \varepsilon_i T_i \right\|_{S_\infty^N} \le C\sqrt{\log n} \left\| \left(\sum_{i=1}^n T_i^* T_i \right)^{1/2} \right\|_{S_{\log n}^N}$$

$$\le C\, e\sqrt{\log n}\, \max_{1\le i\le n} |Y_i|_2 \sup_{y\in S^{N-1}} \left(\sum_{i=1}^n \langle Y_i, y\rangle^2 \right)^{1/2}.$$

□

Remark 5.3.7. — *Since the non-commutative Khinchine inequality holds true for independent Gaussian standard random variables, this result is also valid for Gaussian instead of Bernoulli.*

The proof that we presented here is based on an expression related to some operator norms and our original question can not be expressed with these tools. The original proof of Rudelson used the majorizing measure theory. Several improvements are known and the statements of these results need some definition from the theory of Banach spaces.

Definition 5.3.8. — *A Banach space B is of type 2 if there exists a constant $c > 0$ such that for every n and every $x_1, \dots, x_n \in B$,*

$$\left(\mathbb{E}_\varepsilon \left\| \sum_{i=1}^n \varepsilon_i x_i \right\|^2 \right)^{1/2} \le c \left(\sum_{i=1}^n \|x_i\|^2 \right)^{1/2}.$$

The smallest constant $c > 0$ satisfying this statement is called the type 2 constant of B and is denoted by $T_2(B)$.

Classical examples are Hilbert spaces and L_q space for $2 \le q < +\infty$. From Theorem 1.2.1 in Chapter 1, we know also that L_{ψ_2} has type 2.

Definition 5.3.9. — *A Banach space B has modulus of convexity of power type 2 with constant λ if*

$$\forall x, y \in B, \quad \left\| \frac{x+y}{2} \right\|^2 + \lambda^{-2} \left\| \frac{x-y}{2} \right\|^2 \le \frac{1}{2}\left(\|x\|^2 + \|y\|^2 \right).$$

The modulus of convexity of a Banach space B is defined for every $\varepsilon \in (0, 2]$ by

$$\delta_B(\varepsilon) = \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : \quad \|x\| = \|y\| = 1 \text{ and } \|x - y\| \le \varepsilon \right\}.$$

It is obvious that if B has modulus of convexity of power type 2 with constant λ then $\delta_B(\varepsilon) \ge \varepsilon^2/2\lambda^2$ and it is well known that the reverse holds true (with a different constant than 2). Moreover, for any $1 < p \le 2$, the Clarkson inequality tells that for any $f, g \in L_p$,

$$\left\| \frac{f+g}{2} \right\|_p^2 + \frac{p(p-1)}{8} \left\| \frac{f-g}{2} \right\|_p^2 \le \frac{1}{2}(\|f\|_p^2 + \|g\|_p^2).$$

This proves that for any $p \in (1, 2]$, L_p has modulus of convexity of power type 2 with $\lambda = c\sqrt{p-1}$.

Definition 5.3.10. — *A Banach space B has modulus of smoothness of power type 2 with constant μ if*

$$\forall x, y \in B, \quad \left\| \frac{x+y}{2} \right\|^2 + \mu^2 \left\| \frac{x-y}{2} \right\|^2 \geq \frac{1}{2} \left(\|x\|^2 + \|y\|^2 \right).$$

The modulus of smoothness of a Banach space B is defined for every $\tau > 0$ by

$$\rho_B(\tau) = \sup \left\{ \frac{\|x + \tau y\| + \|x - \tau y\|}{2} - 1 : \quad \|x\| = \|y\| = 1 \right\}.$$

It is clear that if B has modulus of smoothness of power type 2 with constant μ then for every $\tau \in (0, 1), \rho_B(\tau) \leq 2\tau^2\mu^2$ and it is well known that the reverse holds true (with a different constant than 2).
More generally, a Banach space B is uniformly convex if for every $\varepsilon > 0$, $\delta_B(\varepsilon) > 0$ and is uniformly smooth if $\lim_{\tau \to 0} \rho_B(\tau)/\tau = 0$. We have the following simple relationship between these notions.

Proposition 5.3.11. — *For every Banach space B, $B^\star$ being its dual, we have*
(i) *For every $\tau > 0$, $\rho_{B^\star}(\tau) = \sup\{\tau\varepsilon/2 - \delta_B(\varepsilon), 0 < \varepsilon \leq 2\}$.*
(ii) *B is uniformly convex if and only if $B^\star$ is uniformly smooth.*
(iii) *For any Banach space B, if B has modulus of convexity of power type 2 with constant λ then $B^\star$ has modulus of smoothness of power type 2 with constant $c\lambda$ and $T_2(B^\star) \leq c\lambda$.*

Proof. — The proof of (i) is straightforward, using the definition of duality. We have for $\tau > 0$,

$$\begin{aligned}
2\rho_{B^\star}(\tau) &= \sup\{\|x^\star + \tau y^\star\| + \|x^\star - \tau y^\star\| - 2 : \ \|x^\star\| = \|y^\star\| = 1\} \\
&= \sup\{x^\star(x) + \tau y^\star(x) + x^\star(y) - \tau y^\star(y) - 2 : \ \|x^\star\| = \|y^\star\| = \|x\| = \|y\| = 1\} \\
&= \sup\{x^\star(x+y) + \tau y^\star(x-y) - 2 : \ \|x^\star\| = \|y^\star\| = \|x\| = \|y\| = 1\} \\
&= \sup\{\|x+y\| + \tau\|x-y\| - 2 : \ \|x\| = \|y\| = 1\} \\
&= \sup\{\|x+y\| + \tau\varepsilon - 2 : \ \|x\| = \|y\| = 1, \|x-y\| \leq \varepsilon, \varepsilon \in (0,2]\} \\
&= \sup\{\tau\varepsilon - 2\delta_B(\varepsilon) : \ \varepsilon \in (0,2]\}.
\end{aligned}$$

The proof of (ii) follows directly from (i). We will just prove (iii). If B has modulus of convexity of power type 2 with constant λ then $\delta_B(\varepsilon) \geq \varepsilon^2/2\lambda^2$. By (i) we deduce that $\rho_{B^\star}(\tau) \geq \tau^2\lambda^2/4$. It implies that for any $x^\star$, $y^\star \in B^\star$,

$$\left\| \frac{x^\star + y^\star}{2} \right\|_\star^2 + (c\lambda)^2 \left\| \frac{x^\star - y^\star}{2} \right\|_\star^2 \geq \frac{1}{2} \left(\|x^\star\|_\star^2 + \|y^\star\|_\star^2 \right)$$

where c is a positive number. We deduce that for any $u^\star$, $v^\star \in B^\star$,

$$\mathbb{E}_\varepsilon \|\varepsilon u^\star + v^\star\|_\star^2 = \frac{1}{2} \left(\|u^\star + v^\star\|_\star^2 + \|-u^\star + v^\star\|_\star^2 \right) \leq \|v^\star\|_\star^2 + (c\lambda)^2 \|u^\star\|_\star^2.$$

We conclude by induction that for any integer n and any vectors $x_1^\star, \dots, x_n^\star \in B^\star$,

$$\mathbb{E}_\varepsilon \left\| \sum_{i=1}^n \varepsilon_i x_i^\star \right\|_\star^2 \leq (c\lambda)^2 \left(\sum_{i=1}^n \|x_i^\star\|_\star^2 \right)$$

which proves that $T_2(B^\star) \leq c\lambda$. □

It is now possible to state the results about the estimate of the average of the supremum of empirical processes.

