Beurling's Theorem

Let $U : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ be the bilateral shift, i.e.

$$Uf = \zeta f \qquad (f \in L^2(\mathbb{T})),$$

where $\zeta(z) = z$ ($z \in \mathbb{T}$). Recall the definition

$$H^2 = H^2(\mathbb{T}) = \{f \in L^2(\mathbb{T}) : \hat{f}(-k) = 0 \text{ for all } k = 1, 2, \ldots\}.$$

Note that $\{\zeta^n : n \in \mathbb{N}\}$ is an orthonormal basis of H^2. Since $U(\zeta^n) = \zeta^{n+1}$ we have $U(H^2) \subseteq H^2$ and in fact

$$\bigcap_{n \geq 0} U^n(H^2) = \{0\}.$$

Indeed, $\zeta^k \perp U^n(H^2)$ for all $k < n$. Hence if $f \in \bigcap_{n \geq 0} U^n(H^2)$ then $f \perp \zeta^k$ for all $k \in \mathbb{N}$ and so $f = 0$.

Now let $\phi \in L^2$ with $|\phi(z)| = 1$ for almost all $z \in \mathbb{T}$. Note[1] that since $|\phi| = 1$ a.e., ϕ defines a bounded, in fact a unitary operator M_ϕ on L^2; therefore ϕH^2 is a closed subspace of L^2 because M_ϕ is isometric.

Also, ϕH^2 is U-invariant because $\zeta H^2 \subseteq H^2$ and so

$$U(\phi H^2) = \zeta \phi H^2 = \phi(\zeta H^2) \subseteq \phi H^2.$$

In fact, since M_ϕ is isometric,

$$\bigcap_{n \geq 0} U^n(\phi H^2) = \bigcap_{n \geq 0} U^n M_\phi(H^2) = \bigcap_{n \geq 0} M_\phi(U^n(H^2)) = M_\phi\left(\bigcap_{n \geq 0} U^n(H^2)\right) = \{0\}.$$

Theorem 1 *A closed nonzero subspace $E \subseteq L^2 = L^2(\mathbb{T})$ is U-invariant and $U(E) \neq E$ if and only if there exists $\phi \in L^2$ with $|\phi(z)| = 1$ for almost all $z \in \mathbb{T}$ such that $E = \phi H^2$.*

Proof. Suppose that $E \subseteq L^2$ is a closed nonzero U-invariant subspace with $U(E) \neq E$. The space $U(E)$ is a closed subspace of E because U is isometric.

Thus there exists $\phi \in E$ of norm 1, such that $\phi \perp U(E)$.

Claim 1. The sequence $\{\phi, U(\phi), U^2(\phi), \ldots\}$ is an orthonormal sequence in E.
Proof. Since $\phi \in E$ which is U-invariant we have $U^n(\phi) \in E$ for all $n \in \mathbb{N}$. Moreover $\|U^n(\phi)\|_2 = \|\phi\|_2 = 1$. Also, if $m, n \in \mathbb{N}$ with $m > n$ we have

$$U^m(\phi) \in U^m(E) \subseteq U^{n+1}(E) = U^n(U(E)).$$

But $\phi \perp U(E)$ by construction and so $U^n(\phi) \perp U^n(U(\phi))$ since U^n is isometric. Thus

$$U^n(\phi) \perp U^m(\phi). \qquad \square$$

Claim 2. For all nonzero $k \in \mathbb{Z}$ we have $\displaystyle\int \zeta^k |\phi|^2 = 0.$

[1] beur.tex, 26 April 07

Proof. For $k > 0$ write

$$\int \zeta^k |\phi|^2 = \int (\zeta^k \phi)\bar{\phi} = \left\langle \zeta^k \phi, \phi \right\rangle = \left\langle U^k(\phi), \phi \right\rangle = 0$$

by the previous claim, and for $k = -n < 0$,

$$\int \zeta^k |\phi|^2 = \int \phi(\overline{\zeta^n \phi}) = \langle \phi, \zeta^n \phi \rangle = \langle \phi, U^n(\phi) \rangle = 0. \qquad \square$$

It follows from this claim that the function $|\phi|^2$ is orthogonal to all ζ^k except for $k = 0$; thus it is a complex multiple of $\zeta^0 = \mathbf{1}$ and hence a.e. equal to a constant. Hence so is $|\phi|$.

This shows that $|\phi(z)| = 1$ *a.e.*

Claim 3. $\quad E = \phi H^2$.

Proof. First, $\phi H^2 = M_\phi(H^2)$ and M_ϕ is an isometry since $|\phi| = 1$ a.e. Now from Claim 1 we have that the set

$$\{\phi, U(\phi), U^2(\phi), \dots\} = \{\phi, \zeta\phi, \zeta^2\phi, \dots\} = M_\phi(\{\zeta^n : n = 0, 1, 2, \dots\})$$

is contained in E. Since $\{\zeta^n : n = 0, 1, 2, \dots\}$ is an orthonormal basis of H^2 and M_ϕ is an isometry, we conclude that $M_\phi(H^2) \subseteq E$.

