An exercise

Exercise. Let $(X, \mathcal{A}, \mu)$ be a finite measure space and let $\mathcal{F} \subseteq \mathcal{A}$ be an *algebra* of subsets with $\sigma(\mathcal{F}) = \mathcal{A}$. Show that for all $\epsilon > 0$ and $A \in \mathcal{A}$ there exists $F \in \mathcal{F}$ such that $\mu(A \triangle F) < \epsilon$.

Proof. Define

$$\mathcal{D} = \{A \in \mathcal{A} : \text{for all } \epsilon > 0 \text{ there exists } F \in \mathcal{F} \text{ s.t. } \mu(A \triangle F) < \epsilon\}.$$

Clearly $\mathcal{D}$ contains $\mathcal{F}$ (if $A \in \mathcal{F}$, for any $\epsilon > 0$ take $F = A$: then $\mu(A \triangle F) = 0 < \epsilon$).

Hence it suffices to show that $\mathcal{D}$ is a σ-algebra. It clearly contains X, since it contains $\mathcal{F}$.

$\mathcal{D}$ is closed under complements:
If $A \in \mathcal{D}$, given $\epsilon > 0$ choose $F \in \mathcal{F}$ with $\mu(A \triangle F) < \epsilon$. Now $F^c \in \mathcal{F}$ and $A^c \setminus F^c = A^c \cap F = F \setminus A$ and $F^c \setminus A^c = F^c \cap A = A \setminus F$. Hence $A^c \triangle F^c = A \triangle F$ and so $\mu(A^c \triangle F^c) < \epsilon$; so $A^c \in \mathcal{D}$.

It remains to prove that $\mathcal{D}$ is closed under countable unions.

So now let $A_n \in \mathcal{D}$ and $A = \cup_n A_n$. Since $\mu(\cup_{n=1}^m A_n) \to \mu(A)$ *which is finite*, we have $\mu(A \setminus \cup_{n=1}^m A_n) \to 0$ hence given $\epsilon > 0$ we can find $m \in \mathbb{N}$ so that $\mu(A \setminus \cup_{n=1}^m A_n) \leq \epsilon$. We now choose, for each $n = 1, \ldots, m$, elements $F_n \in \mathcal{F}$ so that $\mu(A_n \triangle F_n) < \epsilon/m$. Write $B = \cup_{n=1}^m A_n$ and $F = \cup_{n=1}^m F_n$ and observe that

$$A \triangle F \subseteq (A \triangle B) \cup (B \triangle F). \tag{1}$$

Indeed, writing $A' := A \setminus B = A \triangle B$,

$$\begin{aligned} A \triangle F = (A \cup F) \setminus (A \cap F) &= (A' \cup B \cup F) \setminus (A \cap F) = (A' \setminus (A \cap F)) \cup ((B \cup F) \setminus (A \cap F)) \\ &\subseteq A' \cup ((B \cup F) \setminus (B \cap F)) = (A \triangle B) \cup (B \triangle F). \end{aligned}$$

Observe that

$$(A_1 \cup A_2) \setminus (F_1 \cup F_2) \subseteq (A_1 \setminus F_1) \cup (A_2 \setminus F_2). \tag{2}$$

Indeed,

$$\begin{aligned} (A_1 \cup A_2) \setminus (F_1 \cup F_2) = (A_1 \cup A_2) \cap (F_1^c \cap F_2^c) &= (A_1 \cap F_1^c \cap F_2^c) \cup (A_2 \cap F_1^c \cap F_2^c) \\ &\subseteq (A_1 \cap F_1^c) \cup (A_2 \cap F_2^c) \end{aligned}$$

and similarly

$$(F_1 \cup F_2) \setminus (A_1 \cup A_2) \subseteq (F_1 \setminus A_1) \cup (F_2 \setminus A_2). \tag{3}$$

By (2) and (3) we have

$$(A_1 \cup A_2) \triangle (F_1 \cup F_2) \subseteq (A_1 \triangle F_1) \cup (A_2 \triangle F_2).$$

It follows that

$$A \triangle F \subseteq (A \triangle B) \cup (B \triangle F) \subseteq (A \triangle B) \cup \bigcup_{n=1}^{m} (A_n \triangle F_n)$$

(using (1)). Therefore by subadditivity of μ,

$$\begin{aligned} \mu((A \triangle F) &\leq \mu(A \triangle B) + \mu(B \triangle F) < \epsilon + \mu(B \triangle F) \leq \epsilon + \mu\left(\bigcup_{n=1}^{m} (A_n \triangle F_n)\right) \\ &\leq \epsilon + \sum_{n=1}^{m} (\mu(A_n \triangle F_n) < \epsilon + \epsilon. \qquad \square \end{aligned}$$