A VOLUME INEQUALITY FOR POLAR BODIES

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Abstract

A sharp affine isoperimetric inequality is established which gives a sharp lower bound for the volume of the polar body. It is shown that equality occurs in the inequality if and only if the body is a simplex.

Throughout this paper a convex body K in Euclidean n-space $\mathbb{R}^n$ is a compact convex set that contains the origin in its interior. Its polar body K^* is defined by

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1 \quad \text{for all} \quad y \in K\},$$

where $x \cdot y$ denotes the standard inner product of x and y in $\mathbb{R}^n$. A central quest in convex geometric analysis is finding sharp lower bounds for $|K^*|$, the volume of K^*, in terms of natural geometric invariants of K. In this paper a new sharp lower bound is established, where equality is achieved only for simplices.

Along these lines, the most famous open problem is the Mahler conjecture (see, e.g., [14, 46]). Restated independently by Aleksandrov [1], this is the statement that $1/|K|$, the reciprocal of the volume of K, provides a sharp lower bound for $|K^*|$. Specifically, if K is a convex body, then

$$|K^*| \ \geq \ \frac{(n+1)^{(n+1)}}{(n!)^2}\,\frac{1}{|K|}, \tag{1}$$

with equality conjectured to hold only for simplices. For $n = 2$, Mahler himself proved this inequality in 1939 (see, e.g., [12] for references) and Meyer [39] obtained the equality conditions in 1991. Recently, Meyer and Reisner [40] have proved inequality (1) for polytopes with at most $n + 3$ vertices.

For K positioned so that its centroid is at the origin, the sharp upper bound for $|K^*|$ in terms of $|K|$ is attained only for ellipsoids and this fact is known as the Blaschke-Santaló inequality; see, e.g., [14, 46, 50].

The Mahler conjecture for the class of origin-symmetric bodies is that:

$$|K^*| \ \geq \ \frac{4^n}{n!}\,\frac{1}{|K|}, \tag{2}$$

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with equality holding for parallelepipeds and their polars (and other bodies). For $n = 2$ the inequality is also due to Mahler, and in 1986 Reisner [43] showed that equality holds only for parallelograms. For $n = 2$, a new proof of inequality (2) was obtained by Campi and Gronchi [12]. Reisner [42] established inequality (2) for a class of bodies that have a high degree of symmetry, known as zonoids. See [19] for an excellent survey of these bodies. (For $n = 2$ the class of zonoids coincides with the class of origin-symmetric bodies.) A new proof of Reisner's inequality was given by Gordon, Meyer, and Reisner [20] and later by Campi and Gronchi [12]. Inequality (2) was established by Saint Raymond [45] for bodies which are symmetric with respect to the coordinate hyperplanes (and of course the $\mathrm{GL}(n)$-images of these bodies).

The Bourgain-Milman inequality [7] states that there exists a $c > 0$ independent of the dimension n such that for all origin-symmetric bodies K,

$$|K^*| \;\geq\; c^n \omega_n^2 \frac{1}{|K|}.$$

Here, $\omega_n = |B|$ denotes the volume of the unit ball, B, in $\mathbb{R}^n$. Thus the Bourgain-Milman inequality tells us that the Mahler conjecture for origin-symmetric bodies is at least asymptotically (as $n \to \infty$) correct. Very recently, Kuperberg [23] found a beautiful new approach to the Bourgain-Milman inequality. What's especially remarkable about Kuperberg's inequality is that it provides an explicit value for c.

Let $E_i(K)$ (resp. $E_o(K)$) denote the John (resp. Löwner) ellipsoid of K; i.e., the ellipsoid of largest (resp. smallest) volume contained in (resp. containing) K. Barthe's outer volume ratio inequality [3] (conjectured by Ball [2]) states that

$$|E_o(K)| \;\leq\; \frac{\omega_n n^{\frac{n}{2}} n!}{(n+1)^{(n+1)/2}} |K|,$$

with equality if and only if K is a simplex. From this, together with the definitions of E_i and E_o, the fact that polarity reverses set inclusion, together with the simple observation that $|E||E^*| \geq \omega_n^2$, for any ellipsoid E, it follows immediately that

$$|K^*| \;\geq\; \frac{\omega_n (n+1)^{(n+1)/2}}{n^{\frac{n}{2}} n!} \frac{1}{|E_i(K)|}, \tag{3}$$

with equality if and only if K is a simplex.

