27. Sobolev Inequalities

27.1. Morrey's Inequality.

Notation 27.1. Let S^{d-1} be the sphere of radius one centered at zero inside $\mathbb{R}^d$. For a set $\Gamma \subset S^{d-1}$, $x \in \mathbb{R}^d$, and $r \in (0, \infty)$, let

$$\Gamma_{x,r} \equiv \{x + s\omega : \omega \in \Gamma \text{ such that } 0 \le s \le r\}.$$

So $\Gamma_{x,r} = x + \Gamma_{0,r}$ where $\Gamma_{0,r}$ is a cone based on Γ, see Figure 49 below.

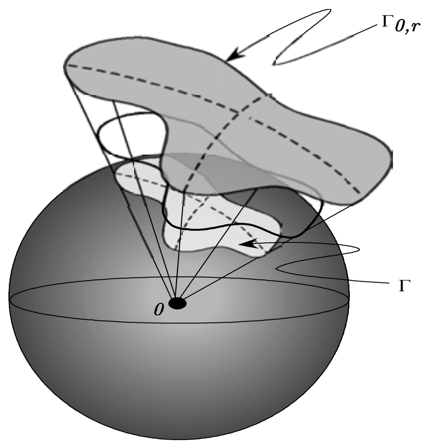

Figure 49. The cone $\Gamma_{0,r}$.

Notation 27.2. If $\Gamma \subset S^{d-1}$ is a measurable set let $|\Gamma| = \sigma(\Gamma)$ be the surface "area" of Γ.

Notation 27.3. If $\Omega \subset \mathbb{R}^d$ is a measurable set and $f : \mathbb{R}^d \to \mathbb{C}$ is a measurable function let

$$f_\Omega := \fint_\Omega f(x)dx := \frac{1}{m(\Omega)} \int_\Omega f(x)dx.$$

By Theorem 8.35,

$$\int_{\Gamma_{x,r}} f(y)dy = \int_{\Gamma_{0,r}} f(x+y)dy = \int_0^r dt\, t^{d-1} \int_\Gamma f(x+t\omega)\, d\sigma(\omega) \tag{27.1}$$

and letting $f = 1$ in this equation implies

$$m(\Gamma_{x,r}) = |\Gamma|\, r^d/d. \tag{27.2}$$

Lemma 27.4. *Let $\Gamma \subset S^{d-1}$ be a measurable set such that $|\Gamma| > 0$. For $u \in C^1(\overline{\Gamma}_{x,r})$,*

$$\fint_{\Gamma_{x,r}} |u(y) - u(x)|dy \le \frac{1}{|\Gamma|} \int_{\Gamma_{x,r}} \frac{|\nabla u(y)|}{|x-y|^{d-1}}\, dy. \tag{27.3}$$

Proof. Write $y = x + s\omega$ with $\omega \in S^{d-1}$, then by the fundamental theorem of calculus,

$$u(x+s\omega) - u(x) = \int_0^s \nabla u(x+t\omega)\cdot\omega\, dt$$

and therefore,

$$\begin{aligned}\int_\Gamma |u(x+s\omega)-u(x)|d\sigma(\omega) &\le \int_0^s \int_\Gamma |\nabla u(x+t\omega)|d\sigma(\omega)dt \\ &= \int_0^s t^{d-1}dt \int_\Gamma \frac{|\nabla u(x+t\omega)|}{|x+t\omega-x|^{d-1}} d\sigma(\omega) \\ &= \int_{\Gamma_{x,s}} \frac{|\nabla u(y)|}{|y-x|^{d-1}}\, dy \le \int_{\Gamma_{x,r}} \frac{|\nabla u(y)|}{|x-y|^{d-1}}\, dy,\end{aligned}$$

wherein the second equality we have used Eq. (27.1). Multiplying this inequality by s^{d-1} and integrating on $s \in [0,r]$ gives

$$\int_{\Gamma_{x,r}} |u(y)-u(x)|dy \le \frac{r^d}{d}\int_{\Gamma_{x,r}} \frac{|\nabla u(y)|}{|x-y|^{d-1}}\, dy = \frac{m(\Gamma_{x,r})}{|\Gamma|}\int_{\Gamma_{x,r}} \frac{|\nabla u(y)|}{|x-y|^{d-1}}\, dy$$

which proves Eq. (27.3). ■

Corollary 27.5. *Suppose $d < p \le \infty$, $\Gamma \in \mathcal{B}_{S^{d-1}}$ such that $|\Gamma| > 0$, $r \in (0,\infty)$ and $u \in C^1(\overline{\Gamma}_{x,r})$. Then*

$$|u(x)| \le C(|\Gamma|, r, d, p)\ \|u\|_{W^{1,p}(\Gamma_{x,r})} \tag{27.4}$$

where

$$C(|\Gamma|, r, d, p) := \frac{1}{|\Gamma|^{1/p}} \max\left(\frac{d^{-1/p}}{r}, \left(\frac{p-1}{p-d}\right)^{1-1/p}\right)\cdot r^{1-d/p}.$$

Proof. For $y \in \Gamma_{x,r}$,

$$|u(x)| \le |u(y)| + |u(y)-u(x)|$$

and hence using Eq. (27.3) and Hölder's inequality,

$$\begin{aligned}|u(x)| &\le ⨍_{\Gamma_{x,r}} |u(y)|dy + \frac{1}{|\Gamma|}\int_{\Gamma_{x,r}} \frac{|\nabla u(y)|}{|x-y|^{d-1}}\, dy \\ &\le \frac{1}{m(\Gamma_{x,r})}\|u\|_{L^p(\Gamma_{x,r})}\|1\|_{L^p(\Gamma_{x,r})} + \frac{1}{|\Gamma|}\|\nabla u\|_{L^p(\Gamma_{x,r})}\|\frac{1}{|x-\cdot|^{d-1}}\|_{L^q(\Gamma_{x,r})} \qquad (27.5)\end{aligned}$$

where $q = \frac{p}{p-1}$ as before. Now

$$\begin{aligned}\|\frac{1}{|\cdot|^{d-1}}\|^q_{L^q(\Gamma_{0,r})} &= \int_0^r dt\, t^{d-1}\int_\Gamma \left(t^{d-1}\right)^{-q}\ d\sigma(\omega) \\ &= |\Gamma|\int_0^r dt\ \left(t^{d-1}\right)^{1-\frac{p}{p-1}} = |\Gamma|\int_0^r dt\, t^{-\frac{d-1}{p-1}}\end{aligned}$$

and since

$$1 - \frac{d-1}{p-1} = \frac{p-d}{p-1}$$

we find

$$\left\| \frac{1}{|\cdot|^{d-1}} \right\|_{L^q(\Gamma_{0,r})} = \left(\frac{p-1}{p-d} |\Gamma| \, r^{\frac{p-d}{p-1}} \right)^{1/q} = \left(\frac{p-1}{p-d} |\Gamma| \right)^{\frac{p-1}{p}} r^{1-\frac{d}{p}}. \tag{27.6}$$

Combining Eqs. (27.5), Eq. (27.6) along with the identity,

$$\frac{1}{m(\Gamma_{x,r})} \|1\|_{L^q(\Gamma_{x,r})} = \frac{1}{m(\Gamma_{x,r})} m(\Gamma_{x,r})^{1/q} = \left(|\Gamma| \, r^d/d \right)^{-1/p}, \tag{27.7}$$

shows

$$\begin{aligned}
|u(x)| &\leq \|u\|_{L^p(\Gamma_{x,r})} \left(|\Gamma| \, r^d/d \right)^{-1/p} + \frac{1}{|\Gamma|} \|\nabla u\|_{L^p(\Gamma_{x,r})} \left(\frac{p-1}{p-d} |\Gamma| \right)^{1-1/p} r^{1-d/p} \\
&= \frac{1}{|\Gamma|^{1/p}} \left[\|u\|_{L^p(\Gamma_{x,r})} \frac{d^{-1/p}}{r} + \|\nabla u\|_{L^p(\Gamma_{x,r})} \left(\frac{p-1}{p-d} \right)^{1-1/p} \right] r^{1-d/p}. \\
&\leq \frac{1}{|\Gamma|^{1/p}} \max \left(\frac{d^{-1/p}}{r}, \left(\frac{p-1}{p-d} \right)^{1-1/p} \right) \|u\|_{W^{1,p}(\Gamma_{x,r})} \cdot r^{1-d/p}.
\end{aligned}$$

■

Corollary 27.6. *For $d \in \mathbb{N}$ and $p \in (d, \infty]$ there are constants $\alpha = \alpha_d$ and $\beta = \beta_d$ such that if $u \in C^1(\mathbb{R}^d)$ then for all $x, y \in \mathbb{R}^d$,*

$$|u(y) - u(x)| \leq 2\beta\alpha^{1/p} \left(\frac{p-1}{p-d} \right)^{\frac{p-1}{p}} \|\nabla u\|_{L^p(B(x,r) \cap B(y,r))} \cdot |x-y|^{\left(1-\frac{d}{p}\right)} \tag{27.8}$$

where $r := |x-y|$.