Theorem 5.3.12. — *If B is a Banach space with modulus of convexity of power type 2 with constant λ then for any integer n and $\xi_1, \dots \xi_n \in B^\star$,*

$$\mathbb{E}_g \sup_{\|x\| \leq 1} \left| \sum_{i=1}^n g_i \langle \xi_i, x\rangle^2 \right| \leq C\, \lambda^5 \, \sqrt{\log n} \, \max_{1 \leq i \leq n} \|\xi_i\|_\star \sup_{\|x\| \leq 1} \left(\sum_{i=1}^n \langle \xi_i, x\rangle^2 \right)^{1/2}$$

where $g_1, \dots, g_n$ are independent $\mathcal{N}(0,1)$ Gaussian random variables and C is a numerical constant.

Corollary 5.3.13. — *Let B be a Banach space with modulus of convexity of power type 2 with constant λ. Let $Y_1, \dots, Y_n \in B^\star$ be independent random vectors and denote*

$$K(n, Y) = 2\sqrt{\frac{2}{\pi}} C\lambda^5 \sqrt{\log n} \left(\mathbb{E} \max_{1 \leq i \leq n} \|Y_i\|_\star^2 \right)^{1/2} \quad \text{and} \quad \sigma^2 = \sup_{\|y\| \leq 1} \sum_{i=1}^n \mathbb{E}\langle Y_i, y\rangle^2$$

where C is the numerical constant of Theorem 5.3.12. Then we have

$$\mathbb{E} \sup_{\|y\| \leq 1} \left| \sum_{i=1}^n \langle Y_i, y\rangle^2 - \mathbb{E}\langle Y_i, y\rangle^2 \right| \leq K(n, Y)^2 \, + \, K(n, Y)\, \sigma.$$

Proof. — Denote by V_2 the expectation of the supremum of the empirical process, that is

$$V_2 = \mathbb{E} \sup_{\|y\| \leq 1} \left| \sum_{i=1}^n \left(\langle Y_i, y\rangle^2 - \mathbb{E}\langle Y_i, y\rangle^2 \right) \right|.$$

We start with a symmetrization argument. By (5.5) and Proposition 5.3.3 we have

$$V_2 \leq 2\, \mathbb{E}\mathbb{E}_\varepsilon \sup_{\|y\| \leq 1} \left| \sum_{i=1}^n \varepsilon_i \langle Y_i, y\rangle^2 \right| \leq 2\sqrt{\frac{2}{\pi}}\, \mathbb{E}\mathbb{E}_g \sup_{\|y\| \leq 1} \left| \sum_{i=1}^n g_i \langle Y_i, y\rangle^2 \right|.$$

In view of Theorem 5.3.12, we observe that the crucial quantity in the estimate is $\sup_{\|x\| \leq 1} \left(\sum_{i=1}^n \langle Y_i, x\rangle^2 \right)^{1/2}$. Indeed, by the triangle inequality,

$$\mathbb{E} \sup_{\|x\| \leq 1} \sum_{i=1}^n \langle Y_i, x\rangle^2 \leq \mathbb{E} \sup_{\|y\| \leq 1} \left| \sum_{i=1}^n \left(\langle Y_i, y\rangle^2 - \mathbb{E}\langle Y_i, y\rangle^2 \right) \right| + \sup_{\|y\| \leq 1} \sum_{i=1}^n \mathbb{E}\langle Y_i, y\rangle^2 = V_2 + \sigma^2.$$

Therefore, applying Theorem 5.3.12 and the Cauchy Schwarz inequality, we get

$$\begin{aligned}
V_2 &\le 2\sqrt{\frac{2}{\pi}}\, C\lambda^5 \sqrt{\log n}\, \mathbb{E}\left(\max_{1\le i\le n} \|Y_i\|_\star \sup_{\|x\|\le 1} \left(\sum_{i=1}^n \langle Y_i, x\rangle^2 \right)^{1/2} \right) \\
&\le 2\sqrt{\frac{2}{\pi}}\, C\lambda^5 \sqrt{\log n} \left(\mathbb{E} \max_{1\le i\le n} \|Y_i\|_\star^2 \right)^{1/2} \left(\mathbb{E} \sup_{\|x\|\le 1} \sum_{i=1}^n \langle Y_i, x\rangle^2 \right)^{1/2} \\
&\le K(n,Y) \left(V_2 + \sigma^2 \right)^{1/2}.
\end{aligned}$$

We get that

$$V_2^2 - K(n,Y)^2 V_2 - K(n,Y)^2\sigma^2 \le 0$$

from which it is easy to conclude that

$$V_2 \le K(n,Y)\ \left(K(n,Y)\, +\, \sigma \right).$$

□

The proof of Theorem 5.3.12 is slightly complicated. It involves a specific construction of majorizing measures and deep results about the duality of covering numbers (it is where the notion of type is used). We will not present it. However, using simpler ideas, we can also prove a general result where the assumption that B has a good modulus of convexity is not needed.

Theorem 5.3.14. — *Let B be a Banach space and $Y_1, \ldots, Y_n$ be independent random vectors in $B^\star$. Let $\mathcal{F}$ be a set of functionals on $B^\star$ with $0 \in \mathcal{F}$. Denote by $d_{\infty,n}$ the random quasi-metric on $\mathcal{F}$ defined for every $f, \overline{f}$ in $\mathcal{F}$ by*

$$d_{\infty,n}(f,\overline{f}) = \max_{1\le i\le n} \left| f(Y_i) - \overline{f}(Y_i) \right|.$$

We have

$$\mathbb{E} \sup_{f\in\mathcal{F}} \left| \sum_{i=1}^n \left(f(Y_i)^2 - \mathbb{E} f(Y_i)^2 \right) \right| \le \max(\sigma_{\mathcal{F}} U_n, U_n^2)$$

where for a numerical constant C,

$$U_n = C\left(\mathbb{E}\gamma_2^2(\mathcal{F}, d_{\infty,n}) \right)^{1/2} \quad \text{and} \quad \sigma_{\mathcal{F}} = \left(\sup_{f\in\mathcal{F}} \sum_{i=1}^n \mathbb{E} f(Y_i)^2 \right)^{1/2}.$$

We refer to Definition 3.1.3 in Chapter 3 for the precise definition of $\gamma_2(\mathcal{F}, d_{\infty,n})$.

