A Short History of Operator Theory

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In the first textbook on operator theory, *Théorie des Opérations Linéaires,* published in Warsaw 1932, Stefan Banach states that the subject of the book is the study of functions on spaces of infinite dimension, especially those he coyly refers to as spaces of type B, otherwise *Banach spaces* (definition).

This was a good description for Banach, but tastes vary. I propose rather the "operational" definition that *operators act like matrices*. And what that means depends on who you are.

If you are an engineering student, matrices are particular symbols you manipulate to solve linear systems. As a working engineer you may instead use Heaviside's operational calculus, in which you are permitted to do all sorts of dangerous manipulations of symbols for derivatives and what not, exactly as if they were matrices, in order to solve linear problems of applied analysis. About 90% of the time you will get the right answer, just like the student; somewhat more with experience. And that is good enough, if the bridges you build aren't where I drive.

In mathematics the student of elementary analysis learns that matrices are linear functions relating finite-dimensional vector spaces, and conversely. As a working mathematician the analyst has lost all fear of minor matters like infinity, and will happy agree with Banach's definition.

For the students of algebra, matrices are fun objects that can be added and multiplied, usually in flagrant disregard for the law (of commutativity). The working algebraist still enjoys adding and multiplying, but feels that the analyst's concern about just what the things being added and multiplied are is, well, limiting.

In this course we'll try to please everyone, except that this is a mathematics course, so we'll always be careful. We'll solve applied problems, we'll analyze, and we'll add and multiply. The book by Arveson is somewhat algebraic, but the lectures will take all three points of view.

We'll start with something completely different, namely history. It is usually instructive to review the history of a branch of mathematics, especially in order to understand how the subject applies and why some parts are considered particularly interesting. Today there are excellent resources making this easy, especially the MacTutor History of Mathematics Archive. Perhaps if Banach had had access to the internet he wouldn't have so carelessly reduced his historical remarks in the introduction to an unsupported repetition of Jacques Hadamard's assertion that it was mainly the creation of Vito Volterra. The most thorough history of operator theory of which I am aware is Jean Dieudonné's *History of Functional Analysis*, on which I draw in this account, along with some other sources in the bibliography you may enjoy.

The concepts whose origins we should seek include: linearity, spaces of infinite dimension, matrices, and the spectrum. (The spectrum comprises eigenvalues and, as we shall learn, other related notions.) As with most of mathematics, these concepts arose in applications.

I. Matrices and Abstract Algebra.

The original model for operator theory is the study of matrices. Although the word "matrix" was only coined by James Sylvester in 1850, matrix methods have been around for over 2000 years, as attested by the use of what we would call Gauß elimination in a Chinese work, *Nine Chapters of the Mathematical Art*, from the Han Dynasty. (Even earlier, around 300 BC, the Babylonians worked with simultaneous linear equations.) Likewise, although Carl Friedrich Gauß gave us the word "determinant," in the 19th Century, determinants had had precursors for centuries, and were explicitly used since their simultaneous discovery in 1683 by Takakazu Seki Kowa in Japan and Gottfried Leibniz in Europe.

Eigenvalues and diagonalization were discovered in 1826 by Augustin Louis Cauchy in the process of finding normal forms for quadratic functions. (An early calculation equivalent to diagonalization is attributed to Johan de Witt in 1660.) Cauchy proved the spectral theorem for self-adjoint matrices, i.e., that every real, symmetric matrix is diagonable. The spectral theorem as generalized by John von Neumann is today the most important result of operator theory. In addition, Cauchy was the first to be systematic about determinants.

All this time, what we regard as linear algebra was embedded in practical calculations. Indeed, although today professional mathematicians intuitively regard our subject as concerned with structures more than with particular realizations of those structures, this idea was absent until the mid-nineteenth century and only came to dominate well into the twentieth century. Abstract algebra can said to have been born with William Rowan Hamilton's discovery of quaternions in 1843, and Hermann Grassmann's introduction of exterior algebra the following year. Grassmann was also responsible for introducing the scalar product. Cauchy and Jean Claude de Saint-Venant also created abstract algebraic structures at about this time. Still, these scholars developed algebras with the idea of modeling something. For Hamilton, quaternions were to give a better algebraic description of space and time, and for Grassmann the goal was geometric.

In 1857 Arthur Cayley introduced the idea of an algebra of matrices, and in 1858 he showed, in modern parlance, that quaternions could be "represented" by matrices. The goal of finding concrete realizations of abstract structures continues to this day to be a salient feature of abstract algebra, and we shall be concerned in this class to see how abstract operator algebras can similarly be represented.

In 1870, Camille Jordan published the full canonical-form analysis of matrices, which is a prototype for the decomposition of compact operators in the infinite-dimensional case.

