

LINEAR PROGRAMMING WITH THE EXCEL SOLVER

Linear programming (or simply *LP*) refers to several related mathematical techniques that are used to allocate limited resources among competing demands in an optimal way. LP is the most popular of the approaches falling under the general heading of mathematical optimization techniques, and, as can be seen in the box “Typical Operations Management Applications of Linear Programming,” has been applied to myriad operations management problems. Our focus here is on how we can quickly set up and solve these problems using the Solver that comes with Microsoft Excel.

There are five essential conditions in a problem situation for linear programming to pertain. First, there must be *limited resources* (such as a limited number of workers, equipment, finances, and material); otherwise there would be no problem. Second, there must be an *explicit objective* (such as maximize profit or minimize cost). Third, there must be *linearity* (two is twice as much as one; if it takes three hours to make a part, then two parts would take

TYPICAL OPERATIONS MANAGEMENT APPLICATIONS OF LINEAR PROGRAMMING

APPLICATIONS

Aggregate production planning: Finding the minimum-cost production schedule, including production rate change costs, given constraints on size of workforce and inventory levels.

Service productivity analysis: Comparing how efficiently different service outlets are using their resources compared to the best-performing unit. (This approach is called data envelopment analysis.)

Product planning: Finding the optimal product mix where several products have different costs and resource requirements (for example, finding the optimal blend of constituents for gasolines, paints, human diets, animal feeds).

Product routing: Finding the optimal routing for a product that must be processed sequentially through several machine centers, with each machine in a center having its own cost and output characteristics.

Process control: Minimizing the amount of scrap material generated by cutting steel, leather, or fabric from a roll or sheet of stock material.

Inventory control: Finding the optimal combination of products to stock in a warehouse or store.

Distribution scheduling: Finding the optimal shipping schedule for distributing products between factories and warehouses or warehouses and retailers.



six hours, and three parts would take nine hours). Fourth, there must be *homogeneity* (the products produced on a machine are identical, or all the hours available from a worker are equally productive). Fifth is *divisibility*: Normal linear programming assumes that products and resources can be subdivided into fractions. If this subdivision is not possible (such as flying half an airplane or hiring one-fourth of a person), a modification of linear programming, called *integer programming*, can be used.

When a single objective is to be maximized (like profit) or minimized (like costs), we can use linear programming. When multiple objectives exist, *goal programming* is used. If a problem is best solved in stages or time frames, this is *dynamic programming*. Other restrictions on the nature of the problem may require that it be solved by other variations of the technique, such as *nonlinear programming* or *quadratic programming*.

THE LINEAR PROGRAMMING MODEL

Stated formally, the linear programming problem entails an optimizing process in which non-negative values for a set of decision variables X_1, X_2, \dots, X_n are selected so as to maximize (or minimize) an objective function in the form

$$\text{Maximize (minimize) } Z = C_1X_1 + C_2X_2 + \dots + C_nX_n$$

Plant location studies: Finding the optimal location of a new plant by evaluating shipping costs between alternative locations and supply and demand sources.

Materials handling: Finding the minimum-cost routings of material handling devices (such as forklift trucks) between departments in a plant and of hauling materials from a supply yard to work sites by trucks, with each truck having different capacity and performance capabilities.

EXAMPLES

AMERICAN RED CROSS BLOOD COLLECTION AND DISTRIBUTION

The blood service of the American Red Cross (ARC) is divided into a number of regions. Each region has responsibility for blood collection, testing, and distribution. The mid-Atlantic region of the American Red Cross (ARC) has responsibility for most of Virginia and northeastern North Carolina. The ARC was concerned with the feasibility and effect of a proposed facility relocation of its permanent blood collection and distribution sites in the mid-Atlantic region. Linear programming models were used to quickly provide insights into its current scheduling of blood collections and distribution as well as the changes imposed by the proposed relocation. As a result

of the analysis, ARC decided to postpone its facility relocation and to optimize the operations of its existing facilities.*

JUDGING PERFORMANCE APPRAISAL CONSISTENCY

Three researchers used linear programming to determine consistency of performance appraisals of Major League Baseball players. The model calculated weights for objective performance measures that best fit the subjective ranking assigned to each player. Comparison of the LP results with the appraisal ranking made comparisons within the league possible. Such an application of LP would be equally applicable to evaluating the consistency of business managers' performance appraisals of employees.[†]