Proof. Let $r := |x-y|$, $V := B_x(r) \cap B_y(r)$ and $\Gamma, \Lambda \subset S^{d-1}$ be chosen so that $x + r\Gamma = \partial B_x(r) \cap B_y(r)$ and $y + r\Lambda = \partial B_y(r) \cap B_x(r)$, i.e.

$$\Gamma = \frac{1}{r} \left(\partial B_x(r) \cap B_y(r) - x \right) \text{ and } \Lambda = \frac{1}{r} \left(\partial B_y(r) \cap B_x(r) - y \right) = -\Gamma.$$

Also let $W = \Gamma_{x,r} \cap \Lambda_{y,r}$, see Figure 50 below. By a scaling,

$$\beta_d := \frac{|\Gamma_{x,r} \cap \Lambda_{y,r}|}{|\Gamma_{x,r}|} = \frac{|\Gamma_{x,1} \cap \Lambda_{y,1}|}{|\Gamma_{x,1}|} \in (0,1)$$

is a constant only depending on d, i.e. we have $|\Gamma_{x,r}| = |\Lambda_{y,r}| = \beta|W|$. Integrating the inequality

$$|u(x) - u(y)| \leq |u(x) - u(z)| + |u(z) - u(y)|$$

over $z \in W$ gives

$$\begin{aligned}
|u(x) - u(y)| &\leq \fint_W |u(x) - u(z)| dz + \fint_W |u(z) - u(y)| dz \\
&= \frac{\beta}{|\Gamma_{x,r}|} \left(\int_W |u(x) - u(z)| dz + \int_W |u(z) - u(y)| dz \right) \\
&\leq \frac{\beta}{|\Gamma_{x,r}|} \left(\int_{\Gamma_{x,r}} |u(x) - u(z)| dz + \int_{\Lambda_{y,r}} |u(z) - u(y)| dz \right).
\end{aligned}$$

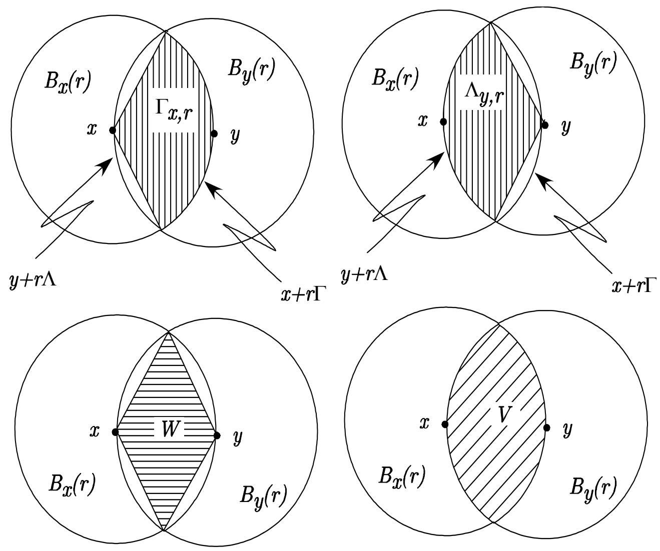

FIGURE 50. The geometry of two intersecting balls of radius $r := |x-y|$. Here $W = \Gamma_{x,r} \cap \Lambda_{y,r}$ and $V = B(x,r) \cap B(y,r)$.

Hence by Lemma 27.4, Hölder's inequality and translation and rotation invariance of Lebesgue measure,

$$\begin{aligned}|u(x)-u(y)| &\le \frac{\beta}{|\Gamma|}\left(\int_{\Gamma_{x,r}} \frac{|\nabla u(z)|}{|x-z|^{d-1}}\,dz + \int_{\Lambda_{y,r}} \frac{|\nabla u(z)|}{|z-y|^{d-1}}\,dz\right) \\ &\le \frac{\beta}{|\Gamma|}\left(\|\nabla u\|_{L^p(\Gamma_{x,r})}\|\frac{1}{|x-\cdot|^{d-1}}\|_{L^q(\Gamma_{x,r})} + \|\nabla u\|_{L^p(\Lambda_{y,r})}\|\frac{1}{|y-\cdot|^{d-1}}\|_{L^q(\Lambda_{y,r})}\right) \\ &\le \frac{2\beta}{|\Gamma|}\|\nabla u\|_{L^p(V)}\|\frac{1}{|\cdot|^{d-1}}\|_{L^q(\Gamma_{0,r})} \end{aligned} \tag{27.9}$$

where $q = \frac{p}{p-1}$ is the conjugate exponent to p. Combining Eqs. (27.9) and (27.6) gives Eq. (27.8) with $\alpha := |\Gamma|^{-1}$. ■

Theorem 27.7 (Morrey's Inequality). *If $d < p \le \infty$, $u \in W^{1,p}(\mathbb{R}^d)$, then there exists a unique version u^* of u (i.e. $u^* = u$ a.e.) such that u^* is continuous. Moreover $u^* \in C^{0,1-\frac{d}{p}}(\mathbb{R}^d)$ and*

$$\|u^*\|_{C^{0,1-\frac{d}{p}}(\mathbb{R}^d)} \le C\|u\|_{W^{1,p}(\mathbb{R}^d)} \tag{27.10}$$

where $C = C(p,d)$ is a universal constant. Moreover, the estimates in Eqs. (27.3), (27.4) and (27.8) still hold when u is replaced by u^*.

Proof. For $p < \infty$ and $u \in C_c^1(\mathbb{R}^d)$, Corollaries 27.5 and 27.6 imply

$$\|u\|_{BC(\mathbb{R}^d)} \le C\|u\|_{W^{1,p}(\mathbb{R}^d)} \text{ and } \frac{|u(y)-u(x)|}{|x-y|^{1-\frac{d}{p}}} \le C\|\nabla u\|_{L^p(\mathbb{R}^d)}$$

which implies $[u]_{1-\frac{d}{p}} \leq C\|\nabla u\|_{L^p(\mathbb{R}^d)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^d)}$and hence

$$\|u\|_{C^{0,1-\frac{d}{p}}(\mathbb{R}^d)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^d)}. \tag{27.11}$$

Now suppose $u \in W^{1,p}(\mathbb{R}^d)$, choose (using Exercise 21.2) $u_n \in C_c^1(\mathbb{R}^d)$ such that $u_n \to u$ in $W^{1,p}(\mathbb{R}^d)$. Then by Eq. (27.11), $\|u_n - u_m\|_{C^{0,1-\frac{d}{p}}(\mathbb{R}^d)} \to 0$ as $m, n \to \infty$ and therefore there exists $u^* \in C^{0,1-\frac{d}{p}}(\mathbb{R}^d)$ such that $u_n \to u^*$ in $C^{0,1-\frac{d}{p}}(\mathbb{R}^d)$. Clearly $u^* = u$ a.e. and Eq. (27.10) holds.

If $p = \infty$ and $u \in W^{1,\infty}\left(\mathbb{R}^d\right)$, then by Proposition 21.29 there is a version u^* of u which is Lipschitz continuous. Now in both cases, $p < \infty$ and $p = \infty$, the sequence $u_m := u * \eta_m = u^* * \eta_m \in C^\infty\left(\mathbb{R}^d\right)$ and $u_m \to u^*$ uniformly on compact subsets of $\mathbb{R}^d$. Using Eq. (27.3) with u replaced by u_m along with a (by now) standard limiting argument shows that Eq. (27.3) still holds with u replaced by u^*. The proofs of Eqs. (27.4) and (27.8) only relied on Eq. (27.3) and hence go through without change. Similarly the argument in the first paragraph only relied on Eqs. (27.4) and (27.8) and hence Eq. (27.10) is also valid for $p = \infty$. ■

Corollary 27.8 (Morrey's Inequality)**.** *Suppose $\Omega \subset_o \mathbb{R}^d$ such that $\bar{\Omega}$ is compact C^1-manifold with boundary and $d < p \leq \infty$. Then for $u \in W^{1,p}(\Omega)$, there exists a unique version u^* of u such that $u^* \in C^{0,1-\frac{d}{p}}(\mathbb{R}^d)$ and we further have*

$$\|u^*\|_{C^{0,1-\frac{d}{p}}(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}, \tag{27.12}$$

where $C = C(p, d, \Omega)$.

Proof. Let U be a precompact open subset of $\mathbb{R}^d$ and $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^d)$ be an extension operator as in Theorem 25.35. For $u \in W^{1,p}(\Omega)$ with $d < p \leq \infty$, Theorem 27.7 implies there is a version $U^* \in C^{0,1-\frac{d}{p}}(\mathbb{R}^d)$ of Eu. Letting $u^* := U^*|_\Omega$, we have and moreover,

$$\|u^*\|_{C^{0,1-\frac{d}{p}}(\Omega)} \leq \|U^*\|_{C^{0,1-\frac{d}{p}}(\mathbb{R}^d)} \leq C\|Eu\|_{W^{1,p}(\mathbb{R}^d)} \leq C\,\|u\|_{W^{1,p}(\Omega)}\,.$$

■

The following example shows that $L^\infty(\mathbb{R}^d) \not\subseteq W^{1,d}(\mathbb{R}^d)$, i.e. $W^{1,d}(\mathbb{R}^d)$ contains unbounded elements. Therefore Theorem 27.7 and Corollary 27.8 are not valid for $p = d$. It turns out that for $p = d$, $W^{1,d}\left(\mathbb{R}^d\right)$ embeds into $BMO(\mathbb{R}^d)$ – the space of functions with "bounded mean oscillation."

