2023-04-24

$\theta = E(X)$, $X \sim F$

$\hat{\theta} = \bar{X}$, $X_1, \dots, X_n$ iid (F)

$Var(\hat{\theta}) = \dfrac{\sigma^2}{n}$, $\sigma^2 = Var(X)$

? $\exists$ $\tilde{X}$:
① προσομοιώνεται
② $E(\tilde{X}) = \theta$
③ $Var(\tilde{X}) < \sigma^2$

$X_1 = h(U_1, \dots, U_m)$

$X_2 = h(1-U_1, \dots, 1-U_m)$

$X_1, X_2 \sim F$

$Cov(X_1, X_2) < 0$.

$$Y = \frac{X_1 + X_2}{2}$$

$$Var(Y) = \frac{Var(X_1) + Var(X_2) + 2\,Cov(X_1, X_2)}{4}$$

$$= \frac{\sigma^2}{2} + Cov(X_1, X_2) < \frac{\sigma^2}{2}$$

$E(Y) = \theta$

Ενώ αν είχαμε $Z = \dfrac{X_1 + X_2}{2}$, X_1, X_2 iid $\sim F$ (κλασσική μέθοδος)

$$Var(Z) = Var(\bar{X}_2) = \sigma^2/2 \qquad E(Z) = \theta.$$

Μέθοδος 2 : Μεταβλητές ελέγχου (Control Variates)

$$X \sim F \quad , \quad \theta = E(X)$$

Έστω Y τμ. : $E(Y) = \mu_Y$ = γνωστή

$$Cov(X, Y) \neq 0$$

$$\tilde{X} = X + c(Y - \mu_Y)$$

$$E(\tilde{X}) = E(X) = \theta.$$

$$Var(\tilde{X}) = Var(X) + c^2 \cdot Var(Y) + 2c\, Cov(X,Y)$$

$$= \sigma_X^2 + c^2\sigma_Y^2 + 2c\sigma_{XY} = \varphi(c)$$

min $\varphi(c)$:

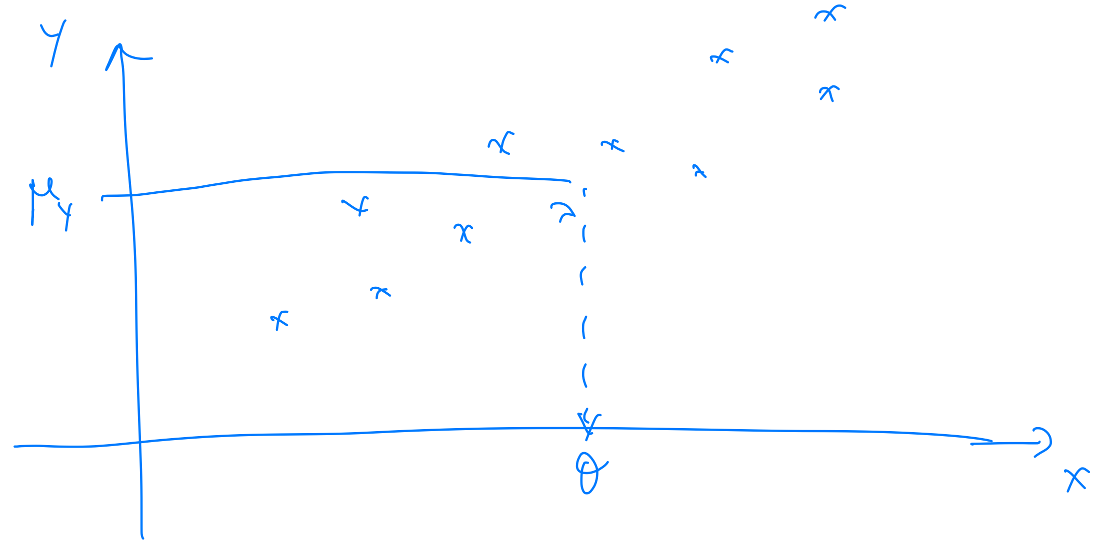
$$\varphi'(c) = 2\sigma_Y^2 \cdot c + 2\sigma_{XY} = 0 \Rightarrow c^* = -\frac{\sigma_{XY}}{\sigma_Y^2}$$

$$\varphi^* = \varphi(c^*) = Var\left(X + c^*(Y - \mu_Y)\right) = \dots = \sigma_X^2 - \frac{\sigma_{XY}^2}{\sigma_Y^2}$$

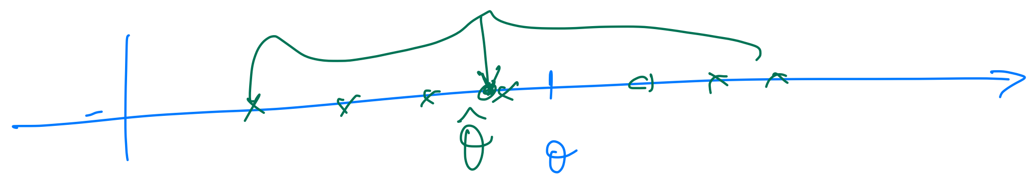
$$= Var(X) - \frac{\left(Cov(X,Y)\right)^2}{\sigma_Y^2} < Var(X)$$