VALUING FOREST RESOURCES

When the New Zealand government was privatizing the state-run forest plantations, they needed to value the expected cash flows from the forests to determine a fair selling price. Using linear programming, a forest estate modeling system was developed that calculated renewable harvesting and log allocation of the 14 districts over a 40- to 70-year time horizon. The expected cash flows from these operations were used to help set reserve prices and indicative valuations. The potential buyers also used the system to help develop their bidding strategies.[‡]

*SOURCE: D. A. JACOBS, M. N. SILAN, AND B. A. CLEMSON, "AN ANALYSIS OF ALTERNATIVE LOCATIONS AND SERVICE AREAS OF AMERICAN RED CROSS BLOOD FACILITIES," *INTERFACES*, MAY-JUNE 1996, PP. 40-50.

†C. ZAPPE, W. WEBSTER, AND I. HOROWITZ, "USING LINEAR PROGRAMMING TO DETERMINE POST-FACTO CONSISTENCY IN PERFORMANCE EVALUATIONS OF MAJOR LEAGUE BASEBALL PLAYERS," *INTERFACES*, NOVEMBER-DECEMBER 1993, PP. 107-13.

‡B. R. MANLEY AND J. A. THREADGILL, "LP USED FOR VALUATION AND PLANNING OF NEW ZEALAND PLANTATION FORESTS," *INTERFACES*, NOVEMBER-DECEMBER 1991, PP. 66-79.

subject to resource constraints in the form

$$\begin{aligned} A_{11}X_1 + A_{12}X_2 + \cdots + A_{1n}X_n &\leq B_1 \\ A_{21}X_1 + A_{22}X_2 + \cdots + A_{2n}X_n &\leq B_2 \\ &\vdots \\ A_{m1}X_1 + A_{m2}X_2 + \cdots + A_{mn}X_n &\leq B_m \end{aligned}$$

where C_n , A_{mn} , and B_m are given constants.

Depending on the problem, the constraints may also be stated with equal signs ($=$) or greater-than-or-equal-to signs (\geq).

EXAMPLE SA.1: Puck and Pawn Company

We describe the steps involved in solving a simple linear programming model in the context of a sample problem, that of the Puck and Pawn Company, which manufactures hockey sticks and chess sets. Each hockey stick yields an incremental profit of \$2, and each chess set, \$4. A hockey stick requires four hours of processing at Machine Center A and two hours at Machine Center B. A chess set requires six hours at Machine Center A, six hours at Machine Center B, and one hour at Machine Center C. Machine Center A has a maximum of 120 hours of available capacity per day, Machine Center B has 72 hours, and Machine Center C has 10 hours.

If the company wishes to maximize profit, how many hockey sticks and chess sets should be produced per day?

SOLUTION

Formulate the problem in mathematical terms. If H is the number of hockey sticks and C is the number of chess sets, to maximize profit the objective function may be stated as

$$\text{Maximize } Z = \$2H + \$4C$$

The maximization will be subject to the following constraints:

$$4H + 6C \leq 120 \quad (\text{Machine Center A constraint})$$

$$2H + 6C \leq 72 \quad (\text{Machine Center B constraint})$$

$$1C \leq 10 \quad (\text{Machine Center C constraint})$$

$$H, C \geq 0 \quad \bullet$$

This formulation satisfies the five requirements for standard LP stated in the second paragraph of this supplement:

- 1 There are limited resources (a finite number of hours available at each machine center).
- 2 There is an explicit objective function (we know what each variable is worth and what the goal is in solving the problem).
- 3 The equations are linear (no exponents or cross products).
- 4 The resources are homogeneous (everything is in one unit of measure, machine hours).
- 5 The decision variables are divisible and nonnegative (we can make a fractional part of a hockey stick or chess set; however, if this were deemed undesirable, we would have to use integer programming).

GRAPHICAL LINEAR PROGRAMMING

Though limited in application to problems involving two decision variables (or three variables for three-dimensional graphing), **graphical linear programming** provides a quick insight into the nature of linear programming.

We describe the steps involved in the graphical method in the context of the Puck and Pawn Company. The following steps illustrate the graphical approach:

1 **Formulate the problem in mathematical terms.** The equations for the problem are given above.