Let $\Gamma_2 K$ denote the Legendre ellipsoid associated with the convex body K; i.e., the origin-centered ellipsoid such that the second moment matrix of the uniform distribution supported in the ellipsoid is equal to the second moment matrix of the uniform distribution supported on K. In [31], the authors showed that associated with K is another ellipsoid $\Gamma_{-2} K$ that is in some sense dual to $\Gamma_2 K$. See §2 for the definition. While the Legendre ellipsoid is related to the second moment matrix, it was shown in [32] that the new Γ_{-2} ellipsoid is closely associated to the Fisher information matrix. Ludwig [25] showed that these two ellipsoids Γ_2 and Γ_{-2}, viewed as suitably normalized matrix-valued operators on the space of convex bodies, are the only linearly invariant operators that satisfy the inclusion-exclusion principle.

In this paper it is shown that the reciprocal of the volume of the new ellipsoid provides a sharp lower bound for the volume of the polar body. Specifically,

Theorem. *If $K \subset \mathbb{R}^n$ is a convex body that contains the origin in its interior, then*

$$|K^*| \;\geq\; \frac{\omega_n (n+1)^{(n+1)/2}}{n!\, n^{n/2}} \frac{1}{|\Gamma_{-2}K|}$$

with equality if and only if K is a simplex whose centroid is at the origin.

Observe that there are cases (for example, the cube in $\mathbb{R}^n$) where both sides of the inequality above blow up as the origin approaches the boundary of the body K. This is in contrast to inequality (3), where the lower bound remains unchanged under translations.

A $\mathrm{GL}(n)$-image of a body is often called a "position" of the body. There has been considerable interest (see, e.g., [2–6, 16–18, 31, 35]) in establishing isoperimetric inequalities for geometric invariants of bodies in special position. The Theorem above may be seen as another contribution to this line of investigation.

The key tool used in establishing the Theorem will be a variant of the Ball-Barthe inequality for isotropic measures (defined in Section 3). To establish the Theorem, the concept of an *isotropic embedding* is introduced and a variant of the Ball-Barthe inequality for these isotropic embeddings is used.

In Section 1 we review some simple facts about support and radial functions of convex bodies and their polars. Good general references for this are Gardner [14], Schneider [46], and Thompson [50]. In Section 2 we recall those elements of the quadratic Brunn-Minkowski theory that are needed here. The basics of isotropic measures are presented in Section 3, and the Theorem is proved in Section 4.

1. Basics and notation regarding convex bodies

Recall that we shall always assume that a convex body in $\mathbb{R}^n$ contains the origin of in its interior. We shall write $e_1, \ldots, e_n$ for the standard Euclidean basis in $\mathbb{R}^n$.

In addition to its denoting absolute value, we shall use $|\cdot|$ to denote the standard Euclidean norm on $\mathbb{R}^n$, on occasion the absolute value of the determinant of an $n \times n$ matrix, and often to denote n-dimensional volume.

For $\phi \in \mathrm{GL}(n)$, write $\phi K = \{\phi x : x \in K\}$ for the image of K under ϕ. If $\lambda > 0$, then write $\lambda K = \{\lambda x : x \in K\}$ for the dilate of K by a factor of λ. Observe that it follows trivially from the definition of the polar K^* of a convex body K that for $\phi \in \mathrm{GL}(n)$,

$$(\phi K)^* = \phi^{-t} K^*, \tag{4}$$

where ϕ^{-t} is the contragradient (inverse of the transpose) of ϕ.