Example 27.9. Let $u(x) = \psi(x) \log\log\left(1 + \frac{1}{|x|}\right)$ where $\psi \in C_c^\infty(\mathbb{R}^d)$ is chosen so that $\psi(x) = 1$ for $|x| \leq 1$. Then $u \notin L^\infty(\mathbb{R}^d)$ while $u \in W^{1,d}(\mathbb{R}^d)$. Let us check this claim. Using Theorem 8.35, one easily shows $u \in L^p(\mathbb{R}^d)$. A short computation shows, for $|x| < 1$, that

$$\begin{aligned}
\nabla u(x) &= \frac{1}{\log\left(1 + \frac{1}{|x|}\right)} \frac{1}{1 + \frac{1}{|x|}} \nabla \frac{1}{|x|} \\
&= \frac{1}{1 + \frac{1}{|x|}} \frac{1}{\log\left(1 + \frac{1}{|x|}\right)} \left(-\frac{1}{|x|}\,\hat{x}\right)
\end{aligned}$$

where $\hat{x} = x/|x|$ and so again by Theorem 8.35,

$$\int_{\mathbb{R}^d} |\nabla u(x)|^d dx \geq \int_{|x|<1} \left(\frac{1}{|x|^2 + |x|} \frac{1}{\log\left(1 + \frac{1}{|x|}\right)} \right)^d dx$$
$$\geq \sigma(S^{d-1}) \int_0^1 \left(\frac{2}{r \log\left(1 + \frac{1}{r}\right)} \right)^d r^{d-1} dr = \infty.$$

27.2. **Rademacher's Theorem.**

Theorem 27.10. *Suppose that $u \in W^{1,p}_{loc}(\Omega)$ for some $d < p \leq \infty$. Then u is differentiable almost everywhere and $w\text{-}\partial_i u = \partial_i u$ a.e. on Ω.*

Proof. We clearly may assume that $p < \infty$. For $v \in W^{1,p}_{loc}(\Omega)$ and $x, y \in \Omega$ such that $\overline{B(x,r) \cap B(y,r)} \subset \Omega$ where $r := |x - y|$, the estimate in Corollary 27.6, gives

$$|v(y) - v(x)| \leq C \|\nabla u\|_{L^p(B(x,r)\cap B(y,r))} \cdot |x - y|^{\left(1-\frac{d}{p}\right)}$$
$$= C \|\nabla v\|_{L^p(B(x,r)\cap B(y,r))} \cdot r^{\left(1-\frac{d}{p}\right)}. \tag{27.13}$$

Let u now denote the unique continuous version of $u \in W^{1,p}_{loc}(\Omega)$. The by the Lebesgue differentiation Theorem 16.12, there exists an exceptional set $E \subset \Omega$ such that $m(E) = 0$ and

$$\lim_{r \downarrow 0} \fint_{B(x,r)} |\nabla u(y) - \nabla u(x)|^p dy = 0 \text{ for } x \in \Omega \setminus E.$$

Fix a point $x \in \Omega \setminus E$ and let $v(y) := u(y) - u(x) - \nabla u(x) \cdot (y - x)$ and notice that $\nabla v(y) = \nabla u(y) - \nabla u(x)$. Applying Eq. (27.13) to v then implies

$$\begin{aligned}
&|u(y) - u(x) - \nabla u(x) \cdot (y - x)| \\
&\quad \leq C \|\nabla u(\cdot) - \nabla u(x)\|_{L^p(B(x,r)\cap B(y,r))} \cdot r^{\left(1-\frac{d}{p}\right)} \\
&\quad \leq C \left(\int_{B(x,r)} |\nabla u(y) - \nabla u(x)|^p dy \right)^{1/p} \cdot r^{\left(1-\frac{d}{p}\right)} \\
&\quad = C\sigma\left(S^{d-1}\right)^{1/p} r^{d/p} \left(\fint_{B(x,r)} |\nabla u(y) - \nabla u(x)|^p dy \right)^{1/p} \cdot r^{\left(1-\frac{d}{p}\right)} \\
&\quad = C\sigma\left(S^{d-1}\right)^{1/p} \left(\fint_{B(x,r)} |\nabla u(y) - \nabla u(x)|^p dy \right)^{1/p} \cdot |x - y|
\end{aligned}$$

which shows u is differentiable at x and $\nabla u(x) = w\text{-}\nabla u(x)$. ■

Theorem 27.11 (Rademacher's Theorem)**.** *Let u be locally Lipschitz continuous on $\Omega \subset_o \mathbb{R}^d$. Then u is differentiable almost everywhere and $w\text{-}\partial_i u = \partial_i u$ a.e. on Ω.*

Proof. By Proposition 21.29 $\partial_i^{(w)} u$ exists weakly and is in $\partial_i u \in L^\infty(\mathbb{R}^d)$ for $i = 1, 2, \ldots, d$. The result now follows from Theorem 27.10. ■

27.3. **Gagliardo-Nirenberg-Sobolev Inequality.** In this section our goal is to prove an inequality of the form:

$$\|u\|_{L^q} \le C\|\nabla u\|_{L^p(\mathbb{R}^d)} \text{ for } u \in C_c^1(\mathbb{R}^d). \tag{27.14}$$

For $\lambda > 0$, let $u_\lambda(x) = u(\lambda x)$. Then

$$\|u_\lambda\|_{L^q}^q = \int_{\mathbb{R}^d} |u(\lambda x)|^q dx = \int_{\mathbb{R}^d} |u(y)|^q \frac{dy}{\lambda^d}$$

and hence $\|u_\lambda\|_{L^q} = \lambda^{-d/q}\|u\|_{L^q}$. Moreover, $\nabla u_\lambda(x) = \lambda(\nabla u)(\lambda x)$ and thus

$$\|\nabla u_\lambda\|_{L^p} = \lambda\|(\nabla u)_\lambda\|_{L^p} = \lambda\,\lambda^{-d/p}\|\nabla u\|_{L^p}.$$

If (27.14) is to hold for all $u \in C_c^1(\mathbb{R}^d)$ then we must have

$$\lambda^{-d/q}\|u\|_{L^q} = \|u_\lambda\|_{L^q} \le C\|\nabla u_\lambda\|_{L^p(\mathbb{R}^d)} = C\lambda^{1-d/p}\|\nabla u\|_{L^p} \text{ for all } \lambda > 0$$

which is only possible if

$$1 - d/p + d/q = 0, \text{ i.e. } 1/p = 1/d + 1/q. \tag{27.15}$$

Notation 27.12. For $p \in [1, d]$, let $p^* := \frac{dp}{d-p}$ with the convention that $p^* = \infty$ if $p = d$. That is $p^* = q$ where q solves Eq. (27.15).

Theorem 27.13. *Let $p = 1$ so $1^* = \frac{d}{d-1}$, then*

$$\|u\|_{1^*} = \|u\|_{\frac{d}{d-1}} \le \prod_{i=1}^d \left(\int_{\mathbb{R}^d} |\partial_i u(x)| dx\right)^{\frac{1}{d}} \le d^{-\frac{1}{2}} \|\nabla u\|_1 \tag{27.16}$$

for all $u \in W^{1,1}(\mathbb{R}^d)$.

Proof. Since there exists $u_n \in C_c^1(\mathbb{R}^d)$ such that $u_n \to u$ in $W^{1,1}(\mathbb{R}^d)$, a simple limiting argument shows that it suffices to prove Eq. (27.16) for $u \in C_c^1(\mathbb{R}^d)$. To help the reader understand the proof, let us give the proof for $d \le 3$ first and with the constant $d^{-1/2}$ being replaced by 1. After that the general induction argument will be given. (The adventurous reader may skip directly to the paragraph containing Eq. (27.17.)

($d = 1$, $p^* = \infty$) By the fundamental theorem of calculus,

$$|u(x)| = \left|\int_{-\infty}^x u'(y)dy\right| \le \int_{-\infty}^x |u'(y)|\, dy \le \int_{\mathbb{R}} |u'(x)|\, dx.$$

Therefore $\|u\|_{L^\infty} \le \|u'\|_{L^1}$, proving the $d = 1$ case.

($d = 2$, $p^* = 2$) Applying the same argument as above to $y_1 \to u(y_1, x_2)$ and $y_2 \to u(x_1, y_2)$,,

$$|u(x_1, x_2)| \le \int_{-\infty}^\infty |\partial_1 u(y_1, x_2)|\, dy_1 \le \int_{-\infty}^\infty |\nabla u(y_1, x_2)|\, dy_1 \text{ and}$$
$$|u(x_1, x_2)| \le \int_{-\infty}^\infty |\partial_2 u(x_1, y_2)|\, dy_2 \le \int_{-\infty}^\infty |\nabla u(x_1, y_2)|\, dy_2$$

and therefore

$$|u(x_1, x_2)|^2 \le \int_{-\infty}^\infty |\partial_1 u(y_1, x_2)| dy_1 \cdot \int_{-\infty}^\infty |\partial_2 u(x_1, y_2)|\, dy_2.$$

Integrating this equation relative to x_1 and x_2 gives

$$\begin{aligned}\|u\|_{L^2}^2 = \int_{\mathbb{R}^2} |u(x)|^2 dx &\le \left(\int_{-\infty}^{\infty} |\partial_1 u(x)|\ dx\right)\left(\int_{-\infty}^{\infty} |\partial_2 u(x)|\ dx\right) \\ &\le \left(\int_{-\infty}^{\infty} |\nabla u(x)|\ dx\right)^2\end{aligned}$$

which proves the $d = 2$ case.