2 **Plot constraint equations.** The constraint equations are easily plotted by letting one variable equal zero and solving for the axis intercept of the other. (The inequality portions of the restrictions are disregarded for this step.) For the machine center A constraint equation when $H = 0$, $C = 20$, and when $C = 0$, $H = 30$. For the machine center B constraint equation, when $H = 0$, $C = 12$, and when $C = 0$, $H = 36$. For the machine center C constraint equation, $C = 10$ for all values of H . These lines are graphed in Exhibit SA.1.

3 **Determine the area of feasibility.** The direction of inequality signs in each constraint determines the area where a feasible solution is found. In this case, all inequalities are of the less-than-or-equal-to variety, which means that it would be impossible to produce any combination of products that would lie to the right of any constraint line on the graph. The region of feasible solutions is unshaded on the graph and forms a convex polygon. A convex polygon exists when a line drawn between any two points in the polygon stays within the boundaries of that polygon. If this condition of convexity does not exist, the problem is either incorrectly set up or is not amenable to linear programming.

4 **Plot the objective function.** The objective function may be plotted by assuming some arbitrary total profit figure and then solving for the axis coordinates, as was done for the constraint equations. Other terms for the objective function, when used in this context, are the *iso-profit* or *equal contribution line*, because it shows all possible production combinations for any given profit figure. For example, from the dotted line closest to the origin on the graph, we can determine all possible combinations of hockey sticks and chess sets that yield \$32 by picking a point on the line and reading the number of each product that can be made at that point. The combination yielding \$32 at point *a* would be 10 hockey sticks and three chess sets. This can be verified by substituting $H = 10$ and $C = 3$ in the objective function:

$$\$2(10) + \$4(3) = \$20 + \$12 = \$32$$

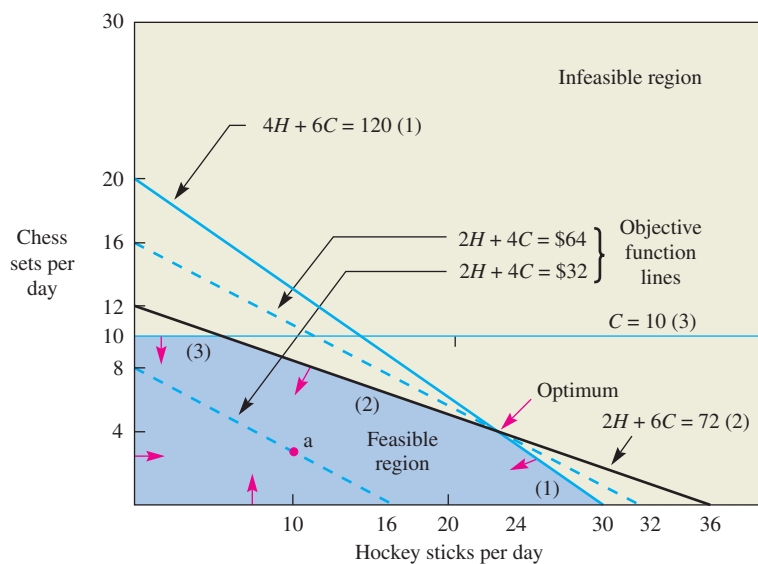


EXHIBIT SA.1

Graph of Hockey Stick and Chess Set Problem

H	C	EXPLANATION
0	$120/6 = 20$	Intersection of Constraint (1) and C axis
$120/4 = 30$	0	Intersection of Constraint (1) and H axis
0	$72/6 = 12$	Intersection of Constraint (2) and C axis
$72/2 = 36$	0	Intersection of Constraint (2) and H axis
0	10	Intersection of Constraint (3) and C axis
0	$32/4 = 8$	Intersection of \$32 iso-point line (objective function) and C axis
$32/2 = 16$	0	Intersection of \$32 iso-profit line and H axis
0	$64/4 = 16$	Intersection of \$64 iso-profit line and C axis
$64/2 = 32$	0	Intersection of \$64 iso-profit line and H axis

5 Find the optimum point. It can be shown mathematically that the optimal combination of decision variables is always found at an extreme point (corner point) of the convex polygon. In Exhibit SA.1 there are four corner points (excluding the origin), and we can determine which one is the optimum by either of two approaches. The first approach is to find the values of the various corner solutions algebraically. This entails simultaneously solving the equations of various pairs of intersecting lines and substituting the quantities of the resultant variables in the objective function. For example, the calculations for the intersection of $2H + 6C = 72$ and $C = 10$ are as follows:

