605: Exercises VI

1. (α) Let $f \in \mathcal{L}^1([-\pi, \pi])$ with $f(t) \in \mathbb{R}$ for all $t \in [-\pi, \pi]$. Show that $\hat{f}(-k) = \overline{f(k)}$ for all $k \in \mathbb{Z}$.

   (β) Let $f \in \mathcal{L}^1([-\pi, \pi])$ with $\hat{f}(-k) = \overline{f(k)}$ for all $k \in \mathbb{Z}$. Is it true that $f(t) \in \mathbb{R}$ for all $t \in [-\pi, \pi]$; For almost all $t \in [-\pi, \pi]$;

2. If $g \in L^1([-\pi, \pi])$ and $m \in \mathbb{N}$, find the Fourier coefficients of the function $f(t) = g(mt)$ in terms of the Fourier coefficients of g.

3. (α) Using the function $f : [-\pi, \pi] \to \mathbb{R}$ given by $f(x) = |x|$ and Parseval's identity, show that

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^4} = \frac{\pi^4}{96} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}.$$

   (β) Using the 2π-periodic odd function $g : [-\pi, \pi] \to \mathbb{R}$ given by $g(x) = x(\pi - x)$ στο $[0, \pi]$ and Parseval's identity, show that

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^6} = \frac{\pi^6}{960} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{k^6} = \frac{\pi^6}{945}.$$

4. Let $f : [-\pi, \pi] \to \mathbb{R}$ be a continuously differentiable function with $f(-\pi) = f(\pi)$.

   (α) Show that there is a constant $C(f) > 0$ so that $|k\hat{f}(k)| \leq C(f)$ for all $k \in \mathbb{Z}$.

   (β) Examine whether $\lim\limits_{|k|\to\infty} |k\hat{f}(k)| = 0$.

   (γ) Examine whether $\sum\limits_{k=-\infty}^{\infty} |f(k)| < +\infty$.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function which is 2π-periodic. If $\int_{-\pi}^{\pi} f(t)dt = 0$, show using Parseval's identity for f and f' that

$$\int_{-\pi}^{\pi} |f(t)|^2 dt \leq \int_{-\pi}^{\pi} |f'(t)|^2 dt.$$

   Show also that equality holds if and only if there are $a, b \in \mathbb{R}$ so that $f(t) = a\cos t + b \sin t$.

6. Let $f \in \mathcal{L}^1([-\pi, \pi])$ be even and bounded with $a_n(f) \geq 0$ for all $n \in \mathbb{Z}_+$.

   Show that $\sum\limits_{n=0}^{\infty} a_k(f) < \infty$.

   What can you conclude about the convergence of the Fourier series of f;

   Conclude that f is almost everywhere equal to a continuous function.