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A Short Solution to the Busemann-Petty Problem

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Abstract. A unified analytic solution to the Busemann-Petty problem was recently found by Gardner, Koldobsky and Schlumprecht. We give an elementary proof of their formulas for the inverse Radon transform of the radial function ρ_K of an origin-symmetric star body K.

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The 1956 Busemann-Petty problem asks the following question: suppose that K and L are origin-symmetric convex bodies in $\mathbb{R}^n$ such that

$$\mathrm{vol}_{n-1}(K \cap H) \leqslant \mathrm{vol}_{n-1}(L \cap H)$$

for every hyperplane H containing the origin; does it follow that

$$\mathrm{vol}_n(K) \leqslant \mathrm{vol}_n(L)?$$

The problem has a long and dramatic history. A negative answer to the problem for $n \geqslant 5$ was established in a series of papers by Larman and Rogers [8] (for $n \geqslant 12$), Ball [1] ($n \geqslant 10$), Giannopoulos [5] and Bourgain [2] (independently; $n \geqslant 7$), Gardner [3] and Papadimitrakis [10] (independently; $n \geqslant 5$). Gardner [4] proved that the answer to the Busemann-Petty problem is affirmative when $n = 3$. A negative answer in the case $n = 4$ was claimed in 1994, but three years later the main argument of that proof was shown to be wrong (for details, see Koldobsky [7]). After that, Zhang [12] showed that the answer is affirmative when $n = 4$, and, a little later, a unified solution to the problem was given by Gardner, Koldobsky and Schlumprecht in [6].

The principal objective of this paper is to present an elementary proof for the main positive result, namely the solution of the Busemann-Petty problem in four dimensions. Gardner, Koldobsky and Schlumprecht proved in [6] that the radial function ρ_K of a smooth symmetric convex body K in $\mathbb{R}^4$ is the Radon transform of an explicit non negative function (see below); according to the 1988 result of

Lutwak [9], the positive solution of the Busemann-Petty problem in $\mathbb{R}^4$ follows. In this paper, we give an elementary proof for the result in [6] about the Radon transform in $\mathbb{R}^4$. Our proof of the four dimensional case extends to an elementary proof of their formulas for the inverse Radon transform of ρ_K in every even dimension, and to a relatively elementary proof in odd dimensions.

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On $\mathbb{R}^n$, we denote the scalar product by $\langle \cdot, \cdot \rangle$ and the Euclidean norm by $| \cdot |$. We write B^n for the unit ball and S^{n-1} for the unit sphere, and v_n, s_{n-1} denote their respective volumes. If $K \subset \mathbb{R}^n$ is a star body, its radial function ρ_K is defined for every $x \in S^{n-1}$ by

$$\rho_K(x) = \sup\{\lambda > 0 ;\ \lambda x \in K\}.$$

The connection between the Busemann-Petty problem and the spherical Radon transform R is due to Lutvak [9]. Recall that R acts on the space of continuous functions on S^{n-1} by setting

$$Rf(\xi) = \int_{S^{n-1}\cap\xi^\perp} f(u)\, d\sigma_{n-2}(u)$$

for every $\xi \in S^{n-1}$; here σ_{n-2} is the Haar measure of total mass s_{n-2} on principal $n-2$ spheres. It follows from Lutvak [9], Zhang [11], that the Busemann-Petty problem has a positive answer in $\mathbb{R}^n$ if and only if every symmetric convex body K in $\mathbb{R}^n$, with positive curvature and C^∞ radial function, is such that $R^{-1}\rho_K$ is a non-negative function. In [6], the authors express $R^{-1}\rho_K$ in terms of

$$A_\xi(t) = \mathrm{Vol}_{n-1}(K \cap (t\xi + \xi^\perp)), \quad \xi \in S^{n-1}$$

as follows:

THEOREM . *Let $n \geqslant 3$. Let $K \subset \mathbb{R}^n$ be an origin-symmetric star body, with C^∞ radial function ρ_K.*

If n is even, then

$$(-1)^{\frac{n-2}{2}} 2^n \pi^{n-2} \rho_K = R\left(\xi \mapsto A_\xi^{(n-2)}(0)\right).$$

If n is odd, then

$$\frac{(-1)^{\frac{n-1}{2}}(2\pi)^{n-1}}{(n-2)!}\rho_K = R\left(\xi \mapsto \int_0^\infty t^{-n+1}\Big(A_\xi(t) - \sum_{k=0}^{\frac{n-3}{2}} A_\xi^{(2k)}(0)\frac{t^{2k}}{(2k)!}\Big)dt\right).$$

REMARK . Let us recall why this solves the case $n = 4$ of the Busemann-Petty problem ([12], [6]). If $n = 4$, then $R^{-1}\rho_K(\xi) = -A_\xi''(0)/16\pi^2$. If K is convex and symmetric, the latter is non-negative (by Brunn-Minkowski, the largest hyperplane section orthogonal to ξ is indeed the one through the origin).