Associated with each convex body K in $\mathbb{R}^n$ is its *support function* $h_K : \mathbb{R}^n \to [0, \infty)$, defined for $x \in \mathbb{R}^n$, by

$$h_K(x) = \max\{y \cdot x : y \in K\},$$

and its *radial function* $\rho_K : \mathbb{R}^n \setminus \{0\} \to (0, \infty)$, defined for $x \neq 0$, by

$$\rho_K(x) = \max\{\lambda \geq 0 \ : \ \lambda x \in K\}.$$

Clearly, the support function is homogeneous of degree 1 while the radial function is homogeneous of degree -1. Note that the support function and the reciprocal of the radial function of a convex body are both convex functions.

From the definitions of the support and radial functions and the definition of the polar body, it follows that

$$\rho_{K^*} = 1/h_K \quad \text{and} \quad h_{K^*} = 1/\rho_K, \tag{5}$$

$$K^* = \{x \in \mathbb{R}^n : h_K(x) \leq 1\}, \tag{6}$$

$$K^{**} = K. \tag{7}$$

We shall make use of:

Lemma. *If ν is a finite, positive Borel measure on S^{n-1}, and Q is a convex body that contains the origin in its interior, then*

$$\int_{S^{n-1}} (v\rho_Q(v), 1)\, d\nu(v) \in r_0 Q \times \{r_0\} \subset \mathbb{R}^{n+1},$$

where $r_0 = \nu(S^{n-1})$.

Proof. The only thing that requires proof is that

$$x_0 := \int_{S^{n-1}} v\rho_Q(v)\, d\nu(v) \in r_0 Q.$$

To see this, observe that by the definition of x_0 and the convexity of h_{Q^*}, the homogeneity of h_{Q^*} and (5) we have,

$$h_{Q^*}(\frac{x_0}{r_0}) \leq \int_{S^{n-1}} h_{Q^*}(\rho_Q(v)v)\, \frac{d\nu(v)}{r_0} = \int_{S^{n-1}} \rho_Q(v) h_{Q^*}(v)\, \frac{d\nu(v)}{r_0} = 1.$$

Hence, by (6) and (7),

$$\frac{x_0}{r_0} \in Q^{**} = Q.$$

q.e.d.

2. Elements from the Quadratic Brunn-Minkowski Theory

The quadratic Brunn-Minkowski theory is the special case $p = 2$ of the evolving L_p-Brunn-Minkowski theory (see e.g., [8–13, 15, 21–38, 41, 44, 45, 47–49, 51]). In this section we list for quick later recall some basic elements of the theory.

If K, L are convex bodies and $\epsilon > 0$, then the quadratic Firey-Minkowski linear combination $K +_2 \sqrt{\varepsilon} L$ is defined as the body whose support function is given by:

$$h^2_{K+_2\sqrt{\varepsilon}L} = h^2_K + \varepsilon h^2_L. \tag{8}$$

The quadratic mixed volume $V_2(K, L)$ of K and L was defined in [27] by

$$\frac{n}{2} V_2(K, L) = \lim_{\varepsilon \to 0^+} \frac{|K +_2 \sqrt{\varepsilon} L| - K}{\varepsilon}. \tag{9}$$

Note that from (8) and (9) it follows immediately that

$$|K| = V_2(K, K). \tag{10}$$

It was shown in [27] that corresponding to each convex body K, there exists a regular Borel measure, $S_2(K, \cdot)$, on S^{n-1}, called the *quadratic surface area measure* of K such that

$$V_2(K, Q) = \frac{1}{n} \int_{S^{n-1}} h^2_Q(u) \, dS_2(K, u), \tag{11}$$

for each convex body Q. It was shown in [27] that the support of the quadratic surface area measure is not contained in any closed hemisphere of S^{n-1} and that

$$\int_{S^{n-1}} u h_K(u) \, dS_2(K, u) = 0. \tag{12}$$

Observe that the support of $S_2(K, \cdot)$ must contain an affinely independent set of vectors (since it is not contained in any closed hemisphere of S^{n-1}).

