A note on the character space

Let [1] $\mathcal{A}$ be an *abelian unital Banach algebra*, and let $\hat{\mathcal{A}}$ be the set of all nonzero morphisms $\phi : \mathcal{A} \to \mathbb{C}$. Note that $\phi(\mathbf{1})^2 = \phi(\mathbf{1}^2) = \phi(\mathbf{1})$ and so $\phi(\mathbf{1}) = 1$ (for if $\phi(\mathbf{1}) = 0$ then $\phi(a) = \phi(a\mathbf{1}) = 0$ for all a, contradiction).

Remark 1 *For each $\phi \in \hat{\mathcal{A}}$ and $a \in \mathcal{A}$ we have $\phi(a) \in \sigma(a)$. Thus $|\phi(a)| \leq \|a\|$ for all a hence $\|\phi\| \leq 1$. But $\phi(\mathbf{1}) = 1$ and so $\|\phi\| = 1$.*

For the proof, notice that the element $b = a - \phi(a)\mathbf{1}$ belongs to $\ker \phi$; but this is an ideal of $\mathcal{A}$ (since ϕ is a morphism) and it is proper (since $\phi \neq 0$) so it cannot contain invertible elements. Thus $a - \phi(a)\mathbf{1} \notin GL(\mathcal{A})$, so $\phi(a) \in \sigma(a)$. □

We have shown that

$$\{\phi(a) : \phi \in \hat{\mathcal{A}}\} \subseteq \sigma(a).$$

We wish to show that equality in fact holds. So fix a $\lambda \in \sigma(a)$ and let $\mathcal{I}_0 = \{x(a-\lambda\mathbf{1}) : x \in \mathcal{A}\}$. One easily sees that $\mathcal{I}_0$ is a proper ideal of $\mathcal{A}$. It is enough to find $\phi \in \hat{\mathcal{A}}$ such that the ideal $\ker \phi$ contains $\mathcal{J}_0$.

We will show that $\mathcal{J}_0$ is contained in a maximal proper ideal of $\mathcal{A}$.

Remark 2 *If $\mathcal{J}$ is a proper ideal of $\mathcal{A}$, then*

$$\|\mathbf{1} - x\| \geq 1 \quad \textit{for all } \ x \in \mathcal{J}.$$

In particular, the closure of a proper ideal is a proper ideal.

Indeed, if $\|\mathbf{1} - x\| < 1$ then $x \in GL(\mathcal{A})$ as we know so x cannot belong to a proper ideal. □

Remark 3 *$\mathcal{J}_0$ is contained in a maximal proper ideal of $\mathcal{A}$, which is therefore closed.*

Proof. Let F be the family of all ideals $\mathcal{J}$ of $\mathcal{A}$ containing $\mathcal{J}_0$ but not $\mathbf{1}$, ordered by inclusion. If $G \subseteq F$ is a totally ordered subset of F, let $\mathcal{J}_G$ be the union of all elements of G. Of course $\mathcal{J}_G$ contains $\mathcal{J}_0$ and does not contain $\mathbf{1}$; it is easy to verify that $\mathcal{J}_G$ is an ideal, hence it is an upper bound for G.

Zorn's lemma shows that there exists $\mathcal{M} \in F$ which is maximal in $(F, \subseteq)$. This is an ideal containing $\mathcal{J}_0$ and it is proper because $\mathbf{1} \notin \mathcal{M}$. In fact it is a maximal proper ideal; for if $\mathcal{N}$ is a proper ideal of $\mathcal{A}$ containing $\mathcal{M}$, then it contains $\mathcal{J}_0$ and, since it is proper, cannot contain $\mathbf{1}$; thus $\mathcal{N} \in F$, hence $\mathcal{N} = \mathcal{M}$ because $\mathcal{M}$ is a maximal member of F.

In particular $\mathcal{M}$ is closed, because its closure is an ideal and does not contain $\mathbf{1}$ by Remark 2. □

Note the essential use of $\mathbf{1}$ in the above argument: in fact the conclusion may fail in non-unital algebras: If for example $\mathcal{A} = c_0$, the Banach algebra of null sequences,

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then it can be shown that the ideal $\mathcal{J} = c_{00}$ (the set of sequences of finite support) is not contained in a maximal ideal.

