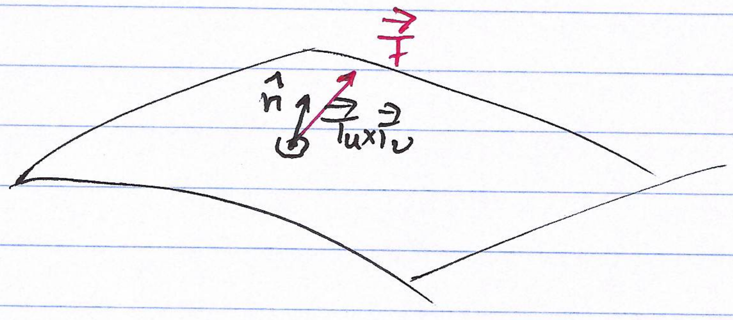
# Επιφανειακά ολοκληρώματα.

- Για βαθμωτή συνάρτηση $f(\vec{r})$:

$$\int_S f\,dS = \int_D f(\vec{\Phi}(u,v))\,\|\vec{T}_u\times\vec{T}_v\|\,du\,dv$$

Αναλλοίωτο σε αναπαραμετροποιήσεις της επιφάνειας.

- Για διανυσματικό πεδίο $\vec{F}(\vec{r})$:




Ροή διανυσματικού πεδίου από επιφάνεια:

$$\int_S \vec{F}\cdot\hat{n}\,dS = \qquad (\hat{n}\,dS = d\vec{S})$$

$$= \int_D \vec{F}(\vec{\Phi}(u,v))\cdot\frac{\vec{T}_u\times\vec{T}_v}{\|\vec{T}_u\times\vec{T}_v\|}\,dS =$$

$\|\vec{T}_u\times\vec{T}_v\|\,du\,dv$

$$\int_D \vec{F}(\vec{\Phi}(u,v))\cdot(\vec{T}_u\times\vec{T}_v)\,du\,dv$$

Το πρόσημο της ροής εξαρτάται από την φορά του $\hat{n}$, αλλά είναι ανεξάρτητο της παραμετροποίησης.

$$\vec{T}_u \times \vec{T}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial y}{\partial v}\right) + \hat{j}\left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial x}{\partial u}\frac{\partial z}{\partial v}\right)$$

$$+ \hat{k}\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right)$$

$$= \hat{i}\,\frac{\partial(y,z)}{\partial(u,v)} + \hat{j}\,\frac{\partial(z,x)}{\partial(u,v)} + \hat{k}\,\frac{\partial(x,y)}{\partial(u,v)}.$$

## Αναπαραμετροποίηση:

$$\vec{\Psi}(\xi,\eta) = \vec{\Phi}\big(u(\xi,\eta), v(\xi,\eta)\big)$$

$$\vec{T}_\xi = \frac{\partial \vec{\Psi}}{\partial \xi} = \frac{\partial \vec{\Phi}}{\partial u}\frac{\partial u}{\partial \xi} + \frac{\partial \vec{\Phi}}{\partial v}\frac{\partial v}{\partial \xi}$$

$$\vec{T}_\eta = \frac{\partial \vec{\Psi}}{\partial \eta} = \frac{\partial \vec{\Phi}}{\partial u}\frac{\partial u}{\partial \eta} + \frac{\partial \vec{\Phi}}{\partial v}\frac{\partial v}{\partial \eta}$$

$$\vec{T}_\xi \times \vec{T}_\eta = \frac{\partial \vec{\Phi}}{\partial u} \times \frac{\partial \vec{\Phi}}{\partial v}\frac{\partial u}{\partial \xi}\frac{\partial v}{\partial \eta} + \frac{\partial \vec{\Phi}}{\partial v} \times \frac{\partial \vec{\Phi}}{\partial u}\frac{\partial u}{\partial \eta}\frac{\partial v}{\partial \xi} \Rightarrow$$