Substituting $C = 10$ in $2H + 6C = 72$ gives $2H + 6(10) = 72$, $2H = 12$, or $H = 6$. Substituting $H = 6$ and $C = 10$ in the objective function, we get

$$\begin{aligned}\text{Profit} &= \$2H + \$4C = \$2(6) + \$4(10) \\ &= \$12 + \$40 = \$52\end{aligned}$$

A variation of this approach is to read the H and C quantities directly from the graph and substitute these quantities into the objective function, as shown in the previous calculation. The drawback in this approach is that in problems with a large number of constraint equations, there will be many possible points to evaluate, and the procedure of testing each one mathematically is inefficient.

The second and generally preferred approach entails using the objective function or iso-profit line directly to find the optimum point. The procedure involves simply drawing a straight line *parallel* to any arbitrarily selected initial iso-profit line so that the iso-profit line is farthest from the origin of the graph. (In cost-minimization problems, the objective would be to draw the line through the point closest to the origin.) In Exhibit SA.1, the dashed line labeled $\$2H + \$4C = \$64$ intersects the most extreme point. Note that the initial arbitrarily selected iso-profit line is necessary to display the slope of the objective function for the particular problem.* This is important since a different objective function (try profit = $3H + 3C$) might indicate that some other point is farthest from the origin. Given that $\$2H + \$4C = \$64$ is optimal, the amount of each variable to produce can be read from the graph: 24 hockey sticks and four chess sets. No other combination of the products yields a greater profit.



LINEAR PROGRAMMING USING MICROSOFT EXCEL

Spreadsheets can be used to solve linear programming problems. Most spreadsheets have built-in optimization routines that are very easy to use and understand. Microsoft Excel has an optimization tool called *Solver* that we will demonstrate by using it to solve the hockey stick and chess problem. We invoke the Solver from the Tools menu. A dialogue box requests information required by the program.

First, we define the problem by identifying a target cell (or objective function), the changing cells (or decision variables), and the constraints that we want in the analysis.

*The slope of the objective function is -2 . If $P = \text{profit}$, $P = \$2H + \$4C$; $\$2H = P - \$4C$; $H = P/2 - 2C$. Thus the slope is -2 .

Microsoft Excel Solver Screen

EXHIBIT SA.2

The screenshot shows the Microsoft Excel Solver LP dialog box overlaid on a spreadsheet. The spreadsheet data is as follows:

	A	B	C	D	E	F	G	H	I	J	K	L
1												
2												
3		Hockey Sticks	Chess Sets	Total								
4	Decision	24	4	28								
5	Profit	\$2	\$4	\$64								
6												
7												
8		Hockey Sticks	Chess Sets	Used	Capacity							
9	Machine A	4	6	120	<=	120						
10	Machine B	2	6	72	<=	72						
11	Machine C		1	4	<=	10						
12												
13												

The Solver Parameters dialog box is open, showing the following settings:

- Set Target Cell: $\$D\5
- Equal To: Max Min Value of: 0
- By Changing Variable Cells: $\$B\$4:\$C\4
- Subject to the Constraints:
 - $\$B\$4 \geq 0$
 - $\$C\$4 \geq 0$
 - $\$D\$10 \leq \$F\10
 - $\$D\$11 \leq \$F\11
 - $\$D\$9 \leq \$F\9

Exhibit SA.2 shows a spreadsheet with the information required for the problem. Our “Target Cell” (cell D5) contains a formula that multiplies the quantities of hockey sticks and chess sets produced by their corresponding profit. The “Changing Cells” correspond to the decision variables for the problem. The “Subject to the Constraints” box contains cells that calculate the individual machine requirements for a solution. Note that we must indicate that our solution values must be greater than or equal to zero.

In Exhibit SA.2, we do not show the “Options” dialogue box. If you click on this box, though, you will see a box that allows you to indicate this is a “linear problem.” This will speed the solution process for our problem. Many other items can be selected in the “Options” dialogue box that control the solution search mechanism used by Solver. The program can solve problems using search strategies different from the simplex approach.