Now let $\mathcal{B} = \mathcal{A}/\mathcal{M}$. It is well known that (since $\mathcal{M}$ is a closed subspace) $\mathcal{B}$ is a Banach space with respect to the quotient norm

$$\|a + \mathcal{M}\| = \inf\{\|a + x\| : x \in \mathcal{M}\} = \text{dist}(a, \mathcal{M}).$$

Remark 4 *$\mathcal{A}/\mathcal{M}$ is a Banach algebra.*

Proof. We have to prove that

$$\|ab + \mathcal{M}\| \leq \|a + \mathcal{M}\| \, \|b + \mathcal{M}\|, \quad a, b \in \mathcal{A}.$$

If $x, y \in \mathcal{M}$ then

$$\|a + x\| \, \|b + y\| \geq \|(a + x)(b + y)\| = \|ab + xb + ay + xy\|.$$

But $xb + ay + xy \in \mathcal{M}$, so $\|ab + xb + ay + xy\| \geq \|ab + \mathcal{M}\|$. Thus

$$\|a + x\| \, \|b + y\| \geq \|ab + \mathcal{M}\|$$

and the required inequality follows by taking the inf over x and y in $\mathcal{M}$. □

Remark 5 *$\mathcal{B} = \mathcal{A}/\mathcal{M}$ is a division algebra with identity $\mathbf{1} + \mathcal{M}$: that is, if $a + \mathcal{M}$ is not the zero element $0 + \mathcal{M}$ of $\mathcal{B}$, then $a + \mathcal{M}$ is invertible.*

Proof. We need to find $b \in \mathcal{A}$ so that $(a + \mathcal{M})(b + \mathcal{M}) = \mathbf{1} + \mathcal{M}$, equivalently $ab + \mathcal{M} = \mathbf{1} + \mathcal{M}$, i.e. $ab - \mathbf{1} \in \mathcal{M}$. Set

$$\mathcal{J} = a\mathcal{A} + \mathcal{M} = \{ab + x : b \in \mathcal{A}, x \in \mathcal{M}\}.$$

This is easily seen to be an ideal of $\mathcal{A}$ and it clearly contains $\mathcal{M}$. But it also contains a which is not in $\mathcal{M}$; hence, by maximality of $\mathcal{M}$, we must have $\mathcal{J} = \mathcal{A}$. Thus there exists $b \in \mathcal{A}$ and $x \in \mathcal{M}$ so that $ab + x = \mathbf{1}$, in other words $ab - \mathbf{1} = -x \in \mathcal{M}$. □

Remark 6 *If $\mathcal{B}$ is a division Banach algebra, there is an isomorphism $a \to \lambda(a) : \mathcal{B} \to \mathbb{C}$.*

Proof. The spectrum $\sigma(a)$ of each $a \in \mathcal{B}$ is nonempty. Thus there exists $\lambda(a) \in \mathbb{C}$ such that $a - \lambda(a)\mathbf{1}$ is not invertible. By the last remark, $a - \lambda(a)\mathbf{1} = 0$, i.e. $a = \lambda(a)\mathbf{1}$. Now if $\mu \in \sigma(a)$ then $a - \mu\mathbf{1}$ is not invertible, hence $a = \mu\mathbf{1}$ and so $\mu = \lambda(a)$.

Thus $\sigma(a) = \{\lambda(a)\}$ is a singleton. Therefore we have a well defined map

$$a \to \lambda(a) : \mathcal{B} \to \mathbb{C}.$$

This is an injective algebra morphism: for example $a = \lambda(a)\mathbf{1}$ and $b = \lambda(b)\mathbf{1}$ gives $ab = \lambda(a)\lambda(b)\mathbf{1}$, but then $\lambda(a)\lambda(b) \in \sigma(ab) = \{\lambda(ab)\}$ and so $\lambda(a)\lambda(b) = \lambda(ab)$. □

To show that $\{\phi(a) : \phi \in \hat{\mathcal{A}}\} = \sigma(a)$, define $\phi : \mathcal{A} \to \mathbb{C}$ as follows:

$$\begin{array}{cccccc} \phi : & \mathcal{A} & \to & \mathcal{B} & \to & \mathbb{C} \\ & x & \to & x + \mathcal{M} & \to & \lambda(x + \mathcal{M}). \end{array}$$

This is a composition of morphisms, hence a morphism. Its kernel is precisely $\mathcal{M}$, so $\phi \neq 0$ and, since $a - \lambda\mathbf{1} \in \mathcal{J}_0 \subseteq \mathcal{M}$, we have $\phi(a - \lambda\mathbf{1}) = 0$ i.e. $\phi(a) = \lambda$. □