($d = 3$, $p^* = 3/2$) Let $x^1 = (y_1, x_2, x_3)$, $x^2 = (x_1, y_2, x_3)$, and $x^3 = (x_1, x_2, y_3)$. Then as above,

$$|u(x)| \le \int_{-\infty}^{\infty} |\partial_i u(x^i)| dy_i \text{ for } i = 1, 2, 3$$

and hence

$$|u(x)|^{\frac{3}{2}} \le \prod_{i=1}^{3} \left(\int_{-\infty}^{\infty} |\partial_i u(x^i)| dy_i\right)^{\frac{1}{2}}.$$

Integrating this equation on x_1 gives,

$$\begin{aligned}\int_{\mathbb{R}} |u(x)|^{\frac{3}{2}} dx_1 &\le \left(\int_{-\infty}^{\infty} |\partial_1 u(x^1)| dy_1\right)^{\frac{1}{2}} \int \prod_{i=2}^{3} \left(\int_{-\infty}^{\infty} |\partial_i u(x^i)| dy_i\right)^{\frac{1}{2}} dx_1 \\ &\le \left(\int_{-\infty}^{\infty} |\partial_1 u(x)| dx_1\right)^{\frac{1}{2}} \prod_{i=2}^{3} \left(\int_{-\infty}^{\infty} |\partial_i u(x^i)| dx_1 dy_i\right)^{\frac{1}{2}}\end{aligned}$$

wherein the second equality we have used the Hölder's inequality with $p = q = 2$. Integrating this result on x_2 and using Hölder's inequality gives

$$\begin{aligned}\int_{\mathbb{R}^2} |u(x)|^{\frac{3}{2}} dx_1 dx_2 &\le \left(\int_{\mathbb{R}^2} |\partial_2 u(x)| dx_1 dx_2\right)^{\frac{1}{2}} \int_{\mathbb{R}} dx_2 \left(\int_{-\infty}^{\infty} |\partial_1 u(x)| dx_1\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} |\partial_3 u(x^3)| dx_1 dy_3\right)^{\frac{1}{2}} \\ &\le \left(\int_{\mathbb{R}^2} |\partial_2 u(x)| dx_1 dx_2\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} |\partial_1 u(x)| dx_1 dx_2\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} |\partial_3 u(x)| dx\right)^{\frac{1}{2}}.\end{aligned}$$

One more integration of x_3 and application of Hölder's inequality, implies

$$\int_{\mathbb{R}^3} |u(x)|^{\frac{3}{2}} dx \le \prod_{i=1}^{3} \left(\int_{\mathbb{R}^3} |\partial_i u(x)| dx\right)^{\frac{1}{2}} \le \left(\int_{\mathbb{R}^3} |\nabla u(x)| dx\right)^{\frac{3}{2}}$$

proving the $d = 3$ case.

For general d ($p^* = \frac{d}{d-1}$), as above let $x^i = (x_1, \dots, x_{i-1}, y_i, x_{i+1} \dots, x_d)$. Then

$$|u(x)| \le \left(\int_{-\infty}^{\infty} |\partial_i u(x^i)| dy_i\right)$$

and

$$|u(x)|^{\frac{d}{d-1}} \le \prod_{i=1}^{d} \left(\int_{-\infty}^{\infty} |\partial_i u(x^i)| dy_i\right)^{\frac{1}{d-1}}. \tag{27.17}$$

Integrating this equation relative to x_1 and making use of Hölder's inequality in the form

$$\left\|\prod_{i=2}^{d} f_i\right\|_1 \le \prod_{i=2}^{d} \|f_i\|_{d-1} \tag{27.18}$$

(see Corollary 9.3) we find

$$\begin{aligned}\int_{\mathbb{R}} |u(x)|^{\frac{d}{d-1}} dx_1 &\le \left(\int_{\mathbb{R}} \partial_1 u(x) dx_1\right)^{\frac{1}{d-1}} \int_{\mathbb{R}} dx_1 \prod_{i=2}^{d} \left(\int_{\mathbb{R}} |\partial_i u(x^i)| dy_i\right)^{\frac{1}{d-1}} \\ &\le \left(\int_{\mathbb{R}} \partial_1 u(x) dx_1\right)^{\frac{1}{d-1}} \prod_{i=2}^{d} \left(\int_{\mathbb{R}^2} |\partial_i u(x^i)| dx_1 dy_i\right)^{\frac{1}{d-1}} \\ &= \left(\int_{\mathbb{R}} \partial_1 u(x) dx_1\right)^{\frac{1}{d-1}} \left(\int_{\mathbb{R}^2} |\partial_2 u(x)| dx_1 dx_2\right)^{\frac{1}{d-1}} \prod_{i=3}^{d} \left(\int_{\mathbb{R}^2} |\partial_i u(x^i)| dx_1 dy_i\right)^{\frac{1}{d-1}}.\end{aligned}$$

Integrating this equation on x_2 and using Eq. (27.18) once again implies,

$$\begin{aligned}\int_{\mathbb{R}^2} |u(x)|^{\frac{d}{d-1}} dx_1 dx_2 &\le \left(\int_{\mathbb{R}^2} |\partial_2 u(x)| dx_1 dx_2\right)^{\frac{1}{d-1}} \int_{\mathbb{R}} dx_2 \left(\int_{\mathbb{R}} \partial_1 u(x) dx_1\right)^{\frac{1}{d-1}} \\ &\quad \times \prod_{i=3}^{d} \left(\int_{\mathbb{R}^2} |\partial_i u(x^i)| dx_1 dy_i\right)^{\frac{1}{d-1}} \\ &\le \left(\int_{\mathbb{R}^2} |\partial_2 u(x)| dx_1 dx_2\right)^{\frac{1}{d-1}} \left(\int_{\mathbb{R}^2} |\partial_1 u(x)| dx_1 dx_2\right)^{\frac{1}{d-1}} \\ &\quad \times \prod_{i=3}^{d} \left(\int_{\mathbb{R}^3} |\partial_i u(x^i)| dx_1 dx_2 dy_i\right)^{\frac{1}{d-1}}.\end{aligned}$$

Continuing this way inductively, one shows

$$\begin{aligned}\int_{\mathbb{R}^k} |u(x)|^{\frac{d}{d-1}} dx_1 dx_2 \dots dx_k &\le \prod_{i=1}^{k} \left(\int_{\mathbb{R}^k} |\partial_i u(x)| dx_1 dx_2 \dots dx_k\right)^{\frac{1}{d-1}} \\ &\quad \times \prod_{i=k+1}^{d} \left(\int_{\mathbb{R}^3} |\partial_i u(x^i)| dx_1 dx_2 \dots dx_k dy_{k+1}\right)^{\frac{1}{d-1}}\end{aligned}$$

and in particular when $k = d$,

$$\begin{aligned}\int_{\mathbb{R}^d} |u(x)|^{\frac{d}{d-1}} dx &\le \prod_{i=1}^{d} \left(\int_{\mathbb{R}^d} |\partial_i u(x)| dx_1 dx_2 \dots dx_d\right)^{\frac{1}{d-1}} \\ &\le \prod_{i=1}^{d} \left(\int_{\mathbb{R}^d} |\nabla u(x)| dx\right)^{\frac{1}{d-1}} = \left(\int_{\mathbb{R}^d} |\nabla u(x)| dx\right)^{\frac{d}{d-1}}.\end{aligned} \tag{27.19}$$

This estimate may now be improved on by using Young's inequality (see Exercise 27.1) in the form $\prod_{i=1}^{d} a_i \le \frac{1}{d} \sum_{i=1}^{d} a_i^d$. Indeed by Eq. (27.19) and Young's inequality,

$$\begin{aligned}\|u\|_{\frac{d}{d-1}} &\le \prod_{i=1}^{d} \left(\int_{\mathbb{R}^d} |\partial_i u(x)| dx\right)^{\frac{1}{d}} \le \frac{1}{d} \sum_{i=1}^{d} \left(\int_{\mathbb{R}^d} |\partial_i u(x)| dx\right) \\ &= \frac{1}{d} \int_{\mathbb{R}^d} \sum_{i=1}^{d} |\partial_i u(x)| dx \le \frac{1}{d} \int_{\mathbb{R}^d} \sqrt{d}\, |\nabla u(x)|\, dx\end{aligned}$$

wherein the last inequality we have used Hölder's inequality for sums,

$$\sum_{i=1}^d |a_i| \le \left(\sum_{i=1}^d 1\right)^{1/2} \left(\sum_{i=1}^d |a_i|^2\right)^{1/2} = \sqrt{d}\,|a|\,.$$

■

The next theorem generalizes Theorem 27.13 to an inequality of the form in Eq. (27.14).

Theorem 27.14. *If $p \in [1, d)$ then,*

$$\|u\|_{L^{p^*}} \le d^{-1/2}\frac{p(d-1)}{d-p}\|\nabla u\|_{L^p} \text{ for all } u \in W^{1,p}(\mathbb{R}^d). \tag{27.20}$$

Proof. As usual since $C_c^1(\mathbb{R}^d)$ is dense in $W^{1,p}\left(\mathbb{R}^d\right)$ it suffices to prove Eq. (27.20) for $u \in C_c^1(\mathbb{R}^d)$. For $u \in C_c^1(\mathbb{R}^d)$ and $s > 1$, $|u|^s \in C_c^1(\mathbb{R}^d)$ and $\nabla |u|^s = s|u|^{s-1}\mathrm{sgn}(u)\nabla u$. Applying Eq. (27.16) with u replaced by $|u|^s$ and then using Holder's inequality gives

$$\begin{aligned} \||u|^s\|_{1^*} &\le d^{-\frac{1}{2}}\left\|\nabla |u|^s\right\|_1 = sd^{-\frac{1}{2}}\||u|^{s-1}\nabla u\|_{L^1} \\ &\le \frac{s}{\sqrt{d}}\left\|\nabla u\right\|_{L^p}\cdot \||u|^{s-1}\|_{L^q} \end{aligned} \tag{27.21}$$

where $q = \frac{p}{p-1}$. We will now choose s so that $s1^* = (s-1)q$, i.e.

$$\begin{aligned} s &= \frac{q}{q-1^*} = \frac{1}{1-1^*\frac{1}{q}} = \frac{1}{1-\frac{d}{d-1}\left(1-\frac{1}{p}\right)} \\ &= \frac{p\,(d-1)}{p\,(d-1)-d\,(p-1)} = \frac{p\,(d-1)}{d-p} = p^*\frac{d-1}{d}. \end{aligned}$$