The fully axiomatic treatment of linear spaces is due to Giuseppe Peano in his 1888 book, *Calcolo geometrico secondo l'Ausdehnungslehre di H. Grassmann preceduto dalle operazioni della logica deduttiva* . This is where you will find the theorem that every operator defined on a finite-dimensional vector space is a matrix. Peano defined the sum and product of linear operators abstractly, and at this stage operator theory began to take shape as progress in algebra merged with developments in analysis.

II. Operators in Early Analysis.

Leibniz was the first to think of the algebraic properties of the operations of calculus, for example by considering higher derivatives as successive operations we might write today as D^a f(x). Reportedly he attempted to understand the case where a might be negative or irrational.

Today, many branches of analysis are inseparable from operator theory, notably variational calculus, transform theory, and differential equations. Since all these subjects predated operator theory as such by a century or two, it is no surprise that some of the earliest antecedents of operator theory are to be found in them. Differential equations and variational calculus were largely the creation of Leonhard Euler, Joseph-Louis Lagrange, and the Bernoulli family. For example, we now realize that the technique of calculating the first variation of a functional is a kind of differentiation in a space of functions, and that a derivative in this context is a linear operator. While the early creators of variational calculus did not avail themselves of operators as abstractly conceived, they were implicitly using operators.

So it is with the transforms of Pierre-Simon Laplace, Joseph Fourier and others, which to this day remain some of the most remarkable and most studied kinds of operators on spaces of functions. Integral operators were also implicit in the work of the self-taught British matematician, George Green.

Fourier was a remarkable scientist (and revolutionary, civil engineer, Egyptologist, and politician), who is perhaps less well appreciated by mathematicians today than he should be. The folk history repeated by many mathematicians would have you believe that the contributions attributed to him were known earlier, and that he lacked "rigor." The latter charge, however, is unreasonable, because current standards of mathematical rigor are a creation of the late nineteenth century, under the influence of analysts such as Karl Weierstraß. Fourier's standards of rigor were those of the day. Moreover, when we read early scholars today, our understanding of the concepts they use is often quite different from theirs. In Fourier's day, a function was generally conceived of as a formula, and some of Fourier's contemporaries criticized him for thinking of functions more as we do today.

Although trigonometric expansions were certainly used before Fourier, he can be credited with many innovations, including:

- The Fourier transform, which is now arguably the most important example of a unitary operator on Hilbert space.
- The derivation and first solutions of the heat, or diffusion, equation.
- The invention of the modern symbol for the definite integral.

The most pertinent of Fourier's innovations for the theory of operators, all from his *Théorie de la Chaleur*, written from 1807 to 1822, are:

- The first explicit use of a differential operator, when he wrote D for the Laplacian and D^2 for its square [Cajori];
- the systematic expansion of functions in a basis, and
- the analysis of infinite systems of equations.

The earliest significant appearance of eigenvalues in connection with differential equations was in the theory developed by Charles François Sturm in 1836 and Joseph Liouville in 1838. This is important because, unlike the situation studied by Cauchy, the underlying space is infinite dimensional, which allows phenomena that do not arise in the finite-dimensional case of linear algebra. For example, infinite-dimensional operators can have continuous spectrum, as became evident (though not in that language) when George Hill presented the theory of periodic Sturm-Liouville equations in order to study the stability of the lunar orbit. In his analysis, Hill introduced infinite determinants.

Sturm-Liouville theory was the beginning of what we now refer to as the spectral theory of ordinary differential operators. In the late Nineteenth Century mathematicians were also concerned with the eigenvalues of partial differential operators, particularly the Laplace operator. The Dirichlet problem, named for Gustav Lejeune Dirichlet (the family name was Lejeune Dirichlet), was to find a solution of Laplace's equation with specified boundary conditions. Subtleties in this problem led mathematicians to a better and more rigorous understanding of convergence of sequences of functions and the nature of what are now termed partial differential operators. Today we recognize this as a a question of topology, as we familiarly treat functions as points in sets usually called *function spaces*, but until the latter part of the Nineteenth Century, this notion was lacking. Grassmann, in 1862 and Salvatore Pincherle seem to have been the first to write functions as abstract entities f, rather than f(x), i.e. as relations between domain and range values. The full idea of a function spaces is of the Twentieth Century, indeed it is the central notion of Twentieth Century analysis, and was influenced by attempts to understand the Dirichlet problem, Fourier series and transforms, and the work of Vito Volterra and Ivar Fredholm on integral equations.

One last Nineteenth Century influence deserving mention is the influence of Oliver Heaviside. Heaviside was a brilliant outsider who with little formal education made substantial contributions to the theory of electricity and magnetism, and between 1880 and 1887 created a systematic *operational calculus*, in which he boldly manipulated symbols, such as the differential operator d/dx, in novel ways. Although he developed efficient ways to solve differential equations, he was disdainful of mathematical rigor and had poor relations with the scholarly community. His influence on mathematics has been correspondingly mixed. In some respects his formal methods were ahead of their time, anticipating Twentieth Century developments such as pseudodifferential operators. On the other hand, the operational calculus can be ambiguous and can interfere with the understanding of important analytical issues. Heaviside's operational calculus has continued to have a following among engineers and scientists to this day, in isolation from modern mathematics, and this situation has been a barrier to good communication among practitioners of different disciplines.

