Operator Theory – Spring 2010 – Summary

Week 3: Mar. 3-4

2 Bounded Operators (continued)

2.3 Invariant subspaces

Definition. Example: eigenspaces.

$$A(M) \subseteq M \iff AP = PAP \iff A \simeq \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$$
$$A^*(M) \subseteq M \iff PA = PAP \iff A \simeq \left(\begin{smallmatrix} * & 0 \\ * & * \end{smallmatrix}\right) \iff A(M^\perp) \subseteq M^\perp$$
$$\text{both: } A \text{ reduces } M \iff AP = PA \iff A \simeq \left(\begin{smallmatrix} * & 0 \\ 0 & * \end{smallmatrix}\right)$$

If $Alg(M) = \{A \in \mathcal{B}(\mathcal{H}) : A(M) \subseteq M\}$ the map $A \to PAP|_M$ preserves products on $Alg(M)$, not *

The map $A \to P^\perp AP^\perp|_{M^\perp}$ preserves products on $Alg(M^\perp)$, not *

If A reduces M then both these maps preserve products AND *, on $Alg(M, M^\perp) = \{A \in \mathcal{B}(\mathcal{H}) : A(M) \subseteq M \text{ and } A(M^\perp) \subseteq M^\perp\}$.

More generally:

Let M_1, M_2 be closed orthogonal subspaces, $M = M_1 \oplus M_2$ and $P = P(M_2)$. If $\mathcal{A} = Alg(M_1, M) = \{A \in \mathcal{B}(\mathcal{H}) : A(M_1) \subseteq M_1 \text{ and } A(M) \subseteq M\}$, the map $A \to PAP|_{M_2}$ preserves products on $\mathcal{A}$, not *. [*Exercise*]

Conversely, if $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is a subalgebra and $P = P(N)$ a projection such that $A \to PAP$ preserves products on $\mathcal{A}$, then the closed subspace N is **semi-invariant** for $\mathcal{A}$, i.e. there are $\mathcal{A}$-invariant subspaces $L \subseteq K$ such that $N = K \cap L^\perp$. [*Exercise*]

2.4 The Spectral Theorem in finite dimensions

Remark A diagonal operator D_a on ℓ^2 is normal. Ditto for a *diagonalisable* operator on a Hilbert space. Partial (!) converse:

Theorem 1 *If* $\dim \mathcal{H} < \infty$, *a normal operator* A *on* $\mathcal{H}$ *is diagonalisable, i.e. there is an* orthonormal basis *of* $\mathcal{H}$ *consisting of e-vectors of* A.

Equivalently, if $\sigma_p(A) = \{\lambda \in \mathbb{C} : \ker(\lambda - A) \neq \{0\}\}$ *is the (nonempty!) finite set of e-values of* A*, and if* P_λ *denotes the projection onto the e-space corresponding to* λ*, then*

$$A = \sum_{\lambda \in \sigma_p(A)} \lambda P_\lambda \quad \textit{and} \quad I = \sum_{\lambda \in \sigma_p(A)} P_\lambda.$$

Uses:

Lemma 2 *If* T *is a normal operator on any Hilbert space, if* $\lambda, \mu \in \sigma_p(T)$ *and* M_λ, M_μ *are the corresponding e-spaces, then*
(i) $Tx = \lambda x$ *implies* $T^* x = \bar{\lambda} x$;
(ii) M_λ *reduces* T;
(iii) if $\lambda \neq \mu$ *then* $M_\lambda \perp M_\mu$.

However if $\dim \mathcal{H} = \infty$ then $\sigma_p(T)$ may be empty.
Example Let $T \in \mathcal{B}(L^2([0,1])$ be given by $Tf(t) = tf(t)$ [*Exercise*].