It was shown in [27] that

$$\operatorname{supp} S_2(K, \ \cdot \ ) = \{u_1, \dots, u_r\},$$

if and only if K is a polytope whose faces have outer unit normals $\{u_1, \dots, u_r\}$. And if a_i is the area of the face (of the polytope) with outer normal u_i and $h_i > 0$ is the distance of the face from the origin, then

$$S_2(K, \{u_i\}) = a_i / h_i.$$

for all i.

The ellipsoid $\Gamma_{-2}K$ associated with a convex body K was defined in [31] as the body whose radial function at $u \in S^{n-1}$ is given by

$$\rho_{\Gamma_{-2}K}(u)^{-2} = \frac{1}{|K|}\int_{S^{n-1}} |u\cdot v|^2 dS_2(K,v). \tag{13}$$

Note that for the Euclidean unit ball B, we have $\Gamma_{-2}B = B$. It was shown in [31] that for $\phi \in \mathrm{GL}(n)$,

$$\Gamma_{-2}\phi K = \phi\Gamma_{-2}K. \tag{14}$$

3. Isotropic embedding

A positive Borel measure ν on S^{n-1} is said to be *isotropic* provided

$$\int_{S^{n-1}} u\otimes u\, d\nu(u) = I_n,$$

where I_n denotes the identity operator on $\mathbb{R}^n$, and $u\otimes u : \mathbb{R}^n \to \mathbb{R}^n$ is the rank 1 linear operator on $\mathbb{R}^n$ that takes x to $(x\cdot u)u$. Thus, ν is isotropic if

$$\int_{S^{n-1}} |v\cdot u|^2\, d\nu(u) = 1, \qquad \text{for all } v \in S^{n-1}. \tag{15}$$

Summing this equation with $v = e_1, \dots, e_n$ shows that necessarily

$$\nu(S^{n-1}) = n. \tag{16}$$

Observe that from (15) it follows that there does not exist an $x \neq 0$ that is orthogonal to all the vectors in the support of ν.

The Ball-Barthe inequality for isotropic measures is:

Ball-Barthe inequality. *If ν is an isotropic measure on S^{n-1}, then for each continuous $f : S^{n-1} \to (0,\infty)$*

$$\left|\int_{S^{n-1}} f(v)\, v\otimes v\, d\nu(v)\right| \geq \exp\left\{\int_{S^{n-1}} \log f(v)\, d\nu(v)\right\}, \tag{17}$$

with equality if and only if $f(v_1)\cdots f(v_n)$ is constant for linearly independent unit vectors $v_1, \dots, v_n \in \mathrm{supp}(\nu)$.

A proof of the Ball-Barthe inequality (along with its equality conditions) can be found in [34].

The concept of an *isotropic embedding* is critical in establishing the Theorem.

Definition. If (S^{n-1}, ν) is a Borel measure space, then a continuous map $v : S^{n-1} \to S^n$ is said to be an isotropic embedding of the Borel measure space (S^{n-1}, ν) into S^n if

$$\int_{S^{n-1}} |w\cdot v(x)|^2\, d\nu(x) = 1, \qquad \text{for all } w \in S^n. \tag{18}$$

Summing the equation (18) with $w = e_1, \ldots, e_{n+1}$ shows that if (S^{n-1}, ν) is isotropically embeddable into S^n then

$$\nu(S^{n-1}) = n + 1. \tag{19}$$

If $v : S^{n-1} \to S^n$ is an isotropic embedding of the Borel measure space (S^{n-1}, ν) into S^n, then the support of ν must contain at least $n+1$ unit vectors. Otherwise $v(\operatorname{supp} \nu)$ would be contained in a great subsphere of S^n and a $w \in S^n$ orthogonal to the vectors in that great subsphere would contradict (18).

We will make use of the following simple observation. If $\operatorname{supp} \nu = \{u_1, \ldots, u_{n+1}\}$, then $v(u_1), \ldots, v(u_{n+1})$ are orthogonal and $\nu(\{u_i\}) = 1$, for all i. To see this substitute $w = v(u_j) \in S^n$ into (18) to see that necessarily $\nu(\{u_j\}) \leq 1$. But in light of (19), we then must have $\nu(\{u_j\}) = 1$. In light of this, substituting $w = v(u_j)$ into (18) now shows that $v(u_j) \perp v(u_i)$, whenever $i \neq j$.