III. Operator Theory in the First Half of the Twentieth Century.

The subjects of operator theory and its most important subset, spectral theory, came into focus rapidly after 1900. A major event was the appearance of Fredholm's theory of integral equations, which arose as a new approach to the Dirichlet problem. In a preliminary report based on his dissertation published in 1900 and a landmark article in *Acta Mathematica* in 1903, Fredholm gave a complete analysis of an important class of integral equations, now known as Fredholm equations. Notable achievements in this work were:

- The famous Fredholm alternative theorem, which extended a non-trivial result of linear algebra to a wide class of operators.
- A careful analysis of the convergence of a sequence of operators, as Fredholm approximated his equations with Riemann sums and passed to a limit.
- The definition of the determinant to a class of operators (greatly extending the innovation of Hill).
- The first use of the *resolvent* operator (although that term is due to Hilbert).

In 1902, in his dissertation, Lebesgue defined the modern form of the integral and introduced the most important spaces of functions, denoted in his honor L^p.

At about this time, Hilbert founded modern spectral theory in a series of articles inspired by Fredholm's work. (The word "spectrum" seems to have been adopted by Hilbert from an 1897 article by Wilhelm Wirtinger.) Hilbert began like Fredholm, with the specific idea of integral equations, and noticed that he could obtain more precise results when the space of functions considered was L^2, the square-integrable functions, and when the integral operator was symmetric. This was the discovery of Hilbert space and the founding of the general study of self-adjoint operators. In 1906, Hilbert freed his analysis from the connection with integral equations, and discovered the continuous spectrum, which had been present but not recognized in the work of Hill.

The concept of an algebra of operators made its appearance in series of articles culminating in a 1913 book by Frigyes Riesz, where Riesz studied the algebra of bounded operators on the Hilbert space l^2. Riesz representation, orthogonal projectors, and spectral integrals made their first appearance in this work. In 1916 Riesz created the theory of what he called "completely continuous" operators, now more familiarly *compact* operators. Since he wrote this in Hungarian, wide recognition came only two years later with a translation into German. Riesz's spectral theorem for compact operators made abstract, greatly extended, and largely supplanted Fredholm's work.

The definitive spectral theorem of self-adjoint, and more generally normal, operators, was the simultaneous discovery of Marshall Stone and John von Neumann in 1929-1932. Although Stone is more readable today, von Neumann's contributions are somewhat more far-reaching. One of von Neumann's motivations was quantum mechanics, which had been discovered in 1926 in two rather distinct forms by Erwin Schrödinger and Werner Heisenberg. It was von Neumann's insight that the natural language of quantum mechanics was that of self-adjoint operators on Hilbert space. This notion permeates modern physics. Von Neumann introduced or transformed many concepts now at the core of operator theory:

- domains of definition
- extension of operators
- closure of an operator
- adjoint operators
- unbounded operators

He also annihilated with examples the imprecise concept of infinite matrices that had been a popular way to understand operators.

The year 1932 saw the first text on operator theory, by Stefan Banach, in which geometric language was used throughout. Banach was responsible for:

- fixed-point theory
- an understanding of contractions
- the closed-graph theorem, and
- weak convergence.

In a series of articles from 1935, partly with F.J. Murray, von Neumann elaborated the theory of operator algebras, introduced by Riesz. It is this point of view that prevails in Arveson's book. They realized that the sets of operators that commutes with an algebra was an important tool of analysis and classification, and made many contributions to pure algebra as well as algebra.

The final seminal work that will be mentioned here is that of Israil Gel'fand, who in a 1941 article in *Matematicheskii Sbornik* extended thei spectral theorem to elements of normed algebras, and in the process introduced

- the spectral radius formula,
- C^* algebras (though not with that name), and
- the *character* of an algebra

Since Gel'fand's time operator theory has become an enormous branch of pure and applied mathematics, and further developments are beyond the scope of a brief historical sketch.

Bibliography

1. A.D. Alexandrov, A.N. Kolmogorov, and M.A. Lavrent'ev, eds., *Mathematics. Its Content, Methods, and Meaning*, in three volumes. Cambridge, Mass.: MIT Press, 1963 (original publication by Akademiya Nauk, 1956).
2. Florian Cajori, *A History of Mathematical Notations*, New York: Dover, 1993.
3. Jean Dieudonné, *History of Functional Analysis*, Amsterdam, New York, and Oxford: North-Holland, 1981
4. Felix Klein's *Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert*, New York: Chelsea, 1967.
5. The MacTutor History of Mathematics Archive