$$\vec{T}_\xi \times \vec{T}_\eta = \left(\vec{T}_u \times \vec{T}_v\right)\underbrace{\left(\frac{\partial u}{\partial \xi}\frac{\partial v}{\partial \eta} - \frac{\partial u}{\partial \eta}\frac{\partial v}{\partial \xi}\right)}_{\frac{\partial(u,v)}{\partial(\xi,\eta)}}$$

$$\left(\vec{T}_\xi \times \vec{T}_\eta\right) d\xi\, d\eta = \left(\vec{T}_u \times \vec{T}_v\right)\underbrace{\frac{\partial(u,v)}{\partial(\xi,\eta)}\, d\xi\, d\eta}_{du\,dv}$$

## Στοιχείο όγκου.

Εάν $x(u,v,w)$
$y(u,v,w)$
$z(u,v,w)$

$$dV = (dx\,dy\,dz) = \left|\vec{T}_w \cdot (\vec{T}_u \times \vec{T}_v)\right| du\,dv\,dw$$

$$\vec{T}_w \cdot (\vec{T}_u \times \vec{T}_v) = \vec{T}_v \cdot (\vec{T}_w \times \vec{T}_u) = \vec{T}_u \cdot (\vec{T}_v \times \vec{T}_w) =$$

$$\begin{vmatrix} \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} =$$

$$= \frac{\partial x}{\partial w}\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial y}{\partial v}\right) + \frac{\partial y}{\partial w}\left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial x}{\partial u}\frac{\partial z}{\partial v}\right)$$

$$+ \frac{\partial z}{\partial w}\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right).$$

Π.χ.

i. Σφαιρικές συντεταγμένες:

$$x = r\cos\varphi\sin\theta$$
$$y = r\sin\varphi\sin\theta$$
$$z = r\cos\theta$$

$$\vec{T}_r = (\cos\varphi\sin\theta,\ \sin\varphi\sin\theta,\ \cos\theta)$$

$$\vec{T}_\theta = (r\cos\varphi\cos\theta,\ r\sin\varphi\cos\theta,\ -r\sin\theta)$$

$$\vec{T}_\varphi = (-r\sin\varphi\sin\theta,\ r\cos\varphi\sin\theta,\ 0)$$

$$\vec{T}_r \cdot (\vec{T}_\theta \times \vec{T}_r)\, dr\,d\theta\,d\varphi = r^2\sin\theta\, dr\,d\theta\,d\varphi.$$

ii. Περιοχή τόρου:

$$x = (a + r\sin\theta)\cos\varphi,$$
$$y = (a + r\sin\theta)\sin\varphi, \qquad 0 \le r \le b,$$
$$z = r\cos\theta, \qquad 0 \le \theta, \varphi \le 2\pi.$$

$$\begin{vmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ r\cos\theta\cos\varphi & r\cos\theta\sin\varphi & -r\sin\theta \\ -(a+r\sin\theta)\sin\varphi & (a+r\sin\theta)\cos\varphi & 0 \end{vmatrix} =$$

$$-\sin\varphi(a+r\sin\theta)\left[(-r\sin^2\theta\sin\varphi) - r\cos^2\theta\sin\varphi\right] - (a+r\sin\theta)\cos\varphi\left[-r\sin^2\theta\cos\varphi - r\cos^2\theta\cos\varphi\right]$$

$$= (a+r\sin\theta)\, r\sin^2\varphi + (a+r\sin\theta)\, r\cos^2\varphi =$$

$$= (a+r\sin\theta)\, r$$

Όγκος τόρου:

$$\int_0^b \int_0^{2\pi} \int_0^{2\pi} (a+r\sin\theta)\, r \, dr\, d\theta\, d\varphi =$$

$$(2\pi)^2 a \frac{1}{2} b^2 = 2\pi^2 a b^2 = \underbrace{\pi b^2}_{\substack{\text{Εμβαδόν} \\ \text{βάσης} \\ \text{κυλίνδρου ακτίνας} \\ b}} \cdot \underbrace{(2\pi a)}_{\text{ύψος κυλίνδρου}}$$