Ασκ. 20

$f : [0,+\infty) \to \mathbb{R}$ συνεχής. Νδο

$$\lim_{x\to 0^+} \frac{1}{x}\int_0^x f(t)\,dt = f(0) \ .$$

Απόδ. Έστω $\varepsilon > 0$. f συνεχής στο $0 \Rightarrow \exists\, \delta > 0$:
$0 \leq t < \delta \Rightarrow |f(t) - f(0)| < \varepsilon$. Έστω $x \in (0,\delta)$.
$\Rightarrow \forall t \in [0,x]$: $0 \leq t \leq x < \delta$, άρα $|f(t)-f(0)| < \varepsilon$.

Άρα

$$\left|\frac{1}{x}\int_0^x f(t)\,dt - f(0)\right| = \left|\frac{1}{x}\int_0^x f(t)\,dt - \frac{1}{x}\int_0^x f(0)\,dt\right|$$

$$= \frac{1}{x}\left|\int_0^x (f(t)-f(0))\,dt\right| \leq$$

$$\leq \frac{1}{x}\int_0^x |f(t)-f(0)|\,dt \leq$$

$$\leq \frac{1}{x}\int_0^x \varepsilon\,dt = \frac{\varepsilon x}{x} = \varepsilon \ .$$

$\lim_{x\to 0^+} g(x) = \ell \iff$
$\forall \varepsilon > 0 \ \exists \delta > 0$:
$0 \leq x < \delta \Rightarrow$
$|g(x) - \ell| < \varepsilon$.

Ασκ. 21

$f : [0,1] \to \mathbb{R}$ ολοκλ. Νδο η ακολουθία

$$a_n = \frac{1}{n}\sum_{k=1}^{n} f(k/n)$$

συγκλίνει στο $\int_0^1 f(x)\,dx$.

Απόδ. $P_n = \{0, 1/n, 2/n, \dots, n/n = 1\}$.

$\Xi_n = \{1/n, 2/n, \dots, 1\}$.

$\|P_n\| = 1/n \to 0$.

ορσ Riemann $\Rightarrow$ $\forall \varepsilon > 0 \ \exists \delta > 0 : \|P\| < \delta$, Ξ τυχαίο $\Rightarrow$
$\left| \Sigma(f, P, \Xi) - \int_0^1 f(x)\,dx \right| < \varepsilon$. (*)

$\|P_n\| = 1/n \to 0 \quad \exists n_0 \in \mathbb{N} : \|P_n\| < \delta \quad \forall n \geq n_0$

$$\Sigma(f, P, \Xi_n) = \sum_{k=1}^{n} f\left(\frac{k}{n}\right) \cdot 1/n = 1/n \sum_{k=1}^{n} f(k/n) \quad (**)$$

$$(*), (**) \Rightarrow \Big| \underbrace{1/n \sum_{k=1}^{n} f(k/n)}_{a_n} - \underbrace{\int_0^1 f(x)\,dx}_{a = \lim a_n} \Big| < \varepsilon .$$

Ασκ. 22 Ν.δ.ο

$$\lim_{n\to\infty} \frac{\sqrt{1} + \sqrt{2} + \dots + \sqrt{n}}{n\sqrt{n}} = 2/3 .$$

Απόδ. Από Ασκ. 21

$$\frac{\sqrt{1} + \sqrt{2} + \dots + \sqrt{n}}{n\sqrt{n}} = \frac{1}{n} \sum_{k=1}^{n} \sqrt{\frac{k}{n}} \to \int_0^1 \sqrt{x}\,dx = 2/3 .$$

Άσκ 23

$f : [0,1] \to \mathbb{R}$ συνεχής. Θέτουμε $[f(x^n) = f \circ \delta_n (x)]$

$a_n := \int_0^1 f(x^n)\,dx$. Νδο $a_n \to f(0)$.

Απόδ [Αν $\delta_n(x) = x^n \Rightarrow f(x^n) = f \circ \delta_n(x) =$ ολοκληρώσιμη] ...

Έστω $\varepsilon > 0$. Θδο τελικά

$$\left| \int_0^1 f(x^n)\,dx - f(0) \right| < \varepsilon .$$

Παρατηρώ ότι

$$\left| \int_0^1 f(x^n)\,dx - f(0) \right| = \left| \int_0^1 (f(x^n) - f(0))\,dx \right| \leq$$

$$\leq \int_0^1 |f(x^n) - f(0)|\,dx$$

f φραγμ. $\Rightarrow \exists\, M > 0 : \quad |f(y)| < M \quad \forall y \in [0,1]$.