Proof. — As in the proof of Corollary 5.3.13, we need first to get a bound of

$$\mathbb{E}_g \sup_{f\in\mathcal{F}} \left| \sum_{i=1}^n g_i f(Y_i)^2 \right|.$$

Let $(X_f)_{f\in\mathcal{F}}$ be the Gaussian process defined conditionally with respect to the Y_i's, $X_f = \sum_{i=1}^n g_i f(Y_i)^2$ and indexed by $f \in \mathcal{F}$. The quasi-metric d associated to this process is given for any $f, \overline{f} \in \mathcal{F}$ by

$$\begin{aligned}
d(f,\overline{f})^2 = \mathbb{E}_g |X_f - X_{\overline{f}}|^2 &= \sum_{i=1}^n \left(f(Y_i)^2 - \overline{f}(Y_i)^2\right)^2 \\
&= \sum_{i=1}^n \left(f(Y_i) - \overline{f}(Y_i)\right)^2 \left(f(Y_i) + \overline{f}(Y_i)\right)^2 \\
&\le 2 \sum_{i=1}^n \left(f(Y_i) - \overline{f}(Y_i)\right)^2 \left(f(Y_i)^2 + \overline{f}(Y_i)^2\right) \\
&\le 4 \sup_{f\in\mathcal{F}} \left(\sum_{i=1}^n f(Y_i)^2\right) \max_{1\le i\le n} (f(Y_i) - \overline{f}(Y_i))^2.
\end{aligned}$$

In conclusion, we have proved that for any $f, \overline{f} \in \mathcal{F}$,

$$d(f,\overline{f}) \le 2 \sup_{f\in\mathcal{F}} \left(\sum_{i=1}^n f(Y_i)^2\right)^{1/2} d_{\infty,n}(f,\overline{f}).$$

By definition of the γ_2 functionals, see Chapter 3, we conclude that for every vectors $Y_1, \dots, Y_n \in B^\star$,

$$\mathbb{E}_g \sup_{f\in\mathcal{F}} \left|\sum_{i=1}^n g_i f(Y_i)^2\right| \le C \sup_{f\in\mathcal{F}} \left(\sum_{i=1}^n f(Y_i)^2\right)^{1/2} \gamma_2(\mathcal{F}, d_{\infty,n})$$

where C is a universal constant. We repeat the proof of Corollary 5.3.13. Let

$$V_2 = \mathbb{E} \sup_{f\in\mathcal{F}} \left|\sum_{i=1}^n \left(f(Y_i)^2 - \mathbb{E} f(Y_i)^2\right)\right|.$$

By a symmetrization argument and the Cauchy-Schwarz inequality,

$$\begin{aligned}
V_2 &\le 2\sqrt{\frac{2}{\pi}} \mathbb{E}\mathbb{E}_g \sup_{f\in\mathcal{F}} \left|\sum_{i=1}^n g_i f(Y_i)^2\right| \le C \left(\mathbb{E}\gamma_2(\mathcal{F}, d_{\infty,n})^2\right)^{1/2} \left(\mathbb{E} \sup_{f\in\mathcal{F}} \sum_{i=1}^n f(Y_i)^2\right)^{1/2} \\
&\le C \left(\mathbb{E}\gamma_2(\mathcal{F}, d_{\infty,n})^2\right)^{1/2} \left(V_2 + \sigma_{\mathcal{F}}^2\right)^{1/2}.
\end{aligned}$$

where the last inequality follows from the triangle inequality: $\mathbb{E} \sup_{f\in\mathcal{F}} \sum_{i=1}^n f(Y_i)^2 \le (V_2 + \sigma_{\mathcal{F}}^2)^2$. This shows that V_2 satisfies an inequality of degree 2 from which it is easy to conclude that

$$V_2 \le \max(\sigma_{\mathcal{F}} U_n, U_n^2), \text{ where } U_n = C \left(\mathbb{E}\gamma_2(\mathcal{F}, d_{\infty,n})^2\right)^{1/2}.$$

□

5.4. Reconstruction property

We are now able to state one main theorem concerning the reconstruction property of a random matrix defined by taking empirical copies of the rows of a fixed bounded orthogonal matrix (or by selecting randomly its rows).

Theorem 5.4.1. — *Let $\phi_1, \dots, \phi_N$ be an orthogonal system in ℓ_2^N such that for a real number K*

$$\forall i \le N, \ |\phi_i|_2 = K \text{ and } |\phi_i|_\infty \le \frac{1}{\sqrt{N}}.$$

Let Y be the random vector defined by $Y = \phi_i$ with probability $1/N$ and $Y_1, \dots, Y_n$ be independent copies of Y. If

$$m \le C_1 K^2 \frac{n}{\log N (\log n)^3}$$

then with probability greater than

$$1 - C_2 \exp(-C_3 K^2 n/m)$$

the matrix $\Phi = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix}$ is a good reconstruction matrix for sparse signals of size m, that is for every $U \in \Sigma_m$, the basis pursuit algorithm (5.1), $\min_{t \in \mathbb{R}^N} \{|t|_1 : \Phi U = \Phi t\}$, *has a unique solution equal to U.*

Remark 5.4.2. — (*i*) *By definition of m, the probability of this event is always greater than $1 - C_2 \exp\left(-C_3 \log N (\log n)^3\right)$.*
(*ii*) *The same result is valid when using the method of selectors.*
(*iii*) *As we already mentioned, this theorem covers the case of a lot of classical systems like the Fourier system or the Walsh system.*
(*iv*) *The result is also valid if the orthogonal system $\phi_1, \dots, \phi_N$ satisfies the weaker condition that for all $i \le N$, $K_1 \le |\phi_i|_2 \le K_2$ and in the statement, K has to be replaced by K_2^2/K_1.*

Proof. — Observe that $\mathbb{E}\langle Y, y\rangle^2 = K^2\,|y|_2^2/N$. We define the class of functions $\mathcal{F}$ in the following way:

$$\mathcal{F} = \left\{ \begin{array}{rcl} f_y : \mathbb{R}^N & \to & \mathbb{R} \\ Y & \mapsto & \langle Y, y\rangle \end{array}, \quad y \in B_1^N \cap \rho S^{N-1} \right\}.$$

Therefore

$$Z = \sup_{f \in \mathcal{F}} \left| \sum_{i=1}^n (f(Y_i)^2 - \mathbb{E} f(Y_i)^2) \right| = \sup_{y \in B_1^N \cap \rho S^{N-1}} \left| \sum_{i=1}^n \langle Y_i, y\rangle^2 - \frac{K^2 n \rho^2}{N} \right|.$$

