The Stone Weierstrass Theorem

Theorem 1 *Let* [1] *X be a compact Hausdorff space and let $C_{\mathbb{R}}(X)$ be the real algebra of all continuous functions $f : X \to \mathbb{R}$. Suppose*

$$\mathcal{A} \subseteq C_{\mathbb{R}}(X)$$

satisfies

(1) $\mathcal{A}$ is a subalgebra (i.e. closed under sums and products)

(2) $\mathcal{A}$ contains constants (i.e. $\mathbf{1} \in \mathcal{A}$)

(3) $\mathcal{A}$ separates points of X (i.e. $f(x) = f(y)$ for all $f \in \mathcal{A}$ implies $x = y$).

Then $\mathcal{A}$ is uniformly dense in $C_{\mathbb{R}}(X)$.

Proof (a) Let $\mathcal{B}$ be the $\|\cdot\|_\infty$-closure of $\mathcal{A}$. We have to prove that $\mathcal{B} = C_{\mathbb{R}}(X)$.

Note that $\mathcal{B}$ also satisfies (1),(2) and (3): Indeed (2) and (3) are obvious and (1) follows from the norm continuity of the algebraic operations.

(b) Claim: If $f \in \mathcal{B}$, then $|f| \in \mathcal{B}$.

(c) Claim: If $f, g \in \mathcal{B}$, then $f \wedge g \in \mathcal{B}$ and $f \vee g \in \mathcal{B}$. [2]

(A *closed* subalgebra of $C_{\mathbb{R}}(X)$ is a sublattice.)

(d) Given $x, y \in X$ and $s, t \in \mathbb{R}$ there exists $g \in \mathcal{B}$ such that $g(x) = s$ and $g(y) = t$.

Now fix $f \in C_{\mathbb{R}}(X)$ and $\epsilon > 0$. To find $g \in \mathcal{B}$ such that $\|f - g\|_\infty < \epsilon$, i.e.

$$\text{for all } \ z \in X, \quad f(z) - \epsilon < g(z) < f(z) + \epsilon.$$

By (d), given any pair $\{x, y\} \subseteq X$ we can find $g \in \mathcal{B}$ such that $g(x) = f(x)$ and $g(y) = f(y)$.

Compactness will allow *uniform approximation* on all of X, in two steps, first from above, then from below. For the first step, we keep the first equality and relax the second to a lower bound, but uniformly on all of X:

(e) Fix $x \in X$. There exists $g^x \in \mathcal{B}$ such that

$$g^x(x) = f(x) \quad \text{and for all } \ z \in X, \quad f(z) - \epsilon < g^x(z).$$

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[2] $(f \wedge g)(x) = \max\{f(x), g(x)\}$ and$(f \vee g)(x) = \min\{f(x), g(x)\}$.

In the second and final step, we find $g \in \mathcal{B}$ still satisfying the lower bound, and, instead of the first equality, an upper bound uniformly on all of X.

Proof of Claim (b): *If $f \in \mathcal{B}$, then $|f| \in \mathcal{B}$.*

Note that $f(X) \subseteq [a, b]$. Let $\phi : [a, b] \to \mathbb{R} : t \to |t|$. By Weierstrass, or Taylor (!) there is a sequence (p_n) of real polynomials such that $p_n \to \phi$ uniformly in $[a, b]$. Then $p_n \circ f \to \phi \circ f$ uniformly in X. Indeed given $\epsilon > 0$ there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $t \in [a, b]$ we have $|p_n(t) - \phi(t)| < \epsilon$, so for all $x \in X$ we have $|p_n(f(x)) - |f(x)|| < \epsilon$. But, since $p_n(t)$ is a linear combination of powers of t, the function $p_n \circ f$ is a linear combination of powers of f, hence $p_n \circ f \in \mathcal{B}$ since $\mathcal{B}$ is an algebra. Thus $|f| \in \mathcal{B}$ since $\mathcal{B}$ is closed. □