The Ball-Barthe inequality for isotropic embeddings is:

Proposition. *If $v : S^{n-1} \to S^n$ is an isotropic embedding of the Borel measure space (S^{n-1}, ν) into S^n, then for each continuous $f : S^{n-1} \to (0, \infty)$*

$$\left| \int_{S^{n-1}} v(u) \otimes v(u)\, f(u)\, d\nu(u) \right| \;\geq\; \exp\left\{ \int_{S^{n-1}} \log f(u)\, d\nu(u) \right\},$$

with equality if and only if $f(u_1) \cdots f(u_{n+1})$ is constant for $u_1, \ldots, u_{n+1} \in \operatorname{supp}(\nu)$ such that $v(u_1), \ldots, v(u_{n+1})$ are linearly independent.

We shall require and thus establish only a very special case of the Ball-Barthe inequality for isotropically embeddable measures. The special case we will need is one where the function f factors through the embedding v; i.e., there exists a continuous $f_v : S^n \to (0, \infty)$ such that $f = f_v \circ v$. For this special case the Ball-Barthe inequality for embeddable measures is a direct consequence of the Ball-Barthe inequality for isotropic measures applied to the pushforward of ν by v onto S^n; i.e., the Ball-Barthe inequality for isotropic measures applied to the isotropic measure $\bar{\nu}$ on S^n defined by

$$\int_{S^n} g\, d\bar{\nu} = \int_{S^{n-1}} g \circ v\, d\nu,$$

for each continuous function $g : S^n \to \mathbb{R}$.

As in [34] and [37], the ideas of Ball [2] and Barthe [5] (see also [6]) play a critical role throughout this work. A great deal of effort has been expended in making this work both elementary and reasonably self contained.

4. The Theorem

We now establish:

Theorem. *If $K \subset \mathbb{R}^n$ is a convex body that contains the origin in its interior, then*

$$|K^*| \;\geq\; \frac{\omega_n (n+1)^{(n+1)/2}}{n!\, n^{n/2}} \frac{1}{|\Gamma_{-2}K|},$$

with equality if and only if K is a simplex whose centroid is at the origin.

Proof. For the sake of brevity, let

$$h := \frac{1}{\sqrt{n}} h_K = h_{\frac{1}{\sqrt{n}}K} \qquad \text{and} \qquad \mu := \frac{1}{|K|} S_2(K, \cdot). \tag{20}$$

In order to establish our Theorem we may assume, in light of (14) and (4), that the body K has been $\mathrm{GL}(n)$-transformed so that $\Gamma_{-2}K = B$. Now, (13) and (15) show that

the measure μ is isotropic.

From (12) and (20) we see that

$$\int_{S^{n-1}} u h(u)\, d\mu(u) = 0, \tag{21}$$

and from (10), (11), and (20)

$$\int_{S^{n-1}} h(u)^2\, d\mu(u) = 1. \tag{22}$$

Define $q : S^{n-1} \to \mathbb{R}^{n+1}$ by

$$q(u) = (u, h(u)) \tag{23}$$

for $u \in S^{n-1}$, and define $\bar{q} : S^{n-1} \to S^n$ by

$$\bar{q} = q/|q|. \tag{24}$$

We first observe that $\bar{q} : S^{n-1} \to S^n$ is an isotropic embedding of the measure space $(S^{n-1}, |q|^2\, d\mu)$ into S^n. To see this, suppose

$$y = (x, r) \in \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}.$$

By (23) and (24), followed by (15), (21), and (22), we have

$$\begin{aligned}\int_{S^{n-1}} &|y\cdot\bar{q}(u)|^2\,|q(u)|^2\,d\mu(u)\\ &= \int_{S^{n-1}} |(x,r)\cdot(u,h(u))|^2\,d\mu(u)\\ &= \int_{S^{n-1}} |x\cdot u + rh(u)|^2\,d\mu(u)\\ &= \int_{S^{n-1}} |x\cdot u|^2\,d\mu(u) + 2rx\cdot\int_{S^{n-1}} uh(u)\,d\mu(u) + r^2\int_{S^{n-1}} h(u)^2\,d\mu(u)\\ &= |x|^2 + r^2\\ &= |y|^2.\end{aligned}$$

Note that since $\bar{q}$ is an isotropic embedding of the measure $|q|^2 d\mu$, there does not exist a non-zero $y\in\mathbb{R}^{n+1}$ that is orthogonal to every vector in $q(\operatorname{supp}\mu)$.

