Notes for a seminar on the Brascamp-Lieb inequalities

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Abstract

These notes describe Barthe's proof of the forward and reverse Brascamp-Lieb inequalities, recapitulating a presentation delivered at the optimal transportation seminar at the University of Alberta on 2012 March 20 and 27.

1 The inequalities

Fix some vectors $(v_i)_{i=1}^m$ in $\mathbb{R}^n$ and some positive real numbers $(c_i)_{i=1}^m$.

The **Brascamp-Lieb inequality** asserts that, for any nonnegative integrable $\mathbb{R} \to \mathbb{R}$ functions $(f_i)_{i=1}^m$,

$$\int_{\mathbb{R}^n} \prod_{i=1}^m f_i(\langle x, v_i \rangle)^{c_i} \, dx \le F \prod_{i=1}^m \left(\int_{\mathbb{R}} f_i \right)^{c_i} \tag{BL}$$

Here F is a constant depending on the c_i and the v_i, but not on the f_i. (In order for the inequality to be nontrivial, we want $F < \infty$, which depends on the c_i and the v_i satisfying certain conditions which will be specified later.)

A few remarks:

1. (Homogeneity.) If $\lambda > 0$, then replacing one f_i with λf_i changes both sides of (BL) by λ^{c_i}. Thus we can usually assume that all $\int f_i = 1$.

2. (Homogeneity.) If $\lambda > 0$, then replacing each f_i with $f_i(\lambda \cdot)$ (the composition of f_i with multiplication by λ) changes the LHS of (BL) by λ^{-n} and the RHS by $\lambda^{-\sum_i c_i}$. Thus it is necessary to assume

$$\sum_{i=1}^m c_i = n \tag{1}$$

   in order to have $F < \infty$.

3. The special case $n = 1$ and $m = 2$ is Hölder's inequality. (The condition (1) corresponds to the assumption that $\frac{1}{p} + \frac{1}{q} = 1$.)

4. If $\bigcap_{i=1}^m v_i^\perp \neq \{0\}$ then the integral on the LHS of (BL) is usually infinite:
$$\begin{aligned}\int_{\mathbb{R}^n} \prod(\cdots)\,dx &= \int_{\bigcap_{i=1}^m v_i^\perp} \int_{\text{span }\{v_i:\,i=1,\ldots,m\}} \prod_{i=1}^m f_i(\underbrace{\langle y+z, v_i\rangle}_{=\langle y, v_i\rangle})^{c_i}\,dy\,dz \\ &= \left(\int_{\bigcap_{i=1}^m v_i^\perp} dz\right)\left(\int_{\text{span }\{v_i:\,i=1,\ldots,m\}} \prod_{i=1}^m f_i(\langle y, v_i\rangle)^{c_i}\,dy\right) \\ &= \infty\ .\end{aligned}$$
(In short, the functions $f_i(\langle \cdot, v_i\rangle)$ don't decay in the directions of $\bigcap_{i=1}^m v_i^\perp$.) Thus it is necessary to assume
$$\bigcap_{i=1}^m v_i^\perp = \{0\} \tag{2}$$
in order to have $F < \infty$. (Equivalently, we assume that the v_i span $\mathbb{R}^n$. Usually $m > n$ and the v_i are linearly dependent.)

(Conditions (1) and (2) are necessary for $F < \infty$ but not sufficient; see section 3.)

Let F denote the best constant in (BL) (for the given c_i and v_i), that is,
$$F = \sup\left\{ \frac{\int_{\mathbb{R}^n} \prod_{i=1}^m f_i(\langle x, v_i\rangle)^{c_i}\,dx}{\prod_{i=1}^m \left(\int_{\mathbb{R}} f_i\right)^{c_i}} \;\middle|\; (f_i)_{i=1}^m \text{ nonnegative and integrable}\right\}\ .$$
Let F_g denote the best constant when the f_i are required to be centred gaussian functions:
$$F_g = \sup\left\{ \frac{\int_{\mathbb{R}^n} \prod_{i=1}^m f_i(\langle x, v_i\rangle)^{c_i}\,dx}{\prod_{i=1}^m \left(\int_{\mathbb{R}} f_i\right)^{c_i}} \;\middle|\; f_i(t) = e^{-\lambda_i t^2},\ \lambda_i > 0\right\}\ .$$
Clearly $F \geq F_g$; we will show that in fact $F = F_g$. In this sense, the inequality is "saturated" by gaussian functions.

The **reverse Brascamp-Lieb inequality**, also called **Barthe's inequality**, asserts that, for any nonnegative integrable $\mathbb{R} \to \mathbb{R}$ functions $(f_i)_{i=1}^m$, if $h\colon \mathbb{R}^n \to \mathbb{R}$ is a measurable function such that
$$h\Big(\sum_{i=1}^m c_i\theta_i v_i\Big) \geq \prod_{i=1}^m f_i(\theta_i)^{c_i} \qquad \text{for any real numbers } (\theta_i)_{i=1}^m, \tag{3}$$
then
$$\int_{\mathbb{R}^n} h(y)\,dy \geq E \prod_{i=1}^m \left(\int_{\mathbb{R}} f_i\right)^{c_i}\ . \tag{RBL}$$
Here E is a constant depending on the c_i and the v_i, but not on the f_i. For a nontrivial inequality we want $E > 0$, which requires assumptions on the c_i and v_i.