For this choice of s, $s1^* = p^* = (s-1)q$ and Eq. (27.21) becomes

$$\left[\int_{\mathbb{R}^d}|u|^{p^*}dm\right]^{1/1^*} \le \frac{s}{\sqrt{d}}\left\|\nabla u\right\|_{L^p}\cdot\left[\int_{\mathbb{R}^d}|u|^{p^*}dm\right]^{1/q}. \tag{27.22}$$

Since

$$\begin{aligned} \frac{1}{1^*}-\frac{1}{q} &= \frac{d-1}{d}-\frac{p-1}{p} = \frac{p(d-1)-d(p-1)}{dp} \\ &= \frac{d-p}{pd} = \frac{1}{p^*}, \end{aligned}$$

Eq. (27.22) implies Eq. (27.20). ■

Corollary 27.15. *Suppose $\Omega \subset \mathbb{R}^d$ is bounded open set with C^1-boundary, then for all $p \in [1, d)$ and $1 \le q \le p^*$ there exists $C = C(\Omega, p, q)$ such that*

$$\|u\|_{L^q(\Omega)} \le C\|u\|_{W^{1,p}(\Omega)}.$$

Proof. Let U be a precompact open subset of $\mathbb{R}^d$ such that $\bar{\Omega} \subset U$ and $E : W^{1,p}(\Omega) \to W^{1,p}\left(\mathbb{R}^d\right)$ be an extension operator as in Theorem 25.35. Then for $u \in C^1(\overline{\Omega}) \cap W^{1,p}(\Omega)$,

$$\|u\|_{L^{p^*}(\Omega)} \le C\|Eu\|_{L^{p^*}(\mathbb{R}^d)} \le C\|\nabla(Eu)\|_{L^p(\mathbb{R}^d)} \le C\|u\|_{W^{1,p}(\Omega)},$$

i.e.

$$\|u\|_{L^{p^*}(\Omega)} \le C\|u\|_{W^{1,p}(\Omega)} \tag{27.23}$$

Since $C^1(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$, Eq. (27.23) holds for all $u \in W^{1,p}(\Omega)$. Finally for all $1 \leq q < p^*$,

$$\|u\|_{L^q(\Omega)} \leq \|u\|_{L^{P^*}(\Omega)} \cdot \|1\|_{L^r(\Omega)} = \|u\|_{L^{p^*}}(\lambda(\Omega))^{\frac{1}{r}} \leq C(\lambda(\Omega))^{\frac{1}{r}}\|u\|_{W^{1,p}(\Omega)}$$

where $\frac{1}{r} + \frac{1}{p^*} = \frac{1}{q}$. ■

27.4. **Sobolev Embedding Theorems Summary.** Let us summarize what we have proved up to this point in the following theorem.

Theorem 27.16. *Let $p \in [1,\infty]$ and $u \in W^{1,p}\left(\mathbb{R}^d\right)$. Then*

1. ***Morrey's Inequality.*** *If $p > d$, then $W^{1,p} \hookrightarrow C^{0,1-\frac{d}{p}}$ and*
$$\|u^*\|_{C^{0,1-\frac{d}{p}}(\mathbb{R}^d)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^d)}.$$
2. *When $p = d$ there is an L^∞ – like space called BMO (which is **not** defined in these notes) such that $W^{1,p} \hookrightarrow BMO$.*
3. ***GNS Inequality.*** *If $1 \leq p < d$, then $W^{1,p} \hookrightarrow L^{p^*}$*
$$\|u\|_{L^{p^*}} \leq d^{-1/2}\frac{p(d-1)}{d-p}\|\nabla u\|_{L^p}$$
where $p^* = \frac{dp}{d-p}$ or equivalently $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{d}$.

Our next goal is write out the embedding theorems for $W^{k,p}(\Omega)$ for general k and p.

Notation 27.17. Given a number $s \geq 0$, let

$$s_+ = \begin{cases} s & \text{if} \quad n \notin \mathbb{N}_0 \\ s+\delta & \text{if} \quad n \in \mathbb{N}_0 \end{cases}$$

where $\delta > 0$ is some arbitrarily small number. When $s = k + \alpha$ with $k \in \mathbb{N}_0$ and $0 \leq \alpha < 1$ we will write $C^{k,\alpha}(\Omega)$ simply as $C^s(\Omega)$. **Warning,** although $C^{k,1}(\Omega) \subset C^{k+1}(\Omega)$ it is **not true** that $C^{k,1}(\Omega) = C^{k+1}(\Omega)$.

Theorem 27.18 (Sobolev Embedding Theorems)**.** *Suppose $\Omega = \mathbb{R}^d$ or $\Omega \subset \mathbb{R}^d$ is bounded open set with C^1-boundary, $p \in [1,\infty)$, $k,l \in \mathbb{N}$ with $l \leq k$.*

1. *If $p < d/l$ then $W^{k,p}(\Omega) \hookrightarrow W^{k-l,q}(\Omega)$ provided $q := \frac{dp}{d-pl}$, i.e. q solves*
$$\frac{1}{q} = \frac{1}{p} - \frac{l}{d} > 0$$
and there is a constant $C < \infty$ such that
$$\|u\|_{W^{k-l,q}(\Omega)} \leq C\|u\|_{W^{k,p}(\Omega)} \text{ for all } u \in W^{k,p}(\Omega).$$
2. *If $p > d/k$, then $W^{k,p}(\Omega) \hookrightarrow C^{k-(d/p)_+}(\Omega)$ and there is a constant $C < \infty$ such that*
$$\|u\|_{C^{k-(d/p)_+}(\Omega)} \leq C\|u\|_{W^{k,p}(\Omega)} \text{ for all } u \in W^{k,p}(\Omega).$$

Proof. 1. $(p < d/l)$ If $u \in W^{k,p}(\Omega)$, then $\partial^\alpha u \in W^{1,p}(\Omega)$ for all $|\alpha| \leq k-1$. Hence by Corollary 27.15, $\partial^\alpha u \in L^{p^*}(\Omega)$ for all $|\alpha| \leq k-1$ and therefore $W^{k,p}(\Omega) \hookrightarrow W^{k-1,p^*}(\Omega)$ and there exists a constant C_1 such that

$$\|u\|_{W^{k-1,p_1}(\Omega)} \leq C\|u\|_{W^{k,p}(\Omega)} \text{ for all } u \in W^{k,p}(\Omega). \tag{27.24}$$

Define p_j inductively by, $p_1 := p^*$ and $p_j := p_{j-1}^*$. Since $\frac{1}{p_j} = \frac{1}{p_{j-1}} - \frac{1}{d}$ it is easily checked that $\frac{1}{p_l} = \frac{1}{p} - \frac{l}{d} > 0$ since $p < d/l$. Hence using Eq. (27.24) repeatedly we learn that the following inclusion maps are all bounded:

$$W^{k,p}(\Omega) \hookrightarrow W^{k-1,p_1}(\Omega) \hookrightarrow W^{k-2,p_2}(\Omega) \ldots \hookrightarrow W^{k-l,p_l}(\Omega).$$

This proves the first item of the theorem. The following lemmas will be used in the proof of item 2. ■

Lemma 27.19. *Suppose $j \in \mathbb{N}$ and $p \geq d$ and $j > d/p$ (i.e. $j \geq 1$ if $p > d$ and $j \geq 2$ if $p = d$) then*

$$W^{j,p}(\Omega) \hookrightarrow C^{j-(d/p)_+}(\Omega)$$

and there is a constant $C < \infty$ such that

$$\|u\|_{C^{j-(d/p)_+}(\Omega)} \leq C \|u\|_{W^{j,p}(\Omega)}. \tag{27.25}$$

Proof. By the usual methods, it suffices to show that the estimate in Eq. (27.25) holds for all $u \in C^\infty(\bar{\Omega})$.

For $p > d$ and $|\alpha| \leq j-1$,

$$\|\partial^\alpha u\|_{C^{0,1-d/p}(\Omega)} \leq C \|\partial^\alpha u\|_{W^{1,p}(\Omega)} \leq C \|u\|_{W^{j,p}(\Omega)}$$

and hence

$$\|u\|_{C^{j-d/p}(\Omega)} := \|u\|_{C^{j-1,1-d/p}(\Omega)} \leq C \|u\|_{W^{j,p}(\Omega)}$$

which is Eq. (27.25).

When $p = d$ (so now $j \geq 2$), choose $q \in (1,d)$ be close to d so that $j > d/q$ and $q^* = \frac{qd}{d-q} > d$. Then

$$W^{j,d}(\Omega) \hookrightarrow W^{j,q}(\Omega) \hookrightarrow W^{j-1,q^*}(\Omega) \hookrightarrow C^{j-2,1-d/q^*}(\Omega).$$

Since $d/q^* \downarrow 0$ as $q \uparrow d$, we conclude that $W^{j,d}(\Omega) \hookrightarrow C^{j-2,\alpha}(\Omega)$ for any $\alpha \in (0,1)$ which we summarizes by writing

$$W^{j,d}(\Omega) \hookrightarrow C^{j-(d/d)}(\Omega).$$

■

Proof. Continuation of the proof of Theorem 27.18. Item 2., $(p > d/k)$. If $p \geq d$, the result follows from Lemma 27.19. So nos suppose that $d > p > d/k$ and choose the largest l such that $1 \leq l < k$ and $d/l > p$ and let $q = \frac{dp}{d-pl}$, i.e. q solves $q \geq d$ and

$$\frac{1}{q} = \frac{1}{p} - \frac{l}{d} \text{ or } \frac{d}{q} = \frac{d}{p} - l$$

Then

$$W^{k,p}(\Omega) \hookrightarrow W^{k-l,q}(\Omega) \hookrightarrow C^{k-l-(d/q)_+}(\Omega) = C^{k-l-\left(\frac{d}{p}-l\right)_+}(\Omega) = C^{k-\left(\frac{d}{p}\right)_+}(\Omega)$$

as desired. ■

Remark 27.20 (Rule of thumb.). Assign the "degrees of regularity" $k - (d/p)_+$ to the space $W^{k,p}$ and $k + \alpha$ to the space $C^{k,\alpha}$. If

$$X, Y \in \left\{W^{k,p} : k \in \mathbb{N}_0, p \in [1,\infty]\right\} \cup \left\{C^{k,\alpha} : k \in \mathbb{N}_0,\ \alpha \in [0,1]\right\}$$

with $\deg_{\mathrm{reg}}(X) \geq \deg_{\mathrm{reg}}(Y)$, then $X \hookrightarrow Y$.