With the notation of Theorem 5.3.14, we have

$$\sigma_{\mathcal{F}}^2 = \sup_{y \in B_1^N \cap \rho S^{N-1}} \sum_{i=1}^n \mathbb{E}\langle Y_i, y\rangle^2 = \frac{K^2 n \rho^2}{N}. \tag{5.8}$$

Moreover, since $B_1^N \cap \rho S^{N-1} \subset B_1^N$,

$$\gamma_2(B_1^N \cap \rho S^{N-1}, d_{\infty,n}) \le \gamma_2(B_1^N, d_{\infty,n}).$$

It is well known that the γ_2 functional is bounded by the Dudley integral (see (3.7) in Chapter 3):

$$\gamma_2(B_1^N, d_{\infty,n}) \le C \int_0^{+\infty} \sqrt{\log N(B_1^N, \varepsilon, d_{\infty,n})}\, d\varepsilon.$$

However, for every $i \le n$, $|Y_i|_\infty \le 1/\sqrt{N}$ and

$$\sup_{y,\overline{y}\in B_1^N} d_{\infty,n}(y,\overline{y}) = \sup_{y,\overline{y}\in B_1^N} \max_{1\le i\le n} |\langle Y_i, y-\overline{y}\rangle| \le 2 \max_{1\le i\le n} |Y_i|_\infty \le \frac{2}{\sqrt{N}}.$$

The integral is only computed from 0 to $2/\sqrt{N}$ and by the change of variable $t = \varepsilon\sqrt{N}$, we deduce that

$$\int_0^{+\infty} \sqrt{\log N(B_1^N, \varepsilon, d_{\infty,n})} d\varepsilon = \frac{1}{\sqrt{N}} \int_0^2 \sqrt{\log N\left(B_1^N, \frac{t}{\sqrt{N}}, d_{\infty,n}\right)}\, dt.$$

From Theorem 1.4.3, since for every $i \le n$, $|Y_i|_\infty \le 1/\sqrt{N}$, we have

$$\sqrt{\log N\left(B_1^N, \frac{t}{\sqrt{N}}, d_{\infty,n}\right)} \le \left\{ \begin{array}{l} \dfrac{C}{t}\, \sqrt{\log n}\sqrt{\log N}, \\ C\sqrt{n \log\left(1 + \dfrac{3}{t}\right)} \end{array}\right. .$$

We split the integral into two parts, the one when $t \le 1/\sqrt{n}$ and the one when $1/\sqrt{n} \le t \le 2$.

$$\begin{aligned} \int_0^{1/\sqrt{n}} \sqrt{n\log\left(1+\frac{3}{t}\right)} dt &= \int_0^1 \sqrt{\log\left(1 + \frac{3\sqrt{n}}{u}\right)}\, du \\ &\le \int_0^1 \sqrt{\log n + \log\left(\frac{3}{u}\right)}\, du \le C\, \sqrt{\log n} \end{aligned}$$

and since

$$\int_{1/\sqrt{n}}^2 \frac{1}{t} dt \le C \log n,$$

we conclude that

$$\gamma_2(B_1^N \cap \rho S^{N-1}, d_{\infty,n}) \le \gamma_2(B_1^N, d_{\infty,n}) \le C\, \sqrt{\frac{(\log n)^3\, \log N}{N}}. \tag{5.9}$$

Combining this estimate and (5.8) with Theorem 5.3.14, we get that for a real number $C \ge 1$,

$$\mathbb{E}Z \le C \max\left(\frac{(\log n)^3 \log N}{N},\ \rho\, K\, \sqrt{\frac{n}{N}}\sqrt{\frac{(\log n)^3 \log N}{N}}\right).$$

We choose ρ such that

$$(\log n)^3 \log N \le \rho\, K\, \sqrt{n\, (\log n)^3 \log N} \le \frac{1}{3C}\, K^2\, \rho^2\, n$$

which means that ρ satisfies

$$K\,\rho \geq 3\,C\,\sqrt{\frac{(\log n)^3 \log N}{n}}. \tag{5.10}$$

For this choice of ρ, we conclude that

$$\mathbb{E}Z = \mathbb{E} \sup_{y \in B_1^N \cap \rho S^{N-1}} \left| \sum_{i=1}^n \langle Y_i, y\rangle^2 - \frac{K^2 n\rho^2}{N} \right| \leq \frac{1}{3}\frac{K^2 n\rho^2}{N}.$$

We use Proposition 5.3.5 to get a deviation inequality for the random variable Z. With the notations of Proposition 5.3.5, we have

$$u = \sup_{y \in B_1^N \cap \rho S^{N-1}} \max_{1\leq i\leq N} \langle \phi_i, y\rangle^2 \leq \max_{1\leq i\leq N} |\phi_i|_\infty^2 \leq \frac{1}{N}$$

and

$$\begin{aligned} v &= \sup_{y \in B_1^N \cap \rho S^{N-1}} \sum_{i=1}^n \mathbb{E}\left(\langle Y_i, y\rangle^2 - \mathbb{E}\langle Y_i, y\rangle^2\right)^2 \ + \ 32\,u\,\mathbb{E}Z \\ &\leq \sup_{y \in B_1^N \cap \rho S^{N-1}} \sum_{i=1}^n \mathbb{E}\langle Y_i, y\rangle^4 \ + \frac{CK^2 n\rho^2}{N^2} \leq \frac{CK^2 n\rho^2}{N^2} \end{aligned}$$

since for every $y \in B_1^N$, $\mathbb{E}\langle Y, y\rangle^4 \leq \mathbb{E}\langle Y, y\rangle^2/N$. Using Proposition 5.3.5 with $t = \frac{1}{3}\frac{K^2 n\rho^2}{N}$, we conclude that

$$\mathbb{P}\left(Z \geq \frac{2}{3}\frac{K^2 n\rho^2}{N}\right) \leq C\exp(-c\,K^2\,n\,\rho^2).$$

With probability greater than $1 - C\exp(-c\,K^2\,n\,\rho^2)$, we get that

$$\sup_{y \in B_1^N \cap \rho S^{N-1}} \left| \sum_{i=1}^n \langle Y_i, y\rangle^2 - \frac{K^2 n\rho^2}{N} \right| \leq \frac{2}{3}\frac{K^2 n\rho^2}{N}$$

from which it is easy to deduce by Proposition 5.1.2 that

$$\operatorname{rad}\left(\ker \Phi \cap B_1^N\right) < \rho.$$

We choose $m = 1/4\rho^2$ and conclude by Proposition 5.1.1 that with probability greater than $1 - C\exp(-cK^2 n/m)$, the matrix Φ is a good reconstruction matrix for sparse signals of size m, that is for every $U \in \Sigma_m$, the basis pursuit algorithm (5.1) has a unique solution equal to U. The condition on m in Theorem 5.4.1 comes from (5.10). □