Define the smooth, monotone, strictly increasing function $\phi:\mathbb{R}\to(0,\infty)$ by

$$\int_0^{\phi(t)} e^{-\tau}\,d\tau = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{t} e^{-\tau^2}\,d\tau,$$

which satisfies

$$-t^2 = \log\sqrt{\pi} - \phi(t) + \log\phi'(t). \tag{25}$$

Suppose $y\in\mathbb{R}^{n+1}$. For $u\in S^{n-1}$, (25) gives

$$-|y\cdot\bar{q}(u)|^2 = \log\sqrt{\pi} - \phi(y\cdot\bar{q}(u)) + \log\frac{\phi'(y\cdot\bar{q}(u))}{e_{n+1}\cdot\bar{q}(u)} + \log(e_{n+1}\cdot\bar{q}(u)). \tag{26}$$

We now integrate (26) over all $u\in S^{n-1}$ with respect to the measure $|q|^2\,d\mu$. Since $\bar{q}:S^{n-1}\to S^n$ is an isotropic embedding of the measure space $(S^{n-1},|q|^2\,d\mu)$ into S^n, we see that the integral on the left is

$$-\int_{S^{n-1}} |y\cdot\bar{q}(u)|^2|q(u)|^2\,d\mu(u) \;= -|y|^2.$$

Also, from (19), it follows that integral of the first term on the right of (26)

$$\int_{S^{n-1}} \log\sqrt{\pi}\,|q(u)|^2\,d\mu(u) \;= (n+1)\log\sqrt{\pi}.$$

We now estimate the integral of the last term on the right of (26):

$$I_4 := \int_{S^{n-1}} \log(e_{n+1}\cdot\bar{q}(u))\,|q(u)|^2\,d\mu(u).$$

By (19), the measure $\frac{1}{n+1}\,|q|^2\,d\mu$ is a probability measure and since, on a probability space, the L_0-mean of a function never exceeds its L_2-mean,

$$\exp\left(\frac{1}{n+1}I_4\right) \le \left(\frac{1}{n+1}\int_{S^{n-1}} |e_{n+1}\cdot \bar q(u)|^2\,|q(u)|^2\,d\mu(u)\right)^{\frac12},$$

with equality if and only if $e_{n+1}\cdot \bar q(u) = 1/\sqrt{1+h(u)^{-2}}$ is constant for $u \in \operatorname{supp}\mu$. Since $\bar q : S^{n-1} \to S^n$ is an isotropic embedding of the measure space $(S^{n-1}, |q|^2\,d\mu)$ into S^n, we see that

$$\int_{S^{n-1}} |e_{n+1}\cdot \bar q(u)|^2 |q(u)|^2\,d\mu(u) = 1$$

and hence

$$I_4 \le -\log\,(n+1)^{(n+1)/2},$$

with equality if and only if h is constant on $\operatorname{supp}\mu$.

We combine these observations to get

$$\begin{aligned} -|y|^2 \le \log\left(\frac{\pi}{n+1}\right)^{\frac{n+1}{2}} &- e_{n+1}\cdot \int_{S^{n-1}} \bar q(u)\frac{\phi(y\cdot \bar q(u))}{e_{n+1}\cdot \bar q(u)}\,|q(u)|^2 d\mu(u) \\ &+ \int_{S^{n-1}} \log \frac{\phi'(y\cdot \bar q(u))}{e_{n+1}\cdot \bar q(u)}\,|q(u)|^2\,d\mu(u), \end{aligned} \tag{27}$$

with equality if and only if h is constant on $\operatorname{supp}\mu$.

