Operator Theory
Exercises 3
Due: April 28, 2010

Exercise 1 On $H = L^2(\mathbb{R})$, let $D(Q) = \{f \in H : t \to tf(t) \text{ is in } L^2(\mathbb{R})\}$ and $(Qf)(t) = tf(t)$ $(f \in D(Q))$. Show that Q is selfadjoint.

If $D(Q_o) = \{f \in H : f \text{ vanishes a.e. outside a compact set}\}$ show that $D(Q_o)$ is a core for Q.

Exercise 2 Show that, for a densely defined operator A on H, the following are equivalent:
(a) A has a unique selfadjoint extension;
(b) A is essentially selfdjoint;
(c) $A^* = A^{**}$.

Exercise 3 Show that if $D(S_1)$ is the set of absolutely continuous functions in $L^2([0,1]$ and $D(S_2) = \{f \in D(S_1) : f(0) = 0\}$, then the formulae

$$S_k f = if', \ f \in D(S_k), \ (k = 1, 2)$$

define closed, densely defined operators.

Exercise 4 Let T be a densely defined operator on H.
(a) Show that $\ker(T^*)$ is a closed subspace.
(b) Show that $(\text{ran}(T))^\perp = \ker(T^*)$.
(c) Is $\ker(T)$ necessarily a closed subspace?

Exercise 5 Let T be a closed, densely defined operator on H.
(a) If T is invertible, show that T^* is invertible and $(T^*)^{-1} = (T^{-1})^*$.
(b) Show that $\sigma(T^*) = \{\bar{\lambda} : \lambda \in \sigma(T)\}$.

Exercise 6 Let $H = \ell^2$ and $D(A) = \{x = (x(n)) \in c_{00} : \sum_n x(n) = 0\}$ (note that, for example, $e_1 - e_2$ is in $D(A)$, but e_2 is not). If A is defined on $D(A)$ by

$$A\left(\sum_n x(n)e_n\right) = \sum_n nx(n)e_n,$$

show that A is densely defined and symmetric, and calculate its Cayley transform. Is A selfadjoint? Does it have selfadjoint extensions? Is it essentially selfadjoint?