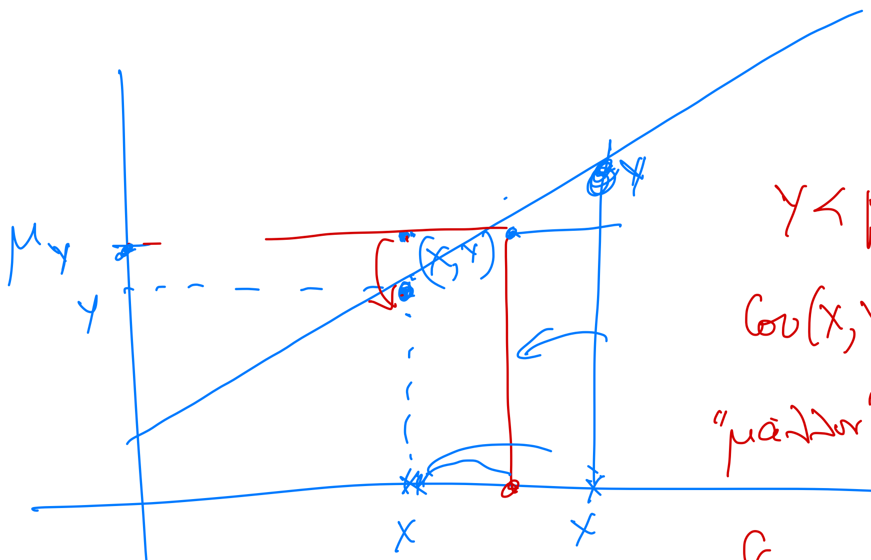
$$\frac{Var\left(X + c^*(Y-\mu_Y)\right)}{Var(X)} = 1 - \rho^2(X,Y)$$

$\rho^2(X,Y)$ = % μείωσης της διασποράς της X.



standard Monte Carlo





$$\left.\begin{array}{l} Y < \mu_Y \\ Cov(X,Y) > 0 \end{array}\right\} \Rightarrow$$

"μάλλον" $X < \theta$

$$Cov = E\left[(X-\mu_X)(Y-\mu_Y)\right]$$

$\tilde{X} = X + c(Y - \mu_Y)$: αν $Y < \mu_Y$ τότε "μεγάλο" $X < \theta$ διόρθωση : $c < 0$

$Cov(X,Y) > 0$

$X + c(Y - \mu_Y)$

Ακραία περίπτωση (1): $|\rho(X,Y)| = 1 \Rightarrow Y = aX + b$ μ.π. 1

$\mu_Y = a\theta + b$

Ακραία περ. (2) : $\rho = 0$

$c^* = -\frac{Cov(X,Y)}{Var(Y)}$ πώς το υπολογίζουμε?

Εκτίμηση.

(1) Pilot study (με "μικρό n")

Εκτιμήστε $\hat{\sigma}_{XY}$, $\hat{\sigma}_Y^2$

$$\hat{c}^* = -\frac{\hat{\sigma}_{XY}}{\hat{\sigma}_Y^2}$$

<u>Παράδειγμα</u> <u>Αξιοπιστία</u>

n συνιστώσες , $S_j = 1$ (συνιστώσα j λειτουργεί)

$$X = \varphi(S_1, \dots, S_n) = 1(\text{σύστημα λειτουργεί}).$$

$\theta = E(X)$ $S_j \sim Ber(p_j)$ <u>ανεξάρτητες</u>

Y ? $Cov(X, Y) \neq 0$

$E(Y)$ = γνωστή

$Y = \sum_{j=1}^{n} S_j$: αρ. συνιστωσών που λειτουργούν.

$$\left.\begin{array}{l} E(Y) = \sum_{j=1}^{n} p_j = \mu_Y \\ Cov(X, Y) > 0 \end{array}\right\} \checkmark$$

Εκτίμηση c^*.

Έστω: $\begin{matrix}(X_1, \dots, X_n) \\ (Y_1, \dots, Y_n)\end{matrix} = \begin{bmatrix} (X_1, Y_1) \\ \vdots \\ (X_n, Y_n) \end{bmatrix}$

$$\widehat{Cov}(X,Y) = \frac{1}{n-1} \sum_{j=1}^{n} (X_j - \bar{X})(Y_j - \bar{Y})$$

$$Var(\hat{Y}) = \frac{1}{n-1} \sum_{j=1}^{n} (Y_j - \bar{Y})^2$$

$$\Rightarrow \boxed{\hat{c}^* = - \frac{\sum_j (X_j - \bar{X})(Y_j - \bar{Y})}{\sum_j (Y_j - \bar{Y})^2}}$$

Από παλινδρόμηση: $Y = b_0 + b_1 X + \varepsilon$

LSE $\quad \hat{b}_1 = \dfrac{\sum (y_j - \bar{y})(x_j - \bar{x})}{\sum (x_j - \bar{x})^2}$

Αν θεωρήσουμε το μοντέλο: $X = a + bY + \varepsilon$

$(\bar{X} = \hat{a} + \hat{b}\,\bar{Y})$

$$\hat{b} = \frac{\sum (x_j - \bar{x})(y_j - \bar{y})}{\sum (y_j - \bar{y})^2}, \quad \hat{a} = \bar{X} - \hat{b}\cdot\bar{Y}$$

$$\Rightarrow c^* = -\hat{b}$$

$$\Rightarrow \tilde{X} = X + c^*(Y - \mu_Y) = X - \hat{b}(Y - \mu_Y)$$

Επίσης: $\bar{\tilde{X}} = \bar{X} + c^*(\bar{Y} - \mu_Y) =$

$\underline{\hat{\theta}}$

$$= \underbrace{\bar{X} - \hat{b}(\bar{Y})}_{\hat{a}} + \hat{b}\mu_Y =$$

$$= \hat{a} + \hat{b}\mu_Y$$

$$\Rightarrow \boxed{\hat{\theta} = \hat{a} + \hat{b}\mu_Y}$$

Αρχικά: $\hat{\theta}_0 = \bar{X}$

$$\left.\begin{array}{l} E(\hat{\theta}_0) = \theta \\ E(\hat{\theta}) = \theta \\ Var(\hat{\theta}) < Var(\hat{\theta}_0) \end{array}\right\}$$