Proof. We first compute the Radon transform of $\xi \to A_\xi(t)$, for any given $t \geqslant 0$. Let $e \in S^{n-1}$ and set $f(t) := R(\xi \mapsto A_\xi(t))(e)$. We identify $e^\perp$ and $\mathbb{R}^{n-1}$, and for $y \in \mathbb{R}^{n-1}$, we set $\phi(y) = \mathrm{Vol}_1(K \cap (y + \mathbb{R}e))$. Then

$$\begin{aligned} f(t) &= \int_{S_{n-1}\cap e^\perp} \int_{x\in\mathbb{R}^n,\ \langle x,\xi\rangle = t} \mathbf{1}_K(x)\, d^{n-1}(x)\, d\sigma_{n-2}(\xi) \\ &= \int_{S_{n-1}\cap e^\perp} \int_{y\in e^\perp,\ \langle y,\xi\rangle = t} \phi(y)\, d^{n-2}(y)\, d\sigma_{n-2}(\xi). \end{aligned}$$

Considered as a function of g, the quantity

$$\int_{S_{n-1}\cap e^\perp} \int_{y\in e^\perp,\ \langle y,\xi\rangle = t} g(y)\, d^{n-2}(y)\, d\sigma_{n-2}(\xi)$$

(where g is defined on $e^\perp \simeq \mathbb{R}^{n-1}$) is linear, continuous and rotation invariant. Hence there exists a measure μ_t on $\mathbb{R}^+$ such that for all g the previous expression is equal to

$$\int_{\mathbb{R}^+} \left(\int_{S^{n-2}} g(ru)\, d\sigma_{n-2}(u) \right) d\mu_t(r).$$

Applying the definition of μ_t with the function $g = \mathbf{1}_{rB^{n-1}}$ yields

$$\begin{aligned} s_{n-2}\ \mu_t([0,r]) &= \int_{S^{n-2}} \int_{\langle y,\xi\rangle = t} \mathbf{1}_{rB^{n-1}}(y)\, d^{n-2}(y)\, d\sigma_{n-2}(\xi) \\ &= s_{n-2}\ v_{n-2}\ \mathbf{1}_{\{t\leqslant r\}}(r^2 - t^2)^{\frac{n-2}{2}}. \end{aligned}$$

Consequently, $d\mu_t(r) = s_{n-3}\ r(r^2 - t^2)^{n-4/2}\mathbf{1}_{\{t\leqslant r\}}\, dr$. Thus we have proved that

$$f(t) = s_{n-3} \int_t^\infty r(r^2 - t^2)^{\frac{n-4}{2}}\, \Phi(r)\, dr,$$

where Φ is defined on $\mathbb{R}$ by

$$\Phi(x) = \int_{S^{n-2}} \phi(xu)\, d\sigma_{n-2}(u).$$

Notice that Φ is even, compactly supported and C^∞ in some neighborhood of the origin. Our aim now is to relate $f(t)$ and $\Phi(0) = 2\rho_K(e)s_{n-2}$. The case $n = 4$ is very simple: $f(t) = 2\pi \int_t^\infty r\Phi(r)dr$, hence $f''(0) = -2\pi\Phi(0) = -16\pi^2\rho_K(e)$. By exchanging the order of the Radon transform and the derivative, we conclude that ρ_K is the Radon transform of $\xi \mapsto -A''_\xi(0)/16\pi^2$.

If n is even:

$$\frac{f(t)}{s_{n-3}} = \int_0^\infty r(r^2 - t^2)^{\frac{n-4}{2}}\, \Phi(r)\, dr - t^{n-2} \int_0^1 u(u^2 - 1)^{\frac{n-4}{2}}\, \Phi(tu)\, du.$$

The first term is a polynomial in t, of degree $n-4$ and Φ is C^∞ in some neighborhood of 0, thus

$$f^{(n-2)}(0) = -s_{n-3}(n-2)! \int_0^1 u(u^2-1)^{\frac{n-4}{2}} \Phi(0)\, du = (-1)^{\frac{n-2}{2}} 2^n \pi^{n-2} \rho_K(e).$$

We conclude by exchanging the order of the Radon transform and the derivative.

If n is odd: the basic principle is still very simple, but the technical details are slightly unpleasant. We shall begin by writing the proof as if Φ were C^∞ on $\mathbb{R}$; but this is not true, because there are points of $e^\perp$ where our initial function ϕ is not differentiable, for example the points of the boundary of the projection of K on $e^\perp$; we shall indicate afterwards the standard approximation argument that fixes this difficulty. Integrating by parts, we get

$$F(t) := -\frac{n-2}{s_{n-3}} f(t) = \int_t^\infty (r^2-t^2)^{\frac{n-2}{2}} \Phi'(r)\, dr.$$

For $k \geqslant 0$, let $a_k = (-1)^k \binom{\frac{n-2}{2}}{k} = \frac{(-1)^k}{k!} \prod_{j=0}^{k-1} (\frac{n-2}{2} - j)$. Notice that $\sum |a_k| < \infty$. Let

$$P(t) = \sum_{k=0}^{\frac{n-3}{2}} a_k t^{2k} \int_0^\infty r^{n-2-2k} \Phi'(r)\, dr.$$