Remark 5.4.3. — *By Proposition 2.7.3, it is clear that the matrix Φ shares also the property of approximate reconstruction. It is enough to change the choice of m by $m = 1/16\rho^2$. Therefore, if U is any unknown signal and x a solution of*

$$\min_{t\in\mathbb{R}^N} \{|t|_1, \Phi U = \Phi t\}$$

then for any subset I of cardinality less than m,

$$|x-U|_2 \le \frac{|x-U|_1}{4\sqrt{m}} \le \frac{|U_{I^c}|_1}{\sqrt{m}}.$$

5.5. Random selection of characters within a coordinate subspace

In this part, we consider the problem presented in section 5.1. We briefly recall the notations. Let μ be a probability measure and let $(\psi_1,\dots,\psi_N)$ be an orthonormal system of $L_2(\mu)$ bounded in L_∞ i.e. such that for every $i \le N$, $\|\psi_i\|_\infty \le 1$ (typically, a system of characters in $L_2(\mu)$ like the Fourier or the Walsh system). For a measurable function f and for $p>0$, we denote its L_p norm and its L_∞ norm by

$$\|f\|_p = \left(\int |f|^p d\mu\right)^{1/p} \quad \text{and} \quad \|f\|_\infty = \sup|f|.$$

In $\mathbb{R}^N$ or $\mathbb{C}^N$, μ is just the counting probability measure so that the L_p-norm of a vector $x=(x_1,\dots,x_N)$ is defined by

$$\|x\|_p = \left(\frac{1}{N}\sum_{i=1}^N |x_i|^p\right)^{1/p}.$$

The spaces ℓ_∞^N and L_∞^N coincide and we observe that if $(\psi_1,\dots,\psi_N)$ is a bounded orthonormal system in L_2^N then $(\psi_1/\sqrt{N},\dots,\psi_N/\sqrt{N})$ is an orthonormal system of ℓ_2^N such that for every $i \le N$, $|\psi_i/\sqrt{N}|_\infty \le 1/\sqrt{N}$. Therefore the setting is exactly the same as in the previous part up to a normalization factor of $\sqrt{N}$.

Of course the notation of the radius of a set T will now be adapted to the $L_2(\mu)$ Euclidean structure. This means that for a set T, its radius is

$$\operatorname{Rad} T = \sup_{t\in T} \|t\|_2.$$

For any $q>0$, we denote by B_q the unit ball of $L_q(\mu)$ and by S_q the unit sphere of $L_q(\mu)$. Our question is to find a very large subset I of $\{1,\dots,N\}$ such that

$$\forall (a_i)_{i\in I}, \quad \left\|\sum_{i\in I} a_i\psi_i\right\|_2 \le \rho \left\|\sum_{i\in I} a_i\psi_i\right\|_1$$

with the smallest possible ρ. We already said that Talagrand showed that there exists a small constant δ_0 such that for any bounded orthonormal system $\{\psi_1,\dots,\psi_N\}$, there exists a subset I of cardinality greater than $\delta_0 N$ such that $\rho \le C\sqrt{\log N\,(\log\log N)}$. The proof involves the construction of specific majorizing measures. Moreover, it was known from Bourgain that the $\sqrt{\log N}$ is necessary in the estimate. We will now explain why the strategy that we developed in the previous part is adapted to this type of question. For example, we will be able to extend the result of Talagrand to a Kashin type setting, that is for example to find I of cardinality greater than $N/2$.

We start with the following simple Proposition concerning some properties of a matrix that we will later define randomly as in Theorem 5.4.1.

Proposition 5.5.1. — *Let μ be a probability measure and let $(\psi_1, \ldots, \psi_N)$ be an orthonormal system of $L_2(\mu)$. Let $Y_1, \ldots, Y_n$ be a family of vectors taking values from the set of vectors $\{\psi_1, \ldots, \psi_N\}$. Let Ψ be the matrix $\Psi = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix}$. Then*
(i) $\ker \Psi = \operatorname{span}\{\{\psi_1, \ldots, \psi_N\} \setminus \{Y_i\}_{i=1}^n\} = \operatorname{span}\{\psi_i\}_{i \in I}$ *where I is a subset of cardinality greater than $N - n$.*
(ii) $(\ker \Psi)^\perp = \operatorname{span}\{\psi_i\}_{i \notin I}$.
(iii) For a star body T, if

$$\sup_{y \in T \cap \rho S_2} \left| \sum_{i=1}^n \langle Y_i, y\rangle^2 - \frac{n\rho^2}{N} \right| \leq \frac{1}{3} \frac{n\rho^2}{N} \tag{5.11}$$

then $\operatorname{Rad}(\ker \Psi \cap T) < \rho$.
(iv) If $n < 3N/4$ and if (5.11) *is satisfied then we also have* $\operatorname{Rad}((\ker \Psi)^\perp \cap T) < \rho$.

Proof. — Since $\{\psi_1, \ldots, \psi_N\}$ is an orthonormal system, the parts (i) and (ii) are obvious. For the proof of (iii), we first remark that if (5.11) holds true then we get from the lower bound that for all $y \in T \cap \rho S_2$,

$$\sum_{i=1}^n \langle Y_i, y\rangle^2 \geq \frac{2}{3} \frac{n\rho^2}{N}$$

and we deduce as in Proposition 5.1.2 that $\operatorname{Rad}(\ker \Psi \cap T) < \rho$.
For the proof of (iv), we deduce from the upper bound of (5.11) that for all $y \in T \cap \rho S_2$,

$$\begin{aligned} \sum_{i \in I} \langle \psi_i, y\rangle^2 &= \sum_{i=1}^N \langle \psi_i, y\rangle^2 - \sum_{i=1}^n \langle Y_i, y\rangle^2 = \|y\|_2^2 - \sum_{i=1}^n \langle Y_i, y\rangle^2 \\ &\geq \rho^2 - \frac{4}{3} \frac{n\rho^2}{N} = \rho^2 \left(1 - \frac{4n}{3N}\right) > 0 \text{ since } n < 3N/4. \end{aligned}$$

This inequality means that for the matrix $\tilde{\Psi}$ defined by $\tilde{\Psi} = \begin{pmatrix} \cdot \\ \psi_i \\ \cdot \end{pmatrix}_{i \in I}$, for every $y \in T \cap \rho S_2$, we have

$$\inf_{y \in T \cap \rho S_2} \|\tilde{\Psi} y\|_2^2 > 0$$

and we conclude as in Proposition 5.1.2 that $\operatorname{Rad}(\ker \tilde{\Psi} \cap T) < \rho$. Moreover, it is obvious that $\ker \tilde{\Psi} = (\ker \Psi)^\perp$. □

The case of L_2^N. — We will now present a result concerning the problem of selection of characters in L_2^N. It is not the most general result but we would like to emphasize the deep similarity between the proofs of this result and the proof of Theorem 5.4.1.