Generalisation:

2.5 The Spectrum of a bounded operator

Definition: If $A \in \mathcal{B}(\mathcal{H})$,

$$\sigma(A) = \{\lambda \in \mathbb{C} : \lambda - A \text{ not invertible}\} \qquad \text{the spectrum of } A$$
$$\rho(A) = \mathbb{C} \setminus \sigma(A) \qquad \text{the resolvent of } A$$

Remark By the open mapping theorem, if T is bijective it is a homeo [requires completeness!]. Thus $\lambda \in \sigma(A)$ iff $\lambda - A$ is not bijective.

Lemma 3 *If $\|T\| < 1$ then $I - T$ is invertible. [Geometric series!]*

Proposition 4 *The spectrum of A is a nonempty and compact subset of $\mathbb{C}$ and the $\mathcal{B}(\mathcal{H})$-valued map $z \to (z - A)^{-1}$ is holomorphic on $\rho(A)$. In fact:*
(i) $\sigma(A) \subseteq \{z \in \mathbb{C} : |z| \leq \|A\|\}$, so $\sigma(A)$ is bounded
(ii) If $\lambda \in \rho(A)$ and $|z| < \|(\lambda - A)^{-1}\|^{-1}$, then $\lambda + z \in \rho(A)$ (so $\sigma(A)$ is closed) and
(iii) the function $z \to (z - A)^{-1}$ has a power series expansion in a disk centered at λ of radius $\|(\lambda - A)^{-1}\|^{-1}$.
(iv) $\sigma(A) \neq \emptyset$.

Remark The *resolvent identity*: If $\lambda, \mu \in \rho(A)$ are distinct,

$$\frac{(\lambda - A)^{-1} - (\mu - A)^{-1}}{\lambda - \mu} = -(\lambda - A)^{-1}(\mu - A)^{-1}$$

shows (again) that the resolvent $\mu \to (\mu - A)^{-1}$ is norm-differentiable on $\rho(A)$ and its derivative is $-(\mu - A)^{-2}$.

Lemma 5 *(i)* $\ker(A) = (A^*(\mathcal{H}))^{\perp}$ *and* $\overline{A(\mathcal{H})} = \ker(A^*)^{\perp}$.
(ii) If $A = A^* \in \mathcal{B}(\mathcal{H})$ then $\sigma(A) \subseteq [a, b]$ where $a = \inf\{\langle Ax, x\rangle : \|x\| = 1\}$ and $b = \sup\{\langle Ax, x\rangle : \|x\| = 1\}$ [Exercise]. *In particular, $\sigma(A) \subseteq \mathbb{R}$.*
(iii) If $U \in \mathcal{B}(\mathcal{H})$ is unitary, then $\sigma(U) \subseteq \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

3 The functional calculus: continuous functions of a bounded selfadjoint operator

Fix $A = A^* \in \mathcal{B}(\mathcal{H})$. We wish to define $f(A)$ for appropriate f.

3.1 Polynomials

If $p(t) = c_0 + c_1 t + \cdots + c_n t^n$ ($c_k \in \mathbb{C}$) is a poly of a real variable, then $p(A) = c_0 I + c_1 A + \cdots + c_n A^n$. The map $\Phi_0 : p \to p(A)$ from the algebra of polynomials into $\mathcal{B}(\mathcal{H})$ preserves the algebraic operations $+, \cdot, *$ where $p^*(t) = \bar{p}(t) = \bar{c}_0 + \bar{c}_1 t + \cdots + \bar{c}_n t^n$.

To extend to the "closure" of the algebra of polynomials, need some sort of "continuity" of Φ_0 (for which topologies?).

Lemma 6 (Spectral Mapping Lemma) $\sigma(p(A)) = p(\sigma(A)) = \{p(\lambda) : \lambda \in \sigma(A)\}$.

Proposition 7 $\|p(A)\|_{\mathcal{B}(\mathcal{H})} = \sup\{|p(\lambda)| : \lambda \in \sigma(A)\} \equiv \|p\|_{\sigma(A)}$.

Note $\|p\|_{\sigma(A)} \leq \sup\{|p(\lambda)| : \lambda \in [a, b]\}$ with a, b as in Lemma 5.