Proof of Claim (c) : *If $f, g \in \mathcal{B}$, then $f \wedge g \in \mathcal{B}$ and $f \vee g \in \mathcal{B}$.*

Indeed, since $\mathcal{B}$ is a linear space and $|f - g| \in \mathcal{B}$ from (b),

$$f \vee g = \frac{1}{2}(f + g + |f - g|) \in \mathcal{B}$$
$$f \wedge g = \frac{1}{2}(f + g - |f - g|) \in \mathcal{B}.$$

Proof of Claim (d): *Given $x, y \in X$ and $s, t \in \mathbb{R}$ there exists $f \in \mathcal{B}$ such that $f(x) = s$ and $f(y) = t$.*

Choose $f_1 \in \mathcal{B}$ such that $f_1(x) := s_0 \neq t_0 := f_1(y)$ (hypothesis (3)). Now find $a, b \in \mathbb{R}$ such that

$$as_0 + b = s \quad \text{and} \quad at_0 + b = t.$$

Then set $f = af_1 + b\mathbf{1} \in \mathcal{B}$ by (1) and (2). Now $f(x) = af_1(x) + b = as_0 + b = s$ and $f(y) = af_1(y) + b = at_0 + b = t$.

Proof of Claim (e): *Fix $x \in X$. There exists $g^x \in \mathcal{B}$ such that*

$$g^x(x) = f(x) \quad \textit{and for all } z \in X, \quad f(z) - \epsilon < g^x(z).$$

Let $y \in X$. Apply (d) to $s = f(x)$ and $t = f(y)$: You obtain $f_y \in \mathcal{B}$ which *interpolates* f exactly at x and y, i.e. $f_y(x) = f(x)$ and $f_y(y) = f(y)$.

The continuous function $f - f_y$ vanishes at y; so there is an open neighbourhood U_y of y such that

$$z \in U_y \Rightarrow f(z) - f_y(z) < \epsilon \Leftrightarrow f_y(z) > f(z) - \epsilon. \qquad (*)$$

The family $\{U_y : y \in X\}$ is an open cover of X; choose a finite subcover: $X = \bigcup_{i=1}^{n} U_{y_i}$ and let

$$g^x = f_{y_1} \vee f_{y_2} \vee \cdots \vee f_{y_n}.$$

Note that $g^x \in \mathcal{B}$ by Claim (c). We have $g^x(x) = f(x)$ since $f_{y_i}(x) = f(x)$ for each i. Also, each $z \in X$ is in some U_{y_i} and so $g^x(z) \geq f_{y_i}(z) > f(z) - \epsilon$ from (*). □

Conclusion of the proof: For each $x \in X$ the continuous function $g^x - f$ from (e) vanishes at x. So there is an open neighbourhood V_x of x such that

$$z \in V_x \;\Rightarrow\; g^x(z) - f(z) < \epsilon \;\Leftrightarrow\; g^x(z) < f(z) + \epsilon. \qquad (\dagger)$$

The family $\{V_x : x \in X\}$ is an open cover of X; choose a finite subcover so that $X = \bigcup_{j=1}^{m} V_{x_j}$ and let

$$g = g^{x_1} \wedge g^{x_2} \wedge \cdots \wedge g^{x_m}$$

Note that $g \in \mathcal{B}$ by Claim (c). From (e), each $g^{x_i}(z) > f(z) - \epsilon$ for all $z \in X$ so that $g(z) > f(z) - \epsilon$ for all $z \in X$. Also, each $z \in X$ is in some V_{x_j} and so $g(z) \le g^{x_j}(z) < f(z) + \epsilon$ from ($\dagger$). It follows that

$$\text{for all } z \in X, \quad f(z) - \epsilon < g(z) < f(z) + \epsilon. \qquad \square$$

The complex case. $C(X)$ is the complex algebra of all continuous functions $f : X \to \mathbb{C}$.