f συνεχής στο $0 \Rightarrow \exists\, 0 < \delta < 1 : \forall\, 0 \leq y < \delta :$

$$|f(y) - f(0)| < \varepsilon/2 .$$

$$\uparrow$$

$$0 \leq x^n < \delta \iff 0 \leq x < \sqrt[n]{\delta} \to 1$$

$\sqrt[n]{\delta} \to 1 \Rightarrow \exists\, n_0 \in \mathbb{N} : \forall\, n \geq n_0 \quad 0 < 1 - \sqrt[n]{\delta} < \varepsilon/4M$

Τότε: $\forall\, n \geq n_0$:

$$\int_0^1 |f(x^n) - f(0)|\,dx = \underbrace{\int_0^{\sqrt[n]{\delta}} |f(x^n) - f(0)|\,dx}_{< \varepsilon/2} + \int_{\sqrt[n]{\delta}}^1 |f(x^n) - f(0)|\,dx \leq$$

$$\leq \sqrt[n]{\delta} \cdot \varepsilon/2 + 2M \cdot (1 - \sqrt[n]{\delta}) < \varepsilon/2 + 2M \cdot \varepsilon/4M =$$

$$= \varepsilon/2 + \varepsilon/2 = \varepsilon .$$

Ασκ. 24

Δείξτε ότι η ακολουθία $\beta_n = 1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}-\int_1^n \frac{1}{x}\,dx$ συγκλίνει.

Απόδ.

Παρατηρούμε ότι $\forall x \in [n, n+1]$, $n\in\mathbb{N}$:

$$\frac{1}{n+1} \leq \frac{1}{x} \leq \frac{1}{n} \Rightarrow$$

$$\int_n^{n+1} \frac{1}{n+1}\,dx \leq \int_n^{n+1} \frac{1}{x}\,dx \leq \int_n^{n+1} \frac{1}{n}\,dx \Rightarrow$$

$$\frac{1}{n+1} \leq \int_n^{n+1} \frac{dx}{x} \leq \frac{1}{n}$$

Οπότε:

$$|\beta_{n+1}-\beta_n| = \left|1+\frac{1}{2}+\dots+\frac{1}{n}+\frac{1}{n+1}-\int_1^{n+1}\frac{dx}{x}-1-\frac{1}{2}-\dots-\frac{1}{n}+\int_1^{n}\frac{dx}{x}\right|$$

$$= \left|\frac{1}{n+1}-\int_n^{n+1}\frac{dx}{x}\right| = \int_n^{n+1}\frac{dx}{x}-\frac{1}{n+1} \leq \frac{1}{n}-\frac{1}{n+1}.$$

Εστω $n>m \Rightarrow$

$$\Rightarrow |\beta_n-\beta_m| = |\beta_n-\beta_{n-1}+\beta_{n-1}-\beta_{n-2}+\dots+\beta_{m+1}-\beta_m| \leq$$

$$\leq |\beta_n-\beta_{n-1}|+\dots+|\beta_{m+1}-\beta_m| \leq$$

$$\leq -\frac{1}{n}+\frac{1}{n-1}-\frac{1}{n-1}+\frac{1}{n-2}+\dots-\frac{1}{m+1}+\frac{1}{m} =$$

$$= -\frac{1}{n}+\frac{1}{m} = |a_n-a_m|,$$

όπου $a_n=\frac{1}{n}\to 0$, άρα είναι βασική $\Rightarrow$

$\Rightarrow (\beta_n)$ βασική, άρα και συγκλίνουσα.

Άσκ. 25

$f : [0,1] \to \mathbb{R}$ Lipschitz-συνεχής με σταθερά $M>0$. Νδο

$$\left| \int_0^1 f(x)dx - \frac{1}{n}\sum_{k=1}^{n} f(k/n) \right| \leq \frac{M}{2n}.$$

Απόδ.

$$\left| \int_0^1 f(x)dx - \frac{1}{n}\sum_{k=1}^{n} f(k/n) \right| = \left| \sum_{k=1}^{n} \int_{(k-1)/n}^{k/n} f(x)dx - \sum_{k=1}^{n} \underbrace{\frac{1}{n} f\left(\frac{k}{n}\right)}_{= \int_{(k-1)/n}^{k/n} f(k/n)dx} \right| =$$

$$= \left| \sum_{k=1}^{n} \int_{(k-1)/n}^{k/n} (f(x) - f(k/n))dx \right| \leq$$

$$\leq \sum_{k=1}^{n} \int_{(k-1)/n}^{k/n} |f(x) - f(k/n)|dx \overset{L\text{-συν.}}{\leq}$$

$$\leq \sum_{k=1}^{n} \int_{(k-1)/n}^{k/n} M \cdot |x - k/n|\, dx = \sum_{k=1}^{n} M \cdot \int_{(k-1)/n}^{k/n} (k/n - x)dx =$$