Usually the v_i are linearly dependent, so any $x \in \mathbb{R}^n$ has many representations $x = \sum_{i=1}^m c_i\theta_i v_i$. The hypothesis (3) on h means that

$$h(x) \geq \sup \left\{ \prod_{i=1}^m f_i(\theta_i)^{c_i} \;\middle|\; x = \sum_{i=1}^m c_i\theta_i v_i \right\} . \tag{4}$$

We could define h to be this supremum; the only reason not to do so is that it might not be measurable. In the proof we will see that when the f_i are centred gaussians, this supremum has a particularly simple form and is measurable; we will also see how this strange hypothesis originates in duality considerations.

Let E be the best constant in (RBL) and let E_g be the best constant when the f_i are required to be centred gaussians. Clearly $E \leq E_g$; we will show that in fact $E = E_g$, and moreover,

$$E = E_g = \sqrt{D} \qquad \text{and} \qquad F = F_g = \frac{1}{\sqrt{D}} \ , \tag{5}$$

where

$$D = \inf \left\{ \frac{\det\left(\sum_{i=1}^m c_i\lambda_i v_i \otimes v_i\right)}{\prod_{i=1}^m \lambda_i^{c_i}} \;\middle|\; \lambda_i > 0 \right\} . \tag{6}$$

Here $x \otimes y$ denotes the map $x \otimes y \colon \mathbb{R}^n \to R$, $x \otimes y(z) = \langle z, x\rangle y$, which is a linear operator of rank one (unless $x = 0$ or $y = 0$), with matrix yx^T and trace $\langle x, y\rangle$. If $x = y$ it is symmetric; if $x = y$ and $|x| = 1$ then it is the orthogonal projection onto the line spanned by x.

2 The proof

We will prove three statements:

$$F_g = 1/\sqrt{D} \tag{7}$$
$$E_g F_g = 1 \tag{8}$$
$$E \geq DF \tag{9}$$

Statement (7) is essentially a classical computation on gaussians; statement (8) uses a duality argument; statement (9) involves optimal transportation. Together these three statements yield

$$\sqrt{D} = E_g \geq E \geq DF \geq DF_g = \sqrt{D} \ ,$$

which establishes (5), as desired.

2.1 First part: $F_g = 1/\sqrt{D}$

First, a classical computation: if A is a symmetric positive definite $n \times n$ matrix, then

$$\int_{\mathbb{R}^n} e^{-\langle Ax, x\rangle}\, dx = \sqrt{\frac{\pi^n}{\det A}} \ . \tag{10}$$

(Note that $\langle Ax, x\rangle$ is a positive definite quadratic form on $\mathbb{R}^n$; thus $e^{-\langle Ax,x\rangle}$ is the density of a centred gaussian.) Indeed, since A is symmetric and positive definite, it has a square root, which is also symmetric and positive definite; so

$$\begin{aligned}\int_{\mathbb{R}^n} e^{-\langle Ax,x\rangle}\,dx &= \int_{\mathbb{R}^n} e^{-\langle \sqrt{A}x,\sqrt{A}x\rangle}\,dx = \frac{1}{\sqrt{\det A}}\int_{\mathbb{R}^n} e^{-\langle y,y\rangle}\,dy \\ &= \frac{1}{\sqrt{\det A}}\int_{\mathbb{R}}\int_{\mathbb{R}}\cdots\int_{\mathbb{R}} e^{-y_1^2}e^{-y_2^2}\cdots e^{-y_n^2}\,dy_1\,dy_2\cdots dy_n \\ &= \frac{1}{\sqrt{\det A}}\left(\int_{\mathbb{R}} e^{-t^2}\,dt\right)^n = \sqrt{\frac{\pi^n}{\det A}}\ .\end{aligned}$$

With the formula (10) in hand, we can now prove statement (7). Let $(\lambda_i)_{i=1}^m$ be positive reals and let $f_i(t) = e^{-\lambda_i t^2}$. Let $A = \sum_{i=1}^m c_i\lambda_i v_i \otimes v_i$. Note that if $x \neq 0$ then $\langle x, v_i\rangle \neq 0$ for some i by our assumption (2), and so $\langle Ax, x\rangle = \sum_{i=1}^m c_i\lambda_i\langle x, v_i\rangle^2 > 0$; thus A is positive definite, and so

$$\int_{\mathbb{R}^n}\prod_{i=1}^m f_i(\langle x, v_i\rangle)^{c_i}\,dx = \int_{\mathbb{R}^n} e^{-\sum_{i=1}^m c_i\lambda_i\langle x,v_i\rangle^2}\,dx = \sqrt{\frac{\pi^n}{\det A}}\ .$$

by (10). On the other hand,

$$\prod_{i=1}^m\left(\int_{\mathbb{R}} f_i\right)^{c_i} = \prod_{i=1}^m\left(\sqrt{\frac{\pi}{\lambda_i}}\right)^{c_i} = \sqrt{\frac{\pi^n}{\prod_{i=1}^m \lambda_i^{c_i}}}$$

(using (1)). Dividing and optimizing over the λ_i yields (7).