After we input all of the information in the dialogue boxes, we can press the Solve button to solve the problem. A number of different reports can be obtained. The most interesting reports for our problem are the Answer Report and the Sensitivity Report, both of which are shown in Exhibit SA.3. The Answer Report shows the final answers for the total profit (\$64) and the amounts produced (24 hockey sticks and four chess sets). In the constraints section of the Answer Report, the status of each resource is given. All of Machine A and Machine B are used, and there are six units of slack for Machine C.

The Sensitivity Report is divided into two parts. The first part, titled “Changing Cells,” corresponds to objective function coefficients. The profit per unit for the hockey sticks can be either up or down \$.67 (between \$2.67 and \$1.33) without having an impact on the solution. Similarly, the profit of the chess sets could be between \$6 and \$3 without changing the solution. In the case of Machine A, the right-hand side could increase to 144 (120 + 24) or decrease to 84 with a resulting \$.33 increase or decrease in the objective function. The right-hand side of Machine B can increase to 90 units or decrease to 60 units with the same \$.33 change in the objective function. For Machine C, the right-hand side

EXHIBIT SA.3

Microsoft Excel Solver Reports

Answer Report

Target Cell (Max)

Cell	Name	Original Value	Final Value
\$D\$5	Profit Total	\$64	\$64

Adjustable Cells

Cell	Name	Original Value	Final Value
\$B\$4	Decision Hockey Sticks	24	24
\$C\$4	Decision Chess Sets	4	4

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
\$D\$9	Machine A Used	120	\$D\$9<=\$F\$9	Binding	0
\$D\$10	Machine B Used	72	\$D\$10<=\$F\$10	Binding	0
\$D\$11	Machine C Used	4	\$D\$11<=\$F\$11	Not Binding	6
\$B\$4	Decision Hockey Sticks	24	\$B\$4>=0	Not Binding	24
\$C\$4	Decision Chess Sets	4	\$C\$4>=0	Not Binding	4

Sensitivity Report

Changing Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
\$B\$4	Decision Hockey Sticks	24	0	2	0.666666667	0.666666667
\$C\$4	Decision Chess Sets	4	0	4	2	1

Constraints

Cell	Name	Final Value	Shadow* Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
\$D\$9	Machine A Used	120	0.333333333	120	24	36
\$D\$10	Machine B Used	72	0.333333333	72	18	12
\$D\$11	Machine C Used	4	0	10	1E+30	6

* Shadow price: The monetary value of one more unit of capacity.

could increase to infinity (1E + 30 is scientific notation for a very large number) or decrease to four units, with no change in the objective function.

GENETIC SOLVER OPTION

On the CD-ROM that is packaged with the book, we have included a great product from Frontline Systems called the *Premium Solver*. The package installs on top of Microsoft's Solver and adds many new features to the Microsoft product. All of the old Solver capabilities are retained, but in Premium Solver mode new capabilities include



- A new evolutionary solver for doing genetic searches. This can be very useful for solving problems that are not linear.
- A linearity report for finding where the calculations in a spreadsheet violate the linearity conditions that are needed to use a standard linear programming algorithm.
- A feasibility report to help identify why it is not possible to find a feasible solution to your problem.
- A population report to help explain solutions that are generated with the new genetic search capability.

If you are interested in the latest tricks for solving problems with spreadsheets, install the Premium Solver.

PROBLEMS

1 Solve the following problem with Excel Solver:

$$\begin{aligned} \text{Maximize } Z &= 3X + Y \\ 12X + 14Y &\leq 85 \\ 3X + 2Y &\leq 18 \\ Y &\leq 4 \end{aligned}$$

$$\begin{aligned} 1. \quad X &= 6; \\ Y &= 0; \\ Z &= 18. \end{aligned}$$

2 Solve the following problem with Excel Solver:

$$\begin{aligned} \text{Minimize } Z &= 2A + 4B \\ 4A + 6B &\geq 120 \\ 2A + 6B &\geq 72 \\ B &\geq 10 \end{aligned}$$

$$\begin{aligned} 2. \quad \text{See ISM;} \\ \text{Optimal combination} \\ \text{is } B = 10; A = 15. \\ Z = 70. \end{aligned}$$

3 A manufacturing firm has discontinued production of a certain unprofitable product line. Considerable excess production capacity was created as a result. Management is considering devoting this excess capacity to one or more of three products: X_1 , X_2 , and X_3 . Machine hours required per unit are

MACHINE TYPE	PRODUCT		
	X_1	X_2	X_3
Milling machine	8	2	3
Lathe	4	3	0
Grinder	2	0	1

The available time in machine hours per week is

	MACHINE HOURS PER WEEK
Milling machines	800
Lathes	480
Grinders	320

The salespeople estimate that they can sell all the units of X_1 and X_2 that can be made. But the sales potential of X_3 is 80 units per week maximum.