Example 27.21. (1) $W^{k,p} \hookrightarrow W^{k-\ell,q}$ iff $k - \frac{d}{p} \geq k - \ell - \frac{d}{q}$ iff $\ell \geq \frac{d}{p} - \frac{d}{q}$ iff $\frac{1}{q} \geq \frac{1}{p} = \frac{\ell}{d}$.

(2) $W^{k,p} \subset C^{0,\alpha}$ iff $k - \left(\frac{d}{p}\right)_+ \geq \alpha$.

27.5. **Compactness Theorems.**

Lemma 27.22. *Suppose $K_m : X \to Y$ are compact operators and $\|K - K_m\|_{L(X,Y)} \to 0$ as $n \to \infty$ then K is compact.*

Proof. Let $\{x_n\}_{n=1}^\infty \subset X$ be given such that $\|x_n\| \leq 1$. By Cantor's diagonalization scheme we may choose $\{x_n'\} \subset \{x_n\}$ such that $y_m := \lim_{n\to\infty} K_m x_n' \in Y$ exists for all m. Hence

$$\begin{aligned}\|Kx_n' - Kx_\ell'\| &= \|K(x_n' - x_\ell')\| \leq \|K(x_n' - x_\ell')\| \\ &\leq \|K - K_m\| \|x_n' - x_\ell'\| + \|K_m(x_n' - x_\ell')\| \\ &\leq \|K - K_m\| + \|K_m(x_n' - x_\ell')\|\end{aligned}$$

and therefore,

$$\limsup_{l,n\to\infty} \|Kx_n' - Kx_\ell'\| \leq \|K - K_m\| \to 0 \text{ as } m \to \infty.$$

■

Lemma 27.23. *Let $\eta \in C_c^\infty(\mathbb{R}^d)$, $C_\eta f = \eta * f$, $\Omega \subset \mathbb{R}^d$ be a bounded open set with C^1-boundary, V be an open precompact subset of $\mathbb{R}^d$ such that $\bar{\Omega} \subset U$ and $E : W^{1,1}(\Omega) \to W^{1,1}(\mathbb{R}^d)$ be an extension operator as in Theorem 25.35. Then to every bounded sequence $\{\tilde{u}_n\}_{n=1}^\infty \subset W^{1,1}(\Omega)$ there has a subsequence $\{u_n'\}_{n=1}^\infty$ such that $C_\eta E u_n'$ is uniformly convergent to a function in $C_c(\mathbb{R}^d)$.*

Proof. Let $u_n := E\tilde{u}_n$ and $C := \sup \|u_n\|_{W^{1,1}(\mathbb{R}^d)}$ which is finite by assumption. So $\{u_n\}_{n=1}^\infty \subset W^{1,1}(\mathbb{R}^d)$ is a bounded sequence such that $\operatorname{supp}(u_n) \subset U \subset \bar{U} \sqsubset\sqsubset \mathbb{R}^d$ for all n. Since η is compactly supported there exists a precompact open set V such that $\bar{U} \subset V$ and $v_n := \eta * u_n \in C_c^\infty(V) \subset C_c^\infty(\mathbb{R}^d)$ for all n. Since,

$$\begin{aligned}\|v_n\|_{L^\infty} &\leq \|\eta\|_{L^\infty} \|u_n\|_{L^1} \leq \|\eta\|_{L^\infty} \|u_n\|_{L^1} \leq C\|\eta\|_{L^\infty} \text{and} \\ \|Dv_n\|_{L^\infty} &= \|\eta * Du_n\|_{L^\infty} \leq \|\eta\|_{L^\infty} \|Du_n\|_{L^1} \leq C\|\eta\|_{L^\infty},\end{aligned}$$

it follows by the Arzela-Ascoli theorem that $\{v_n\}_{n=1}^\infty$ has a uniformly convergent subsequence. ■

Lemma 27.24. *Let $\eta \in C_c^\infty(B(0,1),[0,\infty))$ such that $\int_{\mathbb{R}^d} \eta dm = 1$, $\eta_m(x) = m^n \eta(mx)$ and $K_m u = (C_{\eta_m} Eu)|_\Omega$. Then for all $p \in [1,d)$ and $q \in [1,p^*)$,*

$$\lim_{m\to\infty} \|K_m - i\|_{B(W^{1,p}(\Omega), L^q(\Omega))} = 0$$

where $i : W^{1,p}(\Omega) \to L^q(\Omega)$ is the inclusion map.

Proof. For $u \in C_c^1(U)$ let $v_m := \eta_m * u - u$, then

$$\begin{aligned}|v_m(x)| \leq |\eta_m * u(x) - u(x)| &= \left|\int_{\mathbb{R}^d} \eta_m(y)(u(x-y) - u(x))dy\right| \\ &= \left|\int_{\mathbb{R}^d} \eta(y)\left[u(x - \frac{y}{m}) - u(x)\right] dy\right| \\ &\leq \frac{1}{m}\int_{\mathbb{R}^d} dy\ |y|\,\eta(y) \int_0^1 dt\ \left|\nabla u(x - t\frac{y}{m})\right|\end{aligned}$$

and so by Minikowski's inequality for integrals,

$$\begin{aligned}\|v_m\|_{L^r} &\le \frac{1}{m}\int_{\mathbb{R}^d} dy\ \ |y|\,\eta(y)\int_0^1 dt\ \left\|\nabla u(\cdot - t\frac{y}{m})\right\|_{L^r} \\ &\le \frac{1}{m}\left(\int_{\mathbb{R}^d} |y|\,\eta(y)dy\right)\|\nabla u\|_{L^r} \le \frac{1}{m}\|u\|_{W^{1,r}(\mathbb{R}^d)}\,. \end{aligned} \tag{27.26}$$

By the interpolation inequality in Corollary 9.23, Theorem 27.14 and Eq. (27.26) with $r = 1$,

$$\begin{aligned}\|v_m\|_{L^q} &\le \|v_m\|_{L^1}^{\lambda}\|v_m\|_{L^{p^*}}^{1-\lambda} \\ &\le \frac{1}{m^{\lambda}}\|v_m\|_{W^{1,1}(\mathbb{R}^d)}^{\lambda}\left[d^{-1/2}\frac{p(d-1)}{d-p}\|\nabla v_m\|_{L^p}\right]^{1-\lambda} \\ &\le Cm^{-\lambda}\|v_m\|_{W^{1,1}(\mathbb{R}^d)}^{\lambda}\|v_m\|_{W^{1,p}(\mathbb{R}^d)}^{1-\lambda} \\ &\le Cm^{-\lambda}\|v_m\|_{W^{1,1}(\mathbb{R}^d)}^{\lambda}\|v_m\|_{W^{1,p}(\mathbb{R}^d)}^{1-\lambda} \\ &\le C(p,|U|)m^{-\lambda}\|v_m\|_{W^{1,p}(\mathbb{R}^d)}^{\lambda}\|v_m\|_{W^{1,p}(\mathbb{R}^d)}^{1-\lambda} \\ &\le C(p,|U|)m^{-\lambda}\|v_m\|_{W^{1,p}(\mathbb{R}^d)}\end{aligned}$$

where $\lambda \in (0,1)$ is determined by

$$\frac{1}{q} = \frac{\lambda}{1} + \frac{1-\lambda}{p^*} = \lambda\left(1 - \frac{1}{p^*}\right) + \frac{1}{p^*}.$$

Now using Proposition 11.12,

$$\begin{aligned}\|v_m\|_{W^{1,p}(\mathbb{R}^d)} &= \|\eta_m * u - u\|_{W^{1,p}(\mathbb{R}^d)} \\ &\le \|\eta_m * u\|_{W^{1,p}(\mathbb{R}^d)} + \|u\|_{W^{1,p}(\mathbb{R}^d)} \le 2\|u\|_{W^{1,p}(\mathbb{R}^d)}\,.\end{aligned}$$

Putting this all together shows

$$\begin{aligned}\|K_m u - u\|_{L^q(\Omega)} &\le \|K_m u - Eu\|_{L^q} \le C(p,|U|)m^{-\lambda}\|Eu\|_{W^{1,p}(\mathbb{R}^d)} \\ &\le C(p,|U|)m^{-\lambda}\|u\|_{W^{1,p}(\Omega)}\end{aligned}$$

from which it follows that

$$\|K_m - i\|_{B(W^{1,p}(\Omega),L^q(\Omega))} \le Cm^{-\lambda} \to 0 \text{ as } m \to \infty.$$

■

Theorem 27.25 (Rellich - Kondrachov Compactness Theorem)**.** *Suppose $\Omega \subset \mathbb{R}^d$ is a precompact open subset with C^1-boundary, $p \in [1,d)$ and $1 \le q < p^*$ then $W^{1,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$.*