Theorem 5.5.2. — *Let $(\psi_1, \ldots, \psi_N)$ be an orthonormal system of L_2^N bounded in L_∞^N i.e. such that for every $i \leq N$, $\|\psi_i\|_\infty \leq 1$.*

For any $2 \le n \le N-1$, there exists a subset $I \subset [N]$ of cardinality greater than $N-n$ such that for all $(a_i)_{i\in I}$,

$$\left\| \sum_{i\in I} a_i\psi_i \right\|_2 \le C\sqrt{\frac{N}{n}}\sqrt{\log N}(\log n)^{3/2} \left\| \sum_{i\in I} a_i\psi_i \right\|_1 .$$

Proof. — Let Y be the random vector defined by $Y = \psi_i$ with probability $1/N$ and let $Y_1, \ldots, Y_n$ be independent copies of Y. Observe that $\mathbb{E}\langle Y, y\rangle^2 = \|y\|_2^2/N$ and define

$$Z = \sup_{y\in B_1\cap\rho S_2} \left| \sum_{i=1}^n \langle Y_i, y\rangle^2 - \frac{n\rho^2}{N} \right| .$$

Following the proof of Theorem 5.4.1 (the normalization is different from a factor $\sqrt{N}$), we obtain that if ρ is such that

$$\rho \ge C\sqrt{\frac{N\,(\log n)^3\log N}{n}}$$

then

$$\mathbb{P}\left(Z \ge \frac{1}{3}\frac{n\rho^2}{N} \right) \le C\exp\left(-c\,\frac{n\,\rho^2}{N} \right).$$

Therefore there exists a choice of $Y_1, \ldots, Y_n$ (in fact it is with probability greater than $1 - C\exp(-c\,\frac{n\,\rho^2}{N}))$ such that

$$\sup_{y\in B_1\cap\rho S_2} \left| \sum_{i=1}^n \langle Y_i, y\rangle^2 - \frac{n\rho^2}{N} \right| \le \frac{1}{3}\frac{n\rho^2}{N}$$

and if I is defined by $\{\psi_i\}_{i\in I} = \{\psi_1, \ldots, \psi_N\} \setminus \{Y_1, \ldots, Y_n\}$ then by Proposition 5.5.1 (iii) and (i), we conclude that $\mathrm{Rad}\,(\mathrm{span}\,\{\psi_i\}_{i\in I} \cap B_1) \le \rho$ and $|I| \ge N-n$. This means that for every $(a_i)_{i\in I}$,

$$\left\| \sum_{i\in I} a_i\psi_i \right\|_2 \le \rho \left\| \sum_{i\in I} a_i\psi_i \right\|_1 .$$

□

Remark 5.5.3. — *Theorem 5.5.2 implies Theorem 5.4.1. Indeed, if we write the inequality with the classical ℓ_1^N and ℓ_2^N norms, we get that*

$$\left| \sum_{i\in I} a_i\psi_i \right|_2 \le C\sqrt{\frac{\log N}{n}}(\log n)^{3/2} \left| \sum_{i\in I} a_i\psi_i \right|_1$$

which means that $\mathrm{rad}\,(\ker\Psi \cap B_1^N) \le C\sqrt{\frac{\log N}{n}}(\log n)^{3/2}$. *We conclude about the reconstruction property by using Proposition 5.1.1.*

The general case of $L_2(\mu)$. — We can now state a general result about the problem of selection of characters. It is an extension of (5.3) to the existence of a subset of arbitrary size, with a slightly worse dependence in $\log\log N$.

Theorem 5.5.4. — *Let μ be a probability measure and let $(\psi_1, \ldots, \psi_N)$ be an orthonormal system of $L_2(\mu)$ bounded in $L_\infty(\mu)$ i.e. such that for every $i \leq N$, $\|\psi_i\|_\infty \leq 1$.*
For any $n \leq N-1$, there exists a subset $I \subset [N]$ of cardinality greater than $N-n$ such that for all $(a_i)_{i\in I}$,

$$\left\|\sum_{i\in I} a_i\psi_i\right\|_2 \leq C\,\gamma\,(\log\gamma)^{5/2}\left\|\sum_{i\in I} a_i\psi_i\right\|_1$$

where $\gamma = \sqrt{\frac{N}{n}}\sqrt{\log n}$.

Remark 5.5.5. — *(i) If n is chosen to be proportional to N then $\gamma\,(\log\gamma)^{5/2}$ is of the order of $\sqrt{\log N}\,(\log\log N)^{5/2}$. However, if n is chosen to be a power of N then $\gamma\,(\log\gamma)^{5/2}$ is of the order $\sqrt{\frac{N}{n}}\sqrt{\log n}(\log N)^{5/2}$ which is a worse dependence than in Theorem 5.5.2*
(ii) Exactly as in Theorem 5.4.1 we could assume that $(\psi_1, \ldots, \psi_N)$ is an orthogonal system of L_2 such that for every $i \leq N$, $\|\psi_i\|_2 = K$ and $\|\psi_i\|_\infty \leq 1$ for a fixed real number K.

The second main result is an extension of (5.3) to a Kashin type decomposition.

Theorem 5.5.6. — *With the same assumptions as in Theorem 5.5.4, if N is an even natural integer, there exists a subset $I \subset [N]$ with $\frac{N}{2} - c\sqrt{N} \leq |I| \leq \frac{N}{2} + c\sqrt{N}$ such that for all $(a_i)_{i=1}^N$*

$$\left\|\sum_{i\in I} a_i\psi_i\right\|_2 \leq C\,\sqrt{\log N}\,(\log\log N)^{5/2}\left\|\sum_{i\in I} a_i\psi_i\right\|_1$$

and

$$\left\|\sum_{i\notin I} a_i\psi_i\right\|_2 \leq C\,\sqrt{\log N}\,(\log\log N)^{5/2}\left\|\sum_{i\notin I} a_i\psi_i\right\|_1.$$

In order to be able to use Theorem 5.3.12 and its Corollary 5.3.13, we would like to replace the unit ball B_1 by a ball which has a good modulus of convexity that is for example B_p for $1 < p \leq 2$. We start recalling a classical trick that is used very often when we compare the L_r norms of a measurable functions (for example in the theory of thin sets in Harmonic Analysis).