Then the quantity $\frac{F(t)-P(t)}{t^{n-1}}$ is equal to

$$\int_t^\infty \Big(\sum_{k=\frac{n-1}{2}}^{\infty} a_k \big(t^{-1}r\big)^{n-2-2k} \Big) \Phi'(r) \frac{dr}{t} - \int_0^t \Big(\sum_{k=0}^{\frac{n-3}{2}} a_k \big(t^{-1}r\big)^{n-2-2k} \Big) \Phi'(r) \frac{dr}{t}$$

$$= \int_1^\infty \Big(\sum_{k=\frac{n-1}{2}}^{\infty} a_k u^{n-2-2k} \Big) \Phi'(tu)\, du - \int_0^1 \Big(\sum_{k=0}^{\frac{n-3}{2}} a_k u^{n-2-2k} \Big) \Phi'(tu)\, du.$$

By Fubini's theorem and since $\int_0^\infty \Phi'(tu)\, dt = -\Phi(0)/u$, we get

$$\int_0^\infty \frac{F(t)-P(t)}{t^{n-1}}\, dt = \Phi(0) \Big(\sum_{k=0}^{\infty} \frac{a_k}{n-2-2k} \Big) = c_n \rho_K(e),$$

which is finite. Thus, P is the Taylor polynomial of F of order $n-3$ at zero, and the above integral represents the action of the distribution t_+^{-n+1} on F. We obtain therefore

$$\langle t_+^{-n+1},\ R(\xi \to A_\xi(t))(e) \rangle = -c_n \frac{s_{n-3}}{n-2} \rho_K(e).$$

A soft manner to compute c_n is to replace Φ by $G(x) = \mathrm{e}^{-x^2}$ in the previous computation. Once again, we end the proof by exchanging the order in which the

Radon transform and the distribution t_+^{-n+1} act (we shall give some explanation about this at the end).

We now explain how to deal with the fact that Φ is not C^∞ everywhere. To every continuous and even function Φ_1 on $\mathbb{R}$, which is C^∞ in a neighborhood of 0 and supported on a fixed interval $[-R, R]$ containing the support of Φ, we associate the even function F_1 on $\mathbb{R}$ defined for $t \geqslant 0$ by

$$F_1(t) := -(n-2)\int_t^\infty r(r^2 - t^2)^{\frac{n-4}{2}}\Phi_1(r)\, dr.$$

Let $Q(u)$ be the Taylor polynomial of degree $n-3$ for $(1-u^2)^{(n-4)/2}$ at the origin, and let $P_1(t) := -(n-2)\int_0^\infty r^{n-3}Q(t/r)\Phi_1(r)\, dr$ (of course, $F_1 = F$ and $P_1 = P$ when $\Phi_1 = \Phi$). One can get easily the following estimates (where $C(n, R)$ or $C(a, n, R)$ denote constants depending only upon n, R or a, n, R):

– first, $\|F_1\|_\infty \leqslant R^{n-2}\,\|\Phi_1\|_\infty$;

– for every t, we have $|P_1(t)| \leqslant C(n, R)\,(1 + |t|^{n-3})\|\Phi_1\|_\infty$;

– finally, when Φ_1 vanishes on some neighborhood $(-a, a)$ of 0, one can see that $|F_1(t) - P_1(t)| \leqslant C(a, n, R)\, t^{n-1}\|\Phi_1\|_\infty$ for $0 \leqslant t \leqslant 1$.

These three estimates imply that the integral $\int_0^\infty t^{-n+1}(F_1(t) - P_1(t))\, dt$ converges to $\int_0^\infty t^{-n+1}(F(t) - P(t))\, dt$ when we let Φ_1, equal to Φ on a fixed interval $[-a, a]$ and supported on $[-R, R]$, tend uniformly to Φ.

Let us turn finally to the interchange of the actions of the Radon transform and the distribution t_+^{-n+1} on the function $(\xi, t) \to A_\xi(t)$. It follows from our hypothesis that this function is C^∞ on $S^{n-1} \times (-a, a)$ for some $a > 0$. Let us assume $n = 5$ for example. Since K is symmetric, we may write

$$A_\xi(t) = f_0(\xi) + t^2 f_2(\xi) + t^4 g(\xi, t)$$

where f_0, f_2 and g are continuous and bounded on S^{n-1} and $S^{n-1} \times \mathbb{R}$ respectively. Since A_ξ vanishes for $|t| > R$, we have $g(\xi, t) = -t^{-4} f_0(\xi) - t^{-2} f_2(\xi)$ for $t > R$, and

$$\langle A_\xi, t_+^{-4}\rangle = \int_0^R g(\xi, t)\, dt - \frac{R^{-3}}{3} f_0(\xi) - R^{-1} f_2(\xi),$$

which shows that the interversion with the integral over $\xi \in S^{n-1}$ causes no trouble. □

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