Define $T : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ by

$$Ty = \int_{S^{n-1}} \bar q(u)\,\frac{\phi(y\cdot \bar q(u))}{e_{n+1}\cdot \bar q(u)}\,|q(u)|^2\,d\mu(u), \tag{28}$$

for $y \in \mathbb{R}^{n+1}$. Hence

$$dT(y) = \int_{S^{n-1}} \bar q(u)\otimes \bar q(u)\,\frac{\phi'(y\cdot \bar q(u))}{e_{n+1}\cdot \bar q(u)}\,|q(u)|^2\,d\mu(u). \tag{29}$$

Note that for $z \in \mathbb{R}^{n+1}$

$$z\cdot dT(y) z = \int_{S^{n-1}} |z\cdot q(u)|^2\,\phi'(y\cdot \bar q(u))\,\sqrt{1+h(u)^{-2}}\,\,d\mu(u).$$

Recall that there exists no nonzero $z \in \mathbb{R}^{n+1}$ such that $z\cdot q(u) = 0$ for every $u \in \operatorname{supp}\mu$. This together with the fact that $\phi'(y\cdot \bar q(u))\sqrt{1+h(u)^{-2}} > 0$ shows that $z\cdot dT(y)z > 0$ for all $z \ne 0$. Therefore, a simple application of the mean value theorem shows that $T : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is globally injective.

The Ball-Barthe inequality for isotropic embeddings together with (29) now shows that

$$|dT(y)| \ge \exp\int_{S^{n-1}} \log \frac{\phi'(y\cdot \bar q(u))}{e_{n+1}\cdot \bar q(u)}\,|q(u)|^2\,d\mu(u), \tag{30}$$

with equality if and only if

$$\prod_{i=1}^{n+1} \frac{\phi'(y\cdot \bar{q}(u_i))}{e_{n+1}\cdot \bar{q}(u_i)}$$

is constant for $u_1,\dots,u_{n+1} \in \operatorname{supp}\mu$ such that $\bar{q}(u_1),\dots,\bar{q}(u_{n+1})$ are linearly independent. Substituting (28) and (30) into (27) gives

$$e^{-|y|^2} \leq \left(\frac{\pi}{n+1}\right)^{\frac{n+1}{2}} e^{-e_{n+1}\cdot Ty}|dT(y)|, \tag{31}$$

with equality implying that $e_{n+1}\cdot \bar{q}(u)$ is constant on $\operatorname{supp}\mu$ and

$$\prod_{i=1}^{n+1} \phi'(y\cdot \bar{q}(u_i))$$

is constant for $u_1,\dots,u_{n+1} \in \operatorname{supp}\mu$ such that $\bar{q}(u_1),\dots,\bar{q}(u_{n+1})$ are linearly independent.

Integrating (31) over all $y\in\mathbb{R}^{n+1}$ gives

$$(n+1)^{(n+1)/2} \leq \int_{\mathbb{R}^{n+1}} e^{-e_{n+1}\cdot Ty}|dT(y)|\,dy = \int_{T(\mathbb{R}^{n+1})} e^{-e_{n+1}\cdot z}\,dz, \tag{32}$$

with equality implying that h is constant on $\operatorname{supp}\mu$, and that for each $y\in\mathbb{R}^{n+1}$, there exists a $c_y>0$ such that

$$\prod_{i=1}^{n+1} \phi'(y\cdot \bar{q}(u_i)) = c_y,$$

for $u_1,\dots,u_{n+1} \in \operatorname{supp}\mu$ such that $\bar{q}(u_1),\dots,\bar{q}(u_{n+1})$ are linearly independent.