Lemma 5.5.7. — *Let f be a measurable function with respect to the probability measure μ. For $1 < p < 2$,*

$$\text{if } \|f\|_2 \leq A\|f\|_p \text{ then } \|f\|_2 \leq A^{\frac{p}{2-p}}\|f\|_1.$$

Proof. — This is just an application of Hölder inequality. Let $\theta \in (0,1)$ such that $1/p = (1-\theta) + \theta/2$ that is $\theta = 2(1-1/p)$. By Hölder,

$$\|f\|_p \le \|f\|_1^{1-\theta} \|f\|_2^{\theta}.$$

Therefore if $\|f\|_2 \le A\|f\|_p$ we deduce that $\|f\|_2 \le A^{\frac{1}{1-\theta}} \|f\|_1$. □

Proposition 5.5.8. — *With the same assumptions as in Theorem 5.5.4, the following holds.*

1) For any $p \in (1,2)$ and any $2 \le n \le N-1$ there exists a subset $I \subset \{1,\dots,N\}$ with $|I| \ge N-n$ such that for every $a = (a_i) \in \mathbb{C}^N$,

$$\left\| \sum_{i\in I} a_i \psi_i \right\|_2 \le \frac{C}{(p-1)^{5/2}} \sqrt{N/n}\,\sqrt{\log n} \left\| \sum_{i\in I} a_i \psi_i \right\|_p.$$

2) Moreover, if N is an even natural integer, there exists a subset $I \subset \{1,\dots,N\}$ with $N/2 - c\sqrt{N} \le |I| \le N/2 + c\sqrt{N}$ such that for every $a = (a_i) \in \mathbb{C}^N$,

$$\left\| \sum_{i\in I} a_i \psi_i \right\|_2 \le \frac{C}{(p-1)^{5/2}} \sqrt{N/n}\,\sqrt{\log n} \left\| \sum_{i\in I} a_i \psi_i \right\|_p$$

and

$$\left\| \sum_{i\notin I} a_i \varphi_i \right\|_2 \le \frac{C}{(p-1)^{5/2}} \sqrt{N/n}\,\sqrt{\log n} \left\| \sum_{i\notin I} a_i \psi_i \right\|_p.$$

Combining the first part of Proposition 5.5.8 with Lemma 5.5.7, it is easy to prove Theorem 5.5.4. Indeed, let $\gamma = \sqrt{N/n}\,\sqrt{\log n}$ and choose $p = 1 + 1/\log\gamma$. Using Proposition 5.5.8, there is a subset I of cardinality greater than $N-n$ for which

$$\forall (a_i)_{i\in I}, \quad \left\| \sum_{i\in I} a_i \psi_i \right\|_2 \le C_p\,\gamma \left\| \sum_{i\in I} a_i \psi_i \right\|_p$$

where $C_p = C/(p-1)^{5/2}$. By the choice of p and Lemma 5.5.7,

$$\left\| \sum_{i\in I} a_i \psi_i \right\|_2 \le \gamma\, C_p^{p/(2-p)} \gamma^{2(p-1)/(2-p)} \left\| \sum_{i\in I} a_i \psi_i \right\|_1 \le C\,\gamma\,(\log\gamma)^{5/2} \left\| \sum_{i\in I} a_i \psi_i \right\|_1.$$

The same argument works for the Theorem 5.5.6 using the second part of Proposition 5.5.8.

It remains to prove Proposition 5.5.8.

Proof. — Let Y be the random vector defined by $Y = \psi_i$ with probability $1/N$ and let $Y_1,\dots,Y_n$ be independent copies of Y. Observe that for any $y \in L_2(\mu)$, $\mathbb{E}\langle Y, y\rangle^2 = \|y\|_2^2/N$. Let $E = \operatorname{span}\{\psi_1,\dots,\psi_N\}$ and for $\rho > 0$ let E_p be the vectorial space E endowed with the norm defined by

$$\|y\| = \left(\frac{\|y\|_p^2 + \rho^{-2}\|y\|_2^2}{2} \right)^{1/2}.$$

We restrict our study to the vectorial space E and it is clear that

$$(B_p \cap \rho B_2) \subset B_{E_p} \subset \sqrt{2}(B_p \cap \rho B_2) \tag{5.12}$$

where B_{E_p} is the unit ball of E_p. Moreover, the Clarkson inequality tells that for any $f, g \in L_p$,

$$\left\| \frac{f+g}{2} \right\|_p^2 + \frac{p(p-1)}{8} \left\| \frac{f-g}{2} \right\|_p^2 \leq \frac{1}{2}(\|f\|_p^2 + \|g\|_p^2).$$

It is therefore easy to deduce that E_p is a Banach space with modulus of convexity of power type 2 with constant λ such that $\lambda^{-2} = p(p-1)/8$.

We define the random variable

$$Z = \sup_{y \in B_p \cap \rho S_2} \left| \sum_{i=1}^n \langle Y_i, y \rangle^2 - \frac{n\rho^2}{N} \right|$$

and we deduce from (5.12) that

$$\mathbb{E}Z \leq \mathbb{E} \sup_{y \in B_{E_p}} \left| \sum_{i=1}^n \langle Y_i, y \rangle^2 - \mathbb{E}\langle Y_i, y \rangle^2 \right|.$$

We use Corollary 5.3.13. We deduce from (5.12) that $\sigma^2 = \sup_{y \in B_{E_p}} n\|y\|_2^2/N \leq 2n\rho^2/N$ and that for every $i \leq N$, $\|\psi_i\|_{E_p^\star} \leq \sqrt{2}\|\psi_i\|_\infty \leq \sqrt{2}$. By Corollary 5.3.13, we get

$$\mathbb{E} \sup_{y \in B_{E_p}} \left| \sum_{i=1}^n \langle Y_i, y \rangle^2 - \mathbb{E}\langle Y_i, y \rangle^2 \right| \leq C \max \left(\lambda^{10} \log n, \rho\lambda^5 \sqrt{\frac{n \log n}{N}} \right).$$

We conclude that

$$\text{if} \quad \rho \geq C\lambda^5 \sqrt{\frac{N \log n}{n}} \quad \text{then} \quad \mathbb{E}Z \leq \frac{1}{3} \frac{n\rho^2}{N}$$

and using Proposition 5.1.2 we get that

$$\mathrm{Rad}\,(\ker \Psi \cap B_p) < \rho$$

where $\Psi = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix}$. We choose $\rho = C\lambda^5 \sqrt{\frac{N \log n}{n}}$ and deduce from Proposition 5.5.1 (*iii*) and (*i*) that for I defined by $\{\psi_i\}_{i \in I} = \{\psi_1, \dots, \psi_N\} \setminus \{Y_1, \dots, Y_n\}$ we have

$$\forall (a_i)_{i \in I}, \quad \left\| \sum_{i \in I} a_i \psi_i \right\|_2 \leq \rho \left\| \sum_{i \in I} a_i \psi_i \right\|_p.$$

This ends the proof of the first part of Proposition 5.5.8.