Proof. If $\{u_n\}_{n=1}^{\infty}$ is contained in the unit ball in $W^{1,p}(\Omega)$, then by Lemma 27.23 $\{K_m u_n\}_{n=1}^{\infty}$ has a uniformly convergent subsequence and hence is convergent in $L^q(\Omega)$. This shows $K_m : W^{1,p}(\Omega) \to L^q(\Omega)$ is compact for every m. By Lemma 27.24, $K_m \to i$ in the $L\left(W^{1,p}(\Omega), L^q(\Omega)\right)$ – norm and so by Lemma 27.22 $i : W^{1,p}(\Omega) \to L^q(\Omega)$ is compact. ■

Corollary 27.26. *The inclusion of $W^{k,p}(\Omega)$ into $W^{k-\ell,q}(\Omega)$ is compact provided $l \ge 1$ and $\frac{1}{q} > \frac{1}{p} - \frac{l}{d} = \frac{d-pl}{dp} > 0$, i.e. $q < \frac{dp}{d-pl}$.*

Proof. Case (i) Suppose $\ell = 1$, $q \in [1, p^*)$ and $\{u_n\}_{n=1}^{\infty} \subset W^{k,p}(\Omega)$ is bounded. Then $\{\partial^\alpha u_n\}_{n=1}^{\infty} \subset W^{1,p}(\Omega)$ is bounded for all $|\alpha| \le k-1$ and therefore there exist a subsequence $\{\tilde{u}_n\}_{n=1}^{\infty} \subset \{u_n\}_{n=1}^{\infty}$ such that $\partial^\alpha \tilde{u}_n$ is convergent in $L^q(\Omega)$ for all $|\alpha| \le k-1$. This shows that $\{\tilde{u}_n\}$ is $W^{k-1,q}(\Omega)$ – convergent and so proves this case.

Case (ii) $\ell > 1$. Let $\tilde{p}$ be defined so that $\frac{1}{\tilde{p}} = \frac{1}{p} - \frac{\ell-1}{d}$. Then

$$W^{k,p}(\Omega) \subset W^{k-\ell+1,\tilde{p}}(\Omega) \subset\subset W^{k-\ell,q}(\Omega).$$

and therefore $W^{k,p}(\Omega) \subset\subset W^{k-\ell,q}(\Omega)$. ∎

Example 27.27. It is necessary to assume that The inclusion of $L^2([0,1]) \hookrightarrow L^1([0,1])$ is continuous (in fact a contraction) but not compact. To see this, take $\{u_n\}_{n=1}^{\infty}$ to be the Haar basis for L^2. Then $u_n \to 0$ weakly in both L^2 and L^1 so if $\{u_n\}_{n=1}^{\infty}$ were to have a convergent subsequence the limit would have to be $0 \in L^1$. On the other hand, since $|u_n| = 1$, $\|u_n\|_2 = \|u_n\|_1 = 1$ and any subsequential limit would have to have norm one and in particular not be 0.

Lemma 27.28. *Let Ω be a precompact open set such that $\bar{\Omega}$ is a manifold with C^1 – boundary. Then for all $p \in [1, \infty)$, $W^{1,p}(\Omega)$ is compactly embedded in $L^p(\Omega)$. Moreover if $p > d$ and $0 \le \beta < 1 - \frac{d}{p}$, then $W^{1,p}(\Omega)$ is compactly embedded in $C^{0,\beta}(\Omega)$. In particular, $W^{1,p}(\Omega) \subset\subset L^\infty(\Omega)$ for all $d < p \le \infty$.*

Proof. Case 1, $p \in [1, d)$. By Theorem 27.25, $W^{1,p}(\Omega) \subset\subset L^q(\Omega)$ for all $1 \le q < p^*$. Since $p^* > p$ we may choose $q = p$ to learn $W^{1,p}(\Omega) \subset\subset L^p(\Omega)$.

Case 2, $p \in [d, \infty)$. For any $p_0 \in [1, d)$, we have

$$W^{1,p}(\Omega) \hookrightarrow W^{1,p_0}(\Omega) \subset\subset L^{p_0^*}(\Omega).$$

Since $p_0^* = \frac{p_0 d}{d - p_o} \uparrow \infty$ as $p_0 \uparrow d$, we see that $W^{1,p}(\Omega) \subset\subset L^q(\Omega)$ for all $q < \infty$. Moreover by Morrey's inequality (Corollary 27.8) and Proposition26.13 we have $W^{1,p}(\Omega) \hookrightarrow C^{0,1-\frac{d}{p}}(\Omega) \subset\subset C^{0,\beta}(\Omega)$ which completes the proof. ∎

Remark 27.29. Similar proofs may be given to show $W^{k,p} \subset\subset C^{k-\frac{d}{p}-\delta}$ for all $\delta > 0$ provided $k - \frac{d}{p} > 0$ and $k - \frac{d}{p} - \delta > 0$.

Lemma 27.30 (Poincaré Lemma)**.** *Assume $1 \le p \le \infty$, Ω is a precompact open subset of $\mathbb{R}^d$ such that $\overline{\Omega}$ is a manifold with C^1-boundary. Then exist $C = C(\Omega, \rho)$ such that*

$$\|u - u_\Omega\|_{L^p(\Omega)} \le C \|\nabla u\|_{L^p(\Omega)} \text{ for all } u \in W^{1,p}(\Omega), \tag{27.27}$$

where $u_\Omega := ⨍_\Omega u dm$ is the average of u on Ω as in Notation 27.3.

Proof. For sake of contradiction suppose there is no $C < \infty$ such that Eq. (27.27) holds. Then there exists a sequence $\{u_n\}_{n=1}^{\infty} \subset W^{1,p}(\Omega)$ such that

$$\|u_n - (u_n)_\Omega\|_{L^p(\Omega)} > n \|\nabla u_n\|_{L^p(\Omega)} \text{ for all } n.$$

Let

$$u_n := \frac{u_n - (u_n)_\Omega}{\|u_n - (u_n)_\Omega\|_{L^p(\Omega)}}.$$

Then $u_n \in W^{1,p}(\Omega)$, $(u_n)_\Omega = 0$, $\|u_n\|_{L^p(\Omega)} = 1$ and $1 = \|u_n\|_{L^p(\Omega)} > n \|\nabla u_n\|_{L^p(\Omega)}$ for all n. Therefore $\|\nabla u_n\|_{L^p(\Omega)} < \frac{1}{n}$ and in particular $\sup_n \|u_n\|_{W^{1,p}(\Omega)} < \infty$ and hence by passing to a subsequence if necessary there exists $u \in L^p(\Omega)$ such that

$u_n \to u$ in $L^p(\Omega)$. Since $\nabla u_n \to 0$ in $L^p(\Omega)$, it follows that u_n is convergent in $W^{1,p}(\Omega)$ and hence $u \in W^{1,p}(\Omega)$ and $\nabla u = \lim_{n\to\infty} \nabla u_n = 0$ in $L^p(\Omega)$. Since $\nabla u = 0$, $u \in W^{k,p}(\Omega)$ for all $k \in \mathbb{N}$ and hence $u \in C^\infty(\Omega)$ and $\nabla u = 0$ implies u is constant. Since $u_\Omega = 0$ we must have $u \equiv 0$ which is clearly impossible since $\|u\|_{L^p(\Omega)} = \lim_{n\to\infty} \|u_n\|_{L^p(\Omega)} = 1$. ∎

Theorem 27.31 (Poincaré Lemma). *Let Ω be a precompact open subset of $\mathbb{R}^d$ and $p \in [1,\infty]$. Then*

$$\|u\|_{L^p} \le 2 \operatorname{diam}(\Omega)\|\nabla u\|_{L^p} \text{ for all } u \in W_0^{1,p}(\Omega).$$

Proof. Without loss of generality assume $\Omega = [-m,m]^d$ and $u \in C_c^\infty(\Omega)$. Then by the fundamental theorem of calculus,

$$|u(x)| = |\int_{-M}^{x_1} \partial_1 u(y_1, x_2, \dots, x_d)dy_1| \le \int_{-M}^{M} |\partial_1 u(y_1 x_2, \dots, x_d)|dy$$

and hence by Holder's inequality,

$$|u(x)|^p \le (2M)^{p-1} \int_{-M}^{M} |\partial_1 u(y, x_2, \dots, x_d)|^p dy_1.$$

Integrating this equation over x implies,

$$\|u\|_{L^p}^p \le (2M)(2M)^{p-1} \int |\partial_1 u(x)|^p dx = (2M)^p \int |\partial_1 u(x)|^p dx$$

and hence

$$\|u\|_{L^p} \le 2M\|\partial_1 u\|_{L^p} \le 2 \operatorname{diam}(\Omega)\|\nabla u\|_{L^p}.$$

∎

27.6. **Fourier Transform Method.** See L^2 – Sobolev spaces for another proof of the following theorem.