For $\mathcal{A} \subseteq C(X)$, assumptions (1) to (3) do not suffice to guarantee that $\mathcal{A}$ is dense in $C(X)$

Example. Let $X = \overline{\mathbb{D}}$ and let $\mathcal{A}$ be the algebra of all complex polynomials. It is an algebra, contains complex constants and separates points, because it contains p_1 where $p_1(z) = z$. But the continuous function f where $f(z) = \bar{z}$ cannot be approximated by polynomials uniformly in X. Indeed if there existed a sequence (p_n) of polynomials such that $p_n \to f$ uniformly, then we would have

$$\int_0^{2\pi} p_n(e^{it})e^{it}dt \to \int_0^{2\pi} f(e^{it})e^{it}dt.$$

However the left hand side is 0 (it is a linear combination of terms of the form $\int_0^{2\pi} e^{ikt}dt$, $k > 0$) and the right hand side is 2π.

Complex conjugation is exactly what is missing:

Theorem 2 *Let X be a compact Hausdorff space and let $C(X)$ be the complex algebra of all continuous functions $f : X \to \mathbb{C}$. Suppose*

$$\mathcal{A} \subseteq C(X)$$

satisfies

(1) $\mathcal{A}$ is a subalgebra (i.e. closed under sums and products)

(2) $\mathcal{A}$ contains constants (i.e. $\mathbf{1} \in \mathcal{A}$)

(3) $\mathcal{A}$ separates points of X (i.e. $f(x) = f(y)$ for all $f \in \mathcal{A}$ implies $x = y$)

(4) $\mathcal{A}$ is closed under complex conjugation (i.e. $f \in \mathcal{A} \Rightarrow \bar{f} \in \mathcal{A}$).

Then $\mathcal{A}$ is uniformly dense in $C(X)$.

Proof Let $\mathcal{C} = \{f \in \mathcal{A} : f(X) \subseteq \mathbb{R}\}$, considered as a subset of the real algebra $C_{\mathbb{R}}(X)$. This is clearly a subalgebra of $C_{\mathbb{R}}(X)$: if $f, g \in \mathcal{A}$ take real values, then $f+g, fg$ are in $\mathcal{A}$ and take real values. Also, $\mathcal{C}$ contains (real) constants, because $\mathcal{A}$ contains all constants.

Finally, $\mathcal{C}$ separates points of X. Indeed, if $x \neq y$, by (3) there exists $f \in \mathcal{A}$ so that $f(x) \neq f(y)$. Hence either $(\mathrm{Re}\, f)(x) \neq (\mathrm{Re}\, f)(y)$ or $(\mathrm{Im}\, f)(x) \neq (\mathrm{Im}\, f)(y)$. But $\mathrm{Re}\, f = \frac{1}{2}(f + \bar{f})$ and $\mathrm{Im}\, f = \frac{1}{2i}(f - \bar{f})$ are both in $\mathcal{C}$, because $\bar{f} \in \mathcal{A}$ by (4).

By Theorem 1, $\mathcal{C}$ is uniformly dense in $C_{\mathbb{R}}(X)$. So given $f \in C(X)$ and $\epsilon > 0$, since $\mathrm{Re}\, f, \mathrm{Im}\, f$ are in $C_{\mathbb{R}}(X)$, there are $g, h \in \mathcal{C}$ such that $\|\mathrm{Re}\, f - g\|_\infty < \epsilon$ and $\|\mathrm{Im}\, f - h\|_\infty < \epsilon$. Now $\phi := g + ih$ is in $\mathcal{A}$ and $\|f - \phi\|_\infty < 2\epsilon$.

Sample applications. (i) Let $X = \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. The set $\mathcal{A}$ of *trigonometric polynomials*, i.e. linear combinations of the functions $e_k(z) = z^k$, satisfies the hypotheses of Theorem 4 (to satisfy hypothesis (4), we need to take all integer values of k).

Conclusion: any continuous function on $\mathbb{T}$ can be approximated uniformly by trigonometric polynomials.