$$= M \cdot \sum_{k=1}^{n} \left[\int_{(k-1)/n}^{k/n} \frac{k}{n}dx - \int_{(k-1)/n}^{k/n} x dx \right] =$$

$$= M \cdot \sum_{k=1}^{n} \left[\frac{k}{n}\cdot\frac{1}{n} - \frac{x^2}{2}\Big|_{(k-1)/n}^{k/n} \right] = M \cdot \sum_{k=1}^{n} \left[\frac{k}{n^2} - \frac{k^2}{2n^2} + \frac{(k-1)^2}{2n^2} \right]$$

$$= M \cdot \sum_{k=1}^{n} \left[\frac{2k - k^2 + k^2 - 2k + 1}{2n^2} \right] = M \cdot \sum_{k=1}^{n} \frac{1}{2n^2} =$$

$$= M \cdot n \cdot \frac{1}{2n^2} = \frac{M}{2n}.$$

Ασκ 28

$f:[a,b]\to\mathbb{R}$ συνεχής: $\exists M>0$:

$$|f(x)| \le M\int_a^x |f(t)|\,dt \qquad \forall x\in[a,b].$$

Ν.δ.ο $f(x)=0$, $\forall x\in[a,b]$

Αποδ.

f συνεχής $\Rightarrow$ f φραγμ. $\Rightarrow$ $\exists A>0$: $|f(x)| \overset{(1)}{\le} A$ $\forall t\in[a,b]$ $\Rightarrow$

$$\Rightarrow |f(x)| \le M\int_a^x |f(t)|\,dt \le M\int_a^x A\,dt = MA(x-a) \le MA(b-a)$$

Ξαναχρησιμοποιώντας την $|f(x)| \overset{(2)}{\le} MA(x-a)$:

$$|f(x)| \le M\int_a^x |f(t)|\,dt \le M\int_a^x MA(t-a)\,dt =$$

$$= M^2A\int_a^x (t-a)\,dt = \frac{M^2A}{2}(x-a)^2 \le \frac{M^2A}{2}(b-a)^2.$$

Επαγωγικά: $\forall x\in[a,b]$, $\forall n\in\mathbb{N}$:

$$|f(x)| \le \frac{M^nA}{n!}(x-a)^n \le A\,\frac{(M(b-a))^n}{n!}$$

Όμως $A\cdot\dfrac{(M\cdot(b-a))^n}{n!} \xrightarrow[n\to\infty]{} 0$ $\Rightarrow$ $f(x)=0$. ∎

Ασκ. 30

$f:[a,b]\to\mathbb{R}$ συνεχής, $f\ge 0$. $M:=\max\{f(x): x\in[a,b]\}$.

Ν.δ.ο η ακολουθία

$$\gamma_n = \left(\int_a^b [f(x)]^n\,dx\right)^{1/n} \longrightarrow M.$$

Απόδ. Για $M=0$ προφανές. Για $M>0$:

$$0 \le \gamma_n = \left(\int_a^b f(x)^n dx\right)^{1/n} \le \left(\int_a^b M^n dx\right)^{1/n} = \underbrace{M(b-a)^{1/n}}_{\text{φραγμ.}} \longrightarrow M$$

$\Rightarrow (\gamma_n)$ φραγμ., $\limsup \gamma_n \le M$.

Έστω $\varepsilon>0$, $\varepsilon<M$.

f συνεχής στο $[a,b] \Rightarrow \exists\, x_0 \in [a,b]: f(x_0)=M$, και

$\exists\, \delta>0$ με $a \le x_0-\delta < x_0 < x_0+\delta \le b$: $|x-x_0|<\delta \Rightarrow |f(x)-f(x_0)|<\varepsilon$.

$$\gamma_n = \left(\int_a^b f(x)^n dx\right)^{1/n} = \left(\underbrace{\int_a^{x_0-\delta} f(x)^n dx}_{\ge 0} + \int_{x_0-\delta}^{x_0+\delta} f(x)^n dx + \underbrace{\int_{x_0+\delta}^{b} f(x)^n dx}_{\ge 0}\right)^{1/n}$$

$$\ge \left(\int_{x_0-\delta}^{x_0+\delta} f(x)^n dx\right)^{1/n} \ge$$

$$\ge \left(\int_{x_0-\delta}^{x_0+\delta} (M-\varepsilon)^n dx\right)^{1/n} \ge (M-\varepsilon)(2\delta)^{1/n} \longrightarrow M-\varepsilon \Rightarrow$$

$\Rightarrow \liminf \gamma_n \ge M-\varepsilon$, $\forall \varepsilon>0 \Rightarrow$

$\Rightarrow \liminf \gamma_n \ge M$

$$M \le \liminf \gamma_n \le \limsup \gamma_n \le M$$