Unit profits for the three products are

	UNIT PROFITS
X_1	\$20
X_2	6
X_3	8

- Set up the equations that can be solved to maximize the profit per week.
- Solve these equations using the Excel Solver.
- What is the optimal solution? How many of each product should be made, and what should the resultant profit be?
- What is this situation with respect to the machine groups? Would they work at capacity, or would there be unused available time? Will X_3 be at maximum sales capacity?
- Suppose that an additional 200 hours per week can be obtained from the milling machines by working overtime. The incremental cost would be \$1.50 per hour. Would you recommend doing this? Explain how you arrived at your answer.

- See ISM for a and b .
 - $X_1 = 45$;
 $X_2 = 100$;
 $X_3 = 80$;
 $Z = 2,140$ (maximum profit).
 - Milling machines at capacity; lathes at capacity; grinders have 150 hours' slack; X_3 is at maximum sales capacity.
- Recommend working 200 hrs. O.T. in milling.

4. a. $600A + 900B \leq 3,600$
 $600A + 900B \geq 1,800$
 $200A + 700B \leq 1,400$
 $400A + 100B \geq 400$
 $A \leq 2$
 Minimize $.75A + .15B$
 b. $A = 0.54$, $B = 1.85$, $\text{Obj} = 0.68$.
5. Cost per student:
 $A = 0.75$;
 $B = 1.5$;
 $\text{Obj} = 5.06$.
6. a. $A + B \geq 3,000$
 $A + B \leq 4,000$
 $A \leq 2,000$
 $B \leq 4,000$
 $90A + 75B \geq 80(A + B)$, or
 $10A - 5B \geq 0$
 b. Optimum:
 $A = 1,000$;
 $B = 2,000$.
7. Food = \$300;
 Shelter = \$700;
 Entertainment = \$300;
 Satisfaction value = \$4,200.
4. A diet is being prepared for the University of Arizona dorms. The objective is to feed the students at the least cost, but the diet must have between 1,800 and 3,600 calories. No more than 1,400 calories can be starch, and no fewer than 400 can be protein. The varied diet is to be made of two foods: A and B . Food A costs \$0.75 per pound and contains 600 calories, 400 of which are protein and 200 starch. No more than two pounds of Food A can be used per resident. Food B costs \$0.15 per pound and contains 900 calories, of which 700 are starch, 100 are protein, and 100 are fat.
- a. Write the equations representing this information.
 b. Solve the problem graphically for the amounts of each food that should be used.
5. Do Problem 4 with the added constraint that not more than 150 calories shall be fat, and that the price of food has escalated to \$1.75 per pound for food A and \$2.50 per pound for food B .
6. Logan Manufacturing wants to mix two fuels (A and B) for its trucks to minimize cost. It needs no fewer than 3,000 gallons to run its trucks during the next month. It has a maximum fuel storage capacity of 4,000 gallons. There are 2,000 gallons of Fuel A and 4,000 gallons of Fuel B available. The mixed fuel must have an octane rating of no less than 80.
 When fuels are mixed, the amount of fuel obtained is just equal to the sum of the amounts put in. The octane rating is the weighted average of the individual octanes, weighted in proportion to the respective volumes.
 The following is known: Fuel A has an octane of 90 and costs \$1.20 per gallon. Fuel B has an octane of 75 and costs \$0.90 per gallon.
- a. Write the equations expressing this information.
 b. Solve the problem using the Excel Solver, giving the amount of each fuel to be used. State any assumptions necessary to solve the problem.
7. You are trying to create a budget to optimize the use of a portion of your disposable income. You have a maximum of \$1,500 per month to be allocated to food, shelter, and entertainment. The amount spent on food and shelter combined must not exceed \$1,000. The amount spent on shelter alone must not exceed \$700. Entertainment cannot exceed \$300 per month. Each dollar spent on food has a satisfaction value of 2, each dollar spent on shelter has a satisfaction value of 3, and each dollar spent on entertainment has a satisfaction value of 5.
 Assuming a linear relationship, use the Excel Solver to determine the optimal allocation of your funds.

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