(ii) Let $X \subseteq \mathbb{R}^2$ be any compact nonempty set. The following two sets of functions on X satisfy the hypotheses of Theorem 4 and are therefore uniformly dense in $C(X)$:

$\mathcal{A}_1$: linear combinations of functions h of the form $h(s,t) = f(s)g(t)$ where f and g are continuous functions on $\mathbb{R}$ (or suitable subsets of $\mathbb{R}$).

$\mathcal{A}_2$: polynomials of two variables.

(iii) (variation of (ii) Let $X \subseteq \mathbb{C}$ be any compact nonempty set, and $\mathcal{A}$ the set of all polynomials in z and $\bar{z}$. Then $\mathcal{A}$ is uniformly dense in $C(X)$ (we noted that polynomials in z do not suffice).

(iv) Let X be the direct (Cartesian) product of countably many copies of $\mathbb{T}$. This is a compact space (in fact a compact group with coordinate-wise operations) in the product topology (or any 'metrikh ginomeno'). For any $i \in \mathbb{N}$, let $e_i : X \to \mathbb{C}$ be the i-th coordinate function, $e_i(z_1, z_2, \dots) = z_i$. Let $\mathcal{A}$ be the set of all linear combination of products

$$e_{i_1}^{n_1} e_{i_2}^{n_2} \dots e_{i_m}^{n_m}$$

where $n_k \in \mathbb{Z}$ and $m \in \mathbb{N}$. The set $\mathcal{E}$ of all such (finite) products is closed under multiplication and under complex conjugation and contains the constant function $\mathbf{1}$. Therefore its linear span $\mathcal{A}$ is an algebra containing constants and closed under complex conjugation. Finally, $\mathcal{E}$ separates points of X. Indeed, if $z = (z_1, z_2, \dots) \neq w = (w_1, w_2, \dots)$ then there exists $i \in \mathbb{N}$ such that $z_i \neq w_i$ and then $e_i(z) \neq e_i(w)$. It follows that $\mathcal{A}$ also separates points of X.

Therefore any continuous function on the infinite product X can be uniformly approximated by elements of $\mathcal{A}$, each of which depends on finitely many coordinates.

The locally compact case. A Hausdorff topological space is *locally compact* if every point has a compact neighbourhood (example: $(\mathbb{R}^n, \|\cdot\|_2)$ but not $(\ell^2, \|\cdot\|_2)$). Continuous functions need not be bounded (ex: $f(t) = t$ on $\mathbb{R}$). A continuous function $f : X \to \mathbb{C}$ on a locally compact space X is said to *vanish at infinity* if given $\epsilon > 0$ there is a compact subset $K \subseteq X$ such that $|f(x)| < \epsilon$ for all $x \notin K$. Such a function is necessarily bounded.

The set $C_0(X)$ of all continuous functions $f : X \to \mathbb{C}$ which vanish at infinity, equipped with the supremum norm, becomes a complete normed algebra. When X is not compact, $C_0(X)$ cannot contain nonzero constants; they don't vanish at infinity. However it can be shown that for every $x \in X$ there exists $f \in C_0(X)$ such that $f(x) \neq 0$.

Theorem 3 *Let X be a locally compact Hausdorff space. Suppose*

$$\mathcal{A} \subseteq C_0(X)$$

satisfies

(1) $\mathcal{A}$ is a subalgebra (i.e. closed under sums and products)

(2) $\mathcal{A}$ vanishes at no point of X (i.e. for all $x \in X$ there exists $f \in \mathcal{A}$ such that $f(x) \neq 0$)

(3) $\mathcal{A}$ separates points of X (i.e. $f(x) = f(y)$ for all $f \in \mathcal{A}$ implies $x = y$)

(4) $\mathcal{A}$ is closed under complex conjugation (i.e. $f \in \mathcal{A} \Rightarrow \bar{f} \in \mathcal{A}$).

Then $\mathcal{A}$ is uniformly dense in $C_0(X)$.