For the second part, we add the following observation. By a combinatorial argument, it is not difficult to prove that if $n = [\delta N]$ with $\delta = \log 2 < 3/4$ then with probability greater than $3/4$,

$$N/2 - c\sqrt{N} \leq |I| = N - |\{Y_1, \dots, Y_n\}| \leq N/2 + c\sqrt{N},$$

for some absolute constant $c > 0$. Hence $n < 3N/4$ and we can also use part (iv) of Proposition 5.5.1 which proves that

$$\mathrm{Rad}\,(\ker\Psi \cap B_p) \le \rho \quad \text{and} \quad \mathrm{Rad}\,((\ker\Psi)^\perp \cap B_p) \le \rho.$$

Since $\ker\Psi = \mathrm{span}\,\{\psi_i\}_{i\in I}$ and $(\ker\Psi)^\perp = \mathrm{span}\,\{\psi_i\}_{i\notin I}$, this ends the proof of the Proposition. $\square$

5.6. Notes and comments

For the study of the supremum of an empirical process and the connection with Rademacher averages, we already referred to chapter 4 of [**LT91**]. Theorem 5.3.1 is due to Talagrand and can be found in theorem 4.12 in [**LT91**]. Theorem 5.3.2 is often called a "symmetrization principle". This strategy is already used by Kahane in [**Kah68**] for studying random series on Banach spaces. It was pushed forward by Giné and Zinn in [**GZ84**] for studying limit theorem for empirical processes. The concentration inequality, Theorem 5.3.4, is due to Talagrand [**Tal96b**]. Several improvements and simplifications are known, in particular in the case of independent identically distributed random variables. We refer to [**Rio02, Bou03, Kle02, KR05**] for more precise results. The Proposition 5.3.5 is taken from [**Mas00**].

Theorem 5.3.6 is due to Rudelson [**Rud99**]. The proof that we presented was suggested by Pisier to Rudelson. It used a refined version of non-commutative Khinchine inequality that can be found in [**LP86, LPP91, Pis98**]. However, it is based on an expression related to operator norms and we have seen that in other situations, we need an estimate of the supremum of some empirical processes that can not be expressed in terms of operator norms. The original proof of Rudelson can be found in [**Rud96**] and used the majorizing measure theory. Some improvements of this result are proved in [**GR07**] and in [**GMPTJ08**]. The proof of Theorem 5.3.12 can be found in [**GMPTJ08**] and it is based on the same type of construction of majorizing measures than in [**GR07**] and on deep results about the duality of covering numbers [**BPSTJ89**]. The notions of type and cotype of a Banach space are important in this study and we refer the interested reader to [**Mau03**]. The notions of modulus of convexity and smoothness of a Banach space are classical and we refer the interested reader to [**LT79, Pis75**].

Theorem 5.3.14 comes from [**GMPTJ07**]. It was used to prove some results about the problem of selection of characters like Theorem 5.5.2. As we have seen, the proof is very similar to the proof of Theorem 5.4.1 and this result is due to Rudelson and Vershynin [**RV08b**]. They improved a result due to Candès and Tao [**CT05**] and the strategy of their proofs was to study the RIP condition instead of the size of the radius of sections of B_1^N. Moreover, the probabilistic estimate is slightly better than in [**RV08b**] and it was shown to us by Holger Rauhut [**Rau10**]. We refer to [**Rau10, FR10**] for a deeper presentation of the problem of compressed sensing and for several different points of view. We refer also to [**KT07**] where connections between the Compressed Sensing problem and the problem of estimating the Kolmogorov widhts are discussed and to [**CDD09, KT07**] for the study of approximate reconstruction.

For the classical study of local theory of Banach spaces, we refer to [**MS86**] and to [**Pis89**]. Euclidean sections or projections of a convex body are studied in detail in [**FLM77**] and the Kashin decomposition can be found in [**Kaš77**]. About the question of selection of characters, we refer the interested reader to the paper of Bourgain [**Bou89**] where he proved for $p > 2$ the existence of $\Lambda(p)$ sets which are not $\Lambda(r)$ for $r > p$. This problem was related to the theory of majorizing measure in [**Tal95**]. The existence of a subset of a bounded orthonormal system satisfying the inequality (5.3) is proved by Talagrand in [**Tal98**]. Theorems 5.5.4 and 5.5.6 are taken from [**GMPTJ08**]. We refer also to that paper for a proof of the fact that the factor $\sqrt{\log N}$ is necessary in the estimate.

NOTATIONS

- The sets of numbers are $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$
- For all $x \in \mathbb{R}^N$ and $p > 0$,
$$|x|_p = (|x_1|^p + \cdots + |x_N|^p)^{1/p} \quad \text{and} \quad |x|_\infty = \max_{1 \leqslant i \leqslant N} |x_i|$$
- $B_p^N = \{x \in \mathbb{R}^N \,:\, |x|_p \leq 1\}$
- Scalar product $\langle x, y \rangle$ and $x \perp y$ means $x \cdot y = 0$
- $A^* = \bar{A}^\top$ is the conjugate transpose of the matrix A
- $s_1(A) \geqslant \cdots \geqslant s_n(A)$ are the singular values of the $n \times N$ matrix A where $n \leqslant N$
- $\|A\|_{2\to 2}$ is the operator norm of A ($\ell^2 \to \ell^2$)
- $\|A\|_{\mathrm{HS}}$ is the Hilbert-Schmidt norm of A
- $e_1, \ldots, e_n$ is the canonical basis of $\mathbb{R}^n$
- $\stackrel{d}{=}$ stands for the equality in distribution
- $\stackrel{d}{\to}$ stands for the convergence in distribution
- $\stackrel{w}{\to}$ stands for weak convergence of measures
- $\mathcal{M}_{m,n}(K)$ are the $m \times n$ matrices with entries in K, and $\mathcal{M}_n(K) = \mathcal{M}_{n,n}(K)$
- I is the identity matrix
- $x \wedge y = \min(x, y)$ and $x \vee y = \max(x, y)$
- $|S|$ cardinal of the set S
- $\mathrm{dist}_2(x, E) = \inf_{y \in E} |x - y|_2$
- $\mathrm{supp} x$ is the subset of non-zero coordinate of x
- The vector x is said to be m-sparse if $|\mathrm{supp}\, x| \leq m$.
- $\Sigma_m = \Sigma_m(\mathbb{R}^N)$ m-sparse vectors
- $S_p(\Sigma_m) = \{x \in \mathbb{R}^N \,:\, |x|_p = 1, |\mathrm{supp}\, x| \leq m\}$
- $B_p(\Sigma_m) = \{x \in \mathbb{R}^N \,:\, |x|_p \leq 1, |\mathrm{supp}\, x| \leq m\}$
- $\mathrm{conv}(E)$ is the convex hull of E
- $\mathrm{diam}(F, \|\cdot\|) = \sup\{\|x\| : x \in F\}$
- For a random variable Z and any $\alpha \geqslant 1$, $\|Z\|_{\psi_\alpha} = \inf\left\{s > 0\,;\; \mathbb{E}\exp\left(|Z|/s\right)^\alpha \leqslant e\right\}$
- $\ell_*(T) = \mathbb{E}\sup_{t \in T} |\sum_{i=1}^N g_i t_i|_2$

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