By the definition (23) of q, (20), and (5), we have

$$q(u) = (u, h_{\frac{1}{\sqrt{n}}K}(u)) = (\rho_{(\frac{1}{\sqrt{n}}K)^*}(u)u, 1)\, h_{\frac{1}{\sqrt{n}}K}(u)$$

and hence the Lemma, equation (4) and definition (28) show that

$$Ty \in \bigcup_{r>0} r\sqrt{n}K^* \times \{r\} =: C \subset \mathbb{R}^n\times\mathbb{R}.$$

Hence, from (32) letting $z = (x, r) \in \mathbb{R}^n \times \mathbb{R}$,

$$\begin{aligned}(n+1)^{(n+1)/2} &\leq \int_{T(\mathbb{R}^{n+1})} e^{-e_{n+1}\cdot z}\, dz, \\ &\leq \int_C e^{-e_{n+1}\cdot z}\, dz, \\ &= \int_0^\infty \int_{r\sqrt{n}K^*} e^{-r}\, dx dr \\ &= \int_0^\infty |r\sqrt{n}K^*| e^{-r}\, dr \\ &= n^{n/2}|K^*| \int_0^\infty r^n e^{-r}\, dr \\ &= n^{n/2} n!\, |K^*|,\end{aligned}$$

with equality implying that h is constant on $\operatorname{supp}\mu$, and that for each $y \in \mathbb{R}^{n+1}$, there exists a $c_y > 0$ such that

$$\prod_{i=1}^{n+1} \phi'(y \cdot \bar{q}(u_i)) = c_y, \tag{33}$$

for $u_1, \dots, u_{n+1} \in \operatorname{supp}\mu$ such that $\bar{q}(u_1), \dots, \bar{q}(u_{n+1})$ are linearly independent.

This establishes the inequality when $\Gamma_{-2}K = B$. The inequality for arbitrary K now follows by using (4) and (14).

We now turn to the necessity of the equality conditions. Suppose there is equality in the inequality of the theorem. First, recall (from Section 2) that there exist affinely independent $u_1, \dots, u_{n+1} \in \operatorname{supp}\mu$.

Equality in our inequality implies that there exists a $c > 0$ such that $h(u) = c$, for all $u \in \operatorname{supp}\mu$. Hence, $q(u_i) = (u_i, c)$ and since the u_i are affinely independent, $q(u_1), \dots, q(u_{n+1})$ are linearly independent in $\mathbb{R}^{n+1}$.

Assume there exists a $u_0 \in \operatorname{supp}\mu \backslash \{u_1, \dots, u_{n+1}\}$. Write

$$q(u_0) = \beta_1 q(u_1) + \cdots + \beta_{n+1} q(u_{n+1}),$$

and since at least one $\beta_i \neq 0$, with out loss of generality assume $\beta_1 \neq 0$. Hence $q(u_0), q(u_2), \dots, q(u_{n+1})$ are a linearly independent set of vectors in $\mathbb{R}^{n+1}$ with $u_i \in \operatorname{supp}\mu$. But then, since $\phi' > 0$, the equality conditions for (33) give

$$\phi'(y \cdot \bar{q}(u_0)) = \phi'(y \cdot \bar{q}(u_1))$$

for all $y \in \mathbb{R}^{n+1}$. But since ϕ' is non-constant there exist $t_0, t_1 \in \mathbb{R}$ such that $\phi'(t_0) \neq \phi'(t_1)$. Since $\bar{q}(u_0)$ and $\bar{q}(u_1)$ are not parallel, there exists a $y \in \mathbb{R}^{n+1}$ such that $\bar{q}(u_i) \cdot y = t_i$, producing the contradiction that shows that

$$\operatorname{supp}\mu = \{u_1, \dots, u_{n+1}\}.$$

As shown in Section 3, this implies that $q(u_1), \ldots, q(u_{n+1})$ are orthogonal. Since $q(u_i) = (u_i, c)$, the orthogonality gives

$$u_i \cdot u_j = -c^2$$

whenever $i \neq j$, and hence that K is a regular simplex in $\mathbb{R}^n$. Since h is constant on the support of μ, it is a regular simplex circumscribed about a ball that is centered at the origin.

q.e.d.

An inequality similar to that of the Theorem, but restricted to bodies which are origin-symmetric, can be obtained by combining work of Barthe [5] and that of the authors [31].

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