Theorem 27.32. *Suppose $s > t \ge 0$, $\{u_n\}_{n=1}^\infty$ is a bounded sequence (say by 1) in $H^s(\mathbb{R}^d)$ such that $K = \overline{\cup_n \operatorname{supp}(u_n)} \sqsubset\sqsubset \mathbb{R}^d$. Then there exist a subsequence $\{v_n\}_{n=1}^\infty \subset \{u_n\}_{n=1}^\infty$ which is convergent in $H^t(\mathbb{R}^d)$.*

Proof. Since

$$\begin{aligned} \left|\partial_\xi^\alpha \hat{u}_n(\xi)\right| &= \left|\partial_\xi^\alpha \int_{\mathbb{R}^d} e^{-i\xi\cdot x} u_n(x)dx\right| = \left|\int_{\mathbb{R}^d} (-ix)^\alpha e^{-i\xi\cdot x} u_n(x)dx\right| \\ &\le \|x^\alpha\|_{L^2(K)}\|u_n\|_{L^2} \le C_\alpha \|u_n\|_{H^s(\mathbb{R}^d)} \le C_\alpha \end{aligned}$$

$\hat{u}_n$ and all of it's derivatives are uniformly bounded. By the Arzela-Ascoli theorem and Cantor's Diagonalization argument, there exists a subsequence $\{v_n\}_{n=1}^\infty \subset \{u_n\}_{n=1}^\infty$ such that $\hat{v}_n$ and all of its derivatives converge uniformly on compact subsets in ξ –space. If $\hat{v}(\xi) := \lim_{n\to\infty} \hat{v}_n(\xi)$, then by the dominated convergence theorem,

$$\int_{|\xi|\le R} (1+|\xi|^2)^s |\hat{v}(\xi)|^2 d\xi = \lim_{n\to\infty} \int_{|\xi|\le R} (1+|\xi|^2)^s |\hat{v}_n(\xi)|^2 d\xi \le \limsup_{n\to\infty} \|v_n\|^2_{H^s(\mathbb{R}^d)} \le 1.$$

Since R is arbitrary this implies $\hat{v} \in L^2((1+|\xi|^2)^s d\xi)$ and $\|v\|_{H^s(\mathbb{R}^d)} \le 1$. Set $g_n := v - v_n$ while $v = \mathcal{F}^{-1}\hat{v}$. Then $\{g_n\}_{n=1}^\infty \subset H^s(\mathbb{R}^d)$ and we wish to show

$g_n \to 0$ in $H^t(\mathbb{R}^d)$. Let $d\mu_t(\xi) = (1+|\xi|^2)^t d\xi$, then for any $R < \infty$,

$$\begin{aligned}\|g_n\|_{H^t}^2 &= \int |\hat{g}(\xi) - \hat{g}_n(\xi)|^2 \, d\mu_t(\xi) \\ &= \int_{|\xi|\le R} |\hat{g}(\xi) - \hat{g}_n(\xi)|^2 d\mu_t(\xi) + \int_{|\xi|\ge R} |\hat{g}(\xi) - \hat{g}_n(\xi)|^2 d\mu_t(\xi).\end{aligned}$$

The first term goes to zero by the dominated convergence theorem, hence

$$\begin{aligned}\limsup_{n\to\infty} \|g_n\|_{H^t}^2 &\le \limsup_{n\to\infty} \int_{|\xi|\ge R} |\hat{g}(\xi) - \hat{g}_n(\xi)|^2 d\mu_t(\xi) \\ &= \limsup_{n\to\infty} \int_{|\xi|\ge R} |\hat{g} - \hat{g}_n(\xi)|^2 \frac{(1+|\xi|^2)^s}{(1+|\xi|^2)^{s-t}} d\xi \\ &\le \limsup_{n\to\infty} \frac{1}{(1+R^2)^{s-t}} \int_{|\xi|\ge R} |\hat{g} - \hat{g}_n(\xi)|^2 d\mu_s(\xi) \\ &\le \limsup_{n\to\infty} \frac{1}{(1+R^2)^{s-t}} \|g_n - g\|_{H^t}^2 \\ &\le 4 \left(\frac{1}{1+R^2}\right)^{s-t} \to 0 \text{ as } \ R \to \infty.\end{aligned}$$

■

27.7. **Other theorems along these lines.** Another theorem of this form is derived as follows. Let $\rho > 0$ be fixed and $g \in C_c((0,1),[0,1])$ such that $g(t) = 1$ for $|t| \le 1/2$ and set $\tau(t) := g(t/\rho)$. Then for $x \in \mathbb{R}^d$ and $\omega \in \Gamma$ we have

$$\int_0^\rho \frac{d}{dt} [\tau(t) u(x + t\omega)] \, dt = -u(x)$$

and then by integration by parts repeatedly we learn that

$$\begin{aligned}u(x) &= \int_0^\rho \partial_t^2 [\tau(t) u(x+t\omega)] \, t dt = \int_0^\rho \partial_t^2 [\tau(t) u(x+t\omega)] \, d\frac{t^2}{2} \\ &= -\int_0^\rho \partial_t^3 [\tau(t) u(x+t\omega)] \, d\frac{t^3}{3!} = \dots \\ &= (-1)^m \int_0^\rho \partial_t^m [\tau(t) u(x+t\omega)] \, d\frac{t^m}{m!} \\ &= (-1)^m \int_0^\rho \partial_t^m [\tau(t) u(x+t\omega)] \, \frac{t^{m-1}}{(m-1)!} dt.\end{aligned}$$

Integrating this equation on $\omega \in \Gamma$ then implies

$$
\begin{aligned}
|\Gamma| u(x) &= (-1)^m \int_\gamma d\omega \int_0^\rho \partial_t^m \left[\tau(t) u(x+t\omega)\right] \frac{t^{m-1}}{(m-1)!} dt \\
&= \frac{(-1)^m}{(m-1)!} \int_\gamma d\omega \int_0^\rho t^{m-d} \partial_t^m \left[\tau(t) u(x+t\omega)\right] t^{d-1} dt \\
&= \frac{(-1)^m}{(m-1)!} \int_\gamma d\omega \int_0^\rho t^{m-d} \sum_{k=0}^m \binom{m}{k} \left[\tau^{(m-k)}(t) \left(\partial_\omega^k u\right)(x+t\omega)\right] t^{d-1} dt \\
&= \frac{(-1)^m}{(m-1)!} \int_\gamma d\omega \int_0^\rho t^{m-d} \sum_{k=0}^m \binom{m}{k} \rho^{k-m} \left[g^{(m-k)}(t) \left(\partial_\omega^k u\right)(x+t\omega)\right] t^{d-1} dt \\
&= \frac{(-1)^m}{(m-1)!} \sum_{k=0}^m \binom{m}{k} \rho^{k-m} \int_{\Gamma_{x,\rho}} |y-x|^{m-d} \left[g^{(m-k)}(|y-x|) \left(\partial_{\widehat{y-x}}^k u\right)(y)\right] dy
\end{aligned}
$$

and hence

$$
u(x) = \frac{(-1)^m}{|\Gamma|(m-1)!} \sum_{k=0}^m \binom{m}{k} \rho^{k-m} \int_{\Gamma_{x,\rho}} |y-x|^{m-d} \left[g^{(m-k)}(|y-x|) \left(\partial_{\widehat{y-x}}^k u\right)(y)\right] dy
$$

and hence by the Hölder's inequality,

$$
|u(x)| \le C(g) \frac{(-1)^m}{|\Gamma|(m-1)!} \sum_{k=0}^m \binom{m}{k} \rho^{k-m} \left[\int_{\Gamma_{x,\rho}} |y-x|^{q(m-d)} dy\right]^{1/q} \left[\int_{\Gamma_{x,\rho}} \left|\left(\partial_{\widehat{y-x}}^k u\right)(y)\right|^p dy\right]^{1/p}.
$$

From the same computation as in Eq. (25.4) we find

$$
\begin{aligned}
\int_{\Gamma_{x,\rho}} |y-x|^{q(m-d)} dy &= \sigma(\Gamma) \int_0^\rho r^{q(m-d)} r^{d-1} dr = \sigma(\Gamma) \frac{\rho^{q(m-d)+d}}{q(m-d)+d} \\
&= \sigma(\Gamma) \frac{\rho^{\frac{pm-d}{p-1}}}{pm-d} (p-1).
\end{aligned}
$$

provided that $pm - d > 0$ (i.e. $m > d/p$) wherein we have used

$$
q(m-d) + d = \frac{p}{p-1}(m-d) + d = \frac{p(m-d) + d(p-1)}{p-1} = \frac{pm-d}{p-1}.
$$

This gives the estimate

$$
\left[\int_{\Gamma_{x,\rho}} |y-x|^{q(m-d)} dy\right]^{1/q} \le \left[\frac{\sigma(\Gamma)(p-1)}{pm-d}\right]^{\frac{p-1}{p}} \rho^{\frac{pm-d}{p}} = \left[\frac{\sigma(\Gamma)(p-1)}{pm-d}\right]^{\frac{p-1}{p}} \rho^{m-d/p}.
$$

Thus we have obtained the estimate that

$$
|u(x)| \le \frac{C(g)}{|\Gamma|(m-1)!} \left[\frac{\sigma(\Gamma)(p-1)}{pm-d}\right]^{\frac{p-1}{p}} \rho^{m-d/p} \sum_{k=0}^m \binom{m}{k} \rho^{k-m} \left\|\partial_{\widehat{y-x}}^k u\right\|_{L^p(\Gamma_{x,p})}.
$$

27.8. **Exercises.**

Exercise 27.1. Let $a_i \ge 0$ and $p_i \in [1,\infty)$ for $i = 1, 2, \ldots, d$ satisfy $\sum_{i=1}^d p_i^{-1} = 1$, then

$$
\prod_{i=1}^d a_i \le \sum_{i=1}^d \frac{1}{p_i} a_i^{p_i}.
$$

Hint: This may be proved by induction on d making use of Lemma 2.27. Alternatively see Example 9.11, where this is already proved using Jensen's inequality.

27.1. We may assume that $a_i > 0$, in which case

$$\prod_{i=1}^{d} a_i = e^{\sum_{i=1}^{d} \ln a_i} = e^{\sum_{i=1}^{d} \frac{1}{p_i} \ln a_i^{p_i}} \leq \sum_{i=1}^{d} \frac{1}{p_i} e^{\ln a_i^{p_i}} = \sum_{i=1}^{d} \frac{1}{p_i} a_i^{p_i}.$$

This was already done in Example 9.11. ■