Finally if $f \in E$ is orthogonal to ϕH^2 we show that $f = 0$. Indeed, for all $n = 0, 1, 2, \dots$ we have

$$\left\langle M_\phi^*(f), \zeta^n \right\rangle = \langle f, M_\phi(\zeta^n) \rangle = 0$$

since $M_\phi(\zeta^n) \in \phi H^2$. On the other hand if $k = 1, 2, \dots$ since $\zeta^k f = U^k(f) \in U^k(E) \subseteq U(E)$ while $\phi \perp U(E)$ by construction, we have

$$\left\langle M_\phi^*(f), \zeta^{-k} \right\rangle = \left\langle f, M_\phi(\zeta^{-k}) \right\rangle = \left\langle f, \phi\zeta^{-k} \right\rangle = \left\langle \zeta^k f, \phi \right\rangle = 0.$$

This shows that the L^2 function $\bar{\phi} f = M_\phi^*(f)$ is orthogonal to all ζ^k $(k \in \mathbb{Z})$ and hence must vanish. But M_ϕ is 1-1 and hence $f = 0$.

This concludes the proof of the Theorem. $\square$

If $S : H^2(\mathbb{T}) \to H^2(\mathbb{T})$ is the unilateral shift, i.e. the restriction of U to $H^2(\mathbb{T})$, note that

$$\bigcap_{n \geq 0} S^n(H^2) = \{0\}.$$

Therefore every closed S-invariant subspace $E \subseteq H^2$ satisfies

$$\bigcap_{n \geq 0} S^n(E) \subseteq \bigcap_{n \geq 0} S^n(H^2) = \{0\}.$$

Also, if $\phi \in H^2$ and $|\phi(z)| = 1$ for almost all $z \in \mathbb{T}$ then for all $n \in \mathbb{N}$ we have

$$\phi\zeta^n = \zeta^n \phi = S^n(\phi) \in S^n(H^2),$$
$$\text{so} \quad M_\phi(\{\zeta^n : n \in \mathbb{N}\}) \subseteq H^2.$$

Since $\{\zeta^n : n \in \mathbb{N}\}$ is an orthonormal basis of H^2 and M_ϕ is bounded, it follows that the subspace ϕH^2 is contained in H^2.

Therefore the previous Theorem gives

Theorem 2 (Beurling) *A closed nonzero subspace $E \subseteq H^2 = H^2(\mathbb{T})$ is S-invariant if and only if there exists $\phi \in H^2$ with $|\phi(z)| = 1$ for almost all $z \in \mathbb{T}$ such that $E = \phi H^2$*

A function $\phi \in H^2$ with $|\phi(z)| = 1$ for almost all $z \in \mathbb{T}$ is caled an *inner function* Examples are: ζ^n $(n \in \mathbb{N})$ and $f(z) = \exp \frac{z-1}{z+1}$.

Theorem 3 *If f is any* function *such that $f \in H^2$, the set $\sigma = \{z \in \mathbb{T} : f(z) = 0\}$ has Lebesgue measure zero.*

Remark In fact this is also true if $f \in H^1$.
Proof. Consider the space

$$E = \{g \in H^2 : g|_\sigma = 0 \text{ a.e.}\}.$$

This is clearly a closed, S-invariant subspace and hence there exists an inner function ϕ such that $E = \phi H^2$ Thus $\phi \in E$ and hence $\phi|_\sigma = 0$. Since $|\phi| = 1$ a.e. this shows that σ must have measure zero.

Theorem 4 *A closed subspace $E \subseteq L^2 = L^2(\mathbb{T})$ is U-invariant if and only if* either
(a) there exists a Borel set $\sigma \subseteq \mathbb{T}$ such that

$$E = \{f \in L^2 : f|_\sigma = 0 \ a.e.\}$$

in which case $U(E) = E$ (such an E is called doubly invariant*),*
or
(b) there exists $\phi \in L^2$ with $|\phi(z)| = 1$ for almost all $z \in \mathbb{T}$ such that $E = \phi H^2$, in which case $\bigcap_{n \geq 0} U^n(E) = \{0\}$, in which case (such an E is called simply invariant*).*

Proof. Let E be U-invariant. Since U is unitary, we know that we can decompose

$$E = E_1 \oplus E_2$$

where $U(E_1) = E_1$ and $\bigcap_{n \geq 0} U^n(E_2) = \{0\}$. We have also shown that there is a Borel set $\sigma \subseteq \mathbb{T}$ such that

$$E_1 = E_\sigma = \{f \in L^2 : f|_{\sigma^c} = 0 \text{ a.e.}\}.$$

Thus

$$E_2 \subseteq E_\sigma^\perp = \{f \in L^2 : f|_\sigma = 0 \text{ a.e.}\}.$$

But we know that if $E_2 \neq \{0\}$ then $E_2 = \phi H^2$ where $|\phi| = 1$ a.e. Since $\phi \in E_\sigma^\perp$, it follows that σ must have measure zero. But then $E_1 = \{0\}$. $\square$