N. N. Luzin's Theorem

Theorem 1 *If $X \in \mathcal{M}, \lambda(X) < \infty$ and $f : X \to \mathbb{R}$ measurable, then for every $\epsilon > 0$ there exists a closed set $F_\epsilon \subseteq X$ with $\lambda(X \backslash F_\epsilon) < \epsilon$ such that $f|_{F_\epsilon}$ is continuous.* [1]

Remark 1 *It is not claimed that the function f is continuous at every $x \in F_\epsilon$ (for example, the characteristic function of $\mathbb{Q} \cap [0, 1]$ is nowhere continuous).*
The claim is that the function $f_\epsilon : F_\epsilon \to \mathbb{R}$ defined by $f_\epsilon(x) = f(x)$ for $x \in F_\epsilon$ is continuous on the space F_ϵ. In other words, for every $x \in F_\epsilon$ and every open neighbourhood $V \subseteq \mathbb{R}$ of $f(x)$, there exists an open neighbourhood W of x so that if $y \in W$ and *$y \in F_\epsilon$ then $f_\epsilon(y) \in V$.*

Before the full proof, we treat an easier special case: We will assume that f is the characteristic function of a measurable set $E \subseteq X$.

Given $\epsilon > 0$, since $X \in \mathcal{M}$ and $\lambda(X) < \infty$, we may restrict to a closed subset $X_\epsilon \subseteq X$ with $\lambda(X) - \lambda(X_\epsilon) < \frac{\epsilon}{2}$.

There exists a closed set F and an open set G so that

$$F \subseteq E \cap X_\epsilon \subseteq G \quad \text{and} \quad \lambda(F \backslash G) < \frac{\epsilon}{2}\,.$$

The required set is

$$F_\epsilon := F \cup (X_\epsilon \backslash G)\,.$$

This is closed, since F and X_ϵ are closed and G is open. [2]
Also, $\lambda(X \backslash F_\epsilon) \leq \lambda(X \backslash X_\epsilon) + \lambda(X_\epsilon \backslash F_\epsilon) < \epsilon$.

Let us show that $f_\epsilon := f|_{F_\epsilon}$ is continuous. For $x \in F_\epsilon$, let (x_n) be a sequence *of elements of* F_ϵ such that $x_n \to x$. We show that $f(x_n) \to f(x)$.

There are two cases: either $x \in F$ or $x \in X_\epsilon \backslash G$.

• If $x \in F$ then $x \in G$, an open set, so since $x_n \to x$ there is $n_0 \in \mathbb{N}$ such that $x_n \in G$ for all $n \geq n_0$. But since $x_n \in F_\epsilon = F \cup (X_\epsilon \backslash G)$, this forces $x_n \in F$, hence $f(x_n) = 1$ (since $x \in F \subseteq E$) for all $n \geq n_0$ and so $f(x_n) \to 1 = f(x)$.

• If $x \in X_\epsilon \backslash G$ then $x \in F^c$, an open set, so since $x_n \to x$ there is $n_0 \in \mathbb{N}$ such that $x_n \in F^c$ for all $n \geq n_0$. But since $x_n \in F_\epsilon$, this forces $x_n \in X_\epsilon \backslash G$, hence $x_n \in E^c$ and so $f(x_n) = 0$ for all $n \geq n_0$ and so $f(x_n) \to 0 = f(x)$. □

This argument can be continued to yield a proof for the case where f is a simple measurable function, and then to the general case, using the fact that f is a limit of a sequence of simple measurable functions (see Ap. Giannopoulos' notes).

We give an alternative proof of the general case:

Proof of the Theorem
Since $X \in \mathcal{M}$ and $\lambda(X) < \infty$ there is a closed $X_\epsilon \subseteq X$ with $\lambda(X) - \lambda(X_\epsilon) < \frac{\epsilon}{2}$.

Let $\{V_n : n \in \mathbb{N}\}$ be an enumeration of the open intervals in $\mathbb{R}$ with rational endpoints.

For each $n \in \mathbb{N}$ define

$$B_n := f^{-1}(V_n) \cap X_\epsilon = \{x \in X_\epsilon : f(x) \in V_n\}\,.$$

[1] Σύντομα και στα Ελληνικά...

[2] If $E = \mathbb{Q} \cap [0, 1]$, F can be chosen to be the empty set, and G is the union of intervals of very small length around each element of E; in this case, $f|_{F_\epsilon} = 0$, hence it is trivially continuous...

Note $B_n \in \mathcal{M}$.

By regularity , there exist a compact F_n and an open G_n with

$$F_n \subseteq B_n \subseteq G_n \quad \text{and} \quad \lambda(G_n \backslash F_n) < \frac{\epsilon}{2^{n+1}} \,.$$

Define

$$W := \bigcup_{n=1}^{\infty} (G_n \backslash F_n)$$

(the "bad part"). Note that W is an open set and

$$\lambda(W) \leq \sum_n \lambda(G_n \backslash F_n) \leq \frac{\epsilon}{2} \,.$$

Put $F_\epsilon = X_\epsilon \backslash W$. This is a closed subset of X. Also, $\lambda(X \backslash F_\epsilon) \leq \lambda(X \backslash X_\epsilon) + \lambda(X_\epsilon \backslash F_\epsilon) < \epsilon$.

We will prove that $f|_{F_\epsilon}$ is continuous. Let $x \in F_\epsilon$ and $\eta > 0$. We will show that there is an open neighbourhood G of x such that for all $y \in G$ with $y \in F_\epsilon$ we have

$$|f(x) - f(y)| < \eta \,.$$

Proof. Since $f(x) - \eta < f(x) < f(x) + \eta$, there are rationals a, b with
$f(x) - \eta < a < f(x) < b < f(x) + \eta$. The interval (a, b) is V_n for some $n \in \mathbb{N}$, so for this n,

$$f(x) \in V_n = (a, b) \subseteq (f(x) - \eta, f(x) + \eta) \,.$$

Then $x \in f^{-1}(V_n) \cap X_\epsilon = B_n$ so $x \in G_n$. Thus G_n is an open neighbourhood of x. For each $y \in G_n$ with $y \in F_\epsilon$ we have $y \in G_n \cap F_\epsilon$. But since $G_n \backslash F_n \subset W$ and $W \cap F_\epsilon = \emptyset$, we have $(G_n \backslash F_n) \cap F_\epsilon = \emptyset$ and so $y \in F_n \cap F_\epsilon \subseteq B_n \cap F_\epsilon$. Thus $f(y) \in V_n \subseteq (f(x) - \eta, f(x) + \eta)$, which shows that

$$|f(x) - f(y)| < \eta \,.$$