2.2 Second part: $E_g F_g = 1$

Now we develop the duality tools needed to prove (8). Let $(\lambda_i)_{i=1}^m$ be positive reals and let $A = \sum_{i=1}^m c_i\lambda_i v_i \otimes v_i$, as before. Since A is positive definite (as noted above),

$$\|x\| = \sqrt{\langle Ax, x\rangle} = \sqrt{\sum_{i=1}^m c_i\lambda_i\langle x, v_i\rangle^2}$$

is a norm on $\mathbb{R}^n$. (In fact its unit ball is $A^{-1/2}B_2^n$.) We claim that the dual norm satisfies

$$\|y\|_* = \sup_{x\neq 0}\frac{\langle x,y\rangle}{\|x\|} = \sqrt{\langle A^{-1}y, y\rangle} = \inf\left\{\sqrt{\sum_{i=1}^m\frac{c_i\theta_i^2}{\lambda_i}}\ \middle|\ y = \sum_{i=1}^m c_i\theta_i v_i\right\}\ . \tag{11}$$

The first equality in (11) is the definition of dual norms. The second holds because

$$\begin{aligned}\sup_{x\neq 0}\frac{\langle x,y\rangle}{\|x\|} = \sup_{x\neq 0}\frac{\langle x,y\rangle}{\sqrt{\langle Ax,x\rangle}} &= \sup_{x\neq 0}\frac{\langle A^{1/2}x, A^{-1/2}y\rangle}{\sqrt{\langle A^{1/2}x, A^{1/2}x\rangle}} \\ &= \sqrt{\langle A^{-1/2}y, A^{-1/2}y\rangle} = \sqrt{\langle A^{-1}y, y\rangle}\end{aligned}$$

by the Cauchy-Schwarz inequality (and its equality case).

To prove the third equality in (11), let $y \in \mathbb{R}^n$, and let $y = \sum_{i=1}^m c_i \theta_i v_i$. (There exists such a representation of y because the v_i span $\mathbb{R}^n$ by our assumption (2).) Then, for any $x \in \mathbb{R}^n$,

$$\langle x, y\rangle = \sum_{i=1}^m c_i \theta_i \langle x, v_i\rangle = \sum_{i=1}^m \sqrt{\frac{c_i}{\lambda_i}}\,\theta_i \cdot \sqrt{c_i\lambda_i}\langle x, v_i\rangle \le \left(\sum_{=1}^m \frac{c_i\theta_i^2}{\lambda_i}\right)^{1/2} \|x\|$$

by the Cauchy-Schwarz inequality; therefore

$$\sup_{x\neq 0} \frac{\langle x, y\rangle}{\|x\|} \le \inf\left\{\sqrt{\sum_{i=1}^m \frac{c_i\theta_i^2}{\lambda_i}} \;\middle|\; y = \sum_{i=1}^m c_i\theta_i v_i\right\} .$$

On the other hand, we obtain equality by taking $x = A^{-1}y$ and $\theta_i = \lambda_i\langle x, v_i\rangle$. (Note that

$$\sum_{i=1}^m c_i\theta_i v_i = \sum_{i=1}^m c_i\lambda_i\langle x, v_i\rangle v_i = Ax = y\ ,$$

so these θ_i give one of the representations considered above.) This completes the proof of (11).

Now we can prove (8). We will show that, for any positive real numbers $(\lambda_i)_{i=1}^m$, setting $f_i(t) = e^{-\lambda_i t^2}$ and $\tilde{f}_i(t) = e^{-t^2/\lambda_i}$, we have

$$\underbrace{\frac{\int_{\mathbb{R}^n} \sup\left\{\prod_{i=1}^m \tilde{f}_i(\theta_i)^{c_i} \mid y = \sum_{i=1}^m c_i\theta_i v_i\right\} dy}{\prod_{i=1}^m \left(\int_{\mathbb{R}} \tilde{f}_i\right)^{c_i}}}_{=:E_g((1/\lambda_i)_{i=1}^m)} \cdot \underbrace{\frac{\int_{\mathbb{R}^n} \prod_{i=1}^m f_i(\langle x, v_i\rangle)^{c_i}\, dx}{\prod_{i=1}^m \left(\int_{\mathbb{R}} f_i\right)^{c_i}}}_{=:F_g((\lambda_i)_{i=1}^m)} = 1\ . \tag{12}$$

(At the upper left, we have replaced the generic h in (RBL) with the “optimal” h. As noted after (4), this is not possible in general because that optimal h need not be measurable; we will show that when the $\tilde{f}_i$ are centred gaussians, as here, it is measurable.)

The equality (12) suffices to prove (8) because (12) implies

$$\begin{aligned} E_g = \inf_{(\lambda_i)_{i=1}^m} E_g((\lambda_i)_{i=1}^m) &= \inf_{(\lambda_i)_{i=1}^m} E_g((1/\lambda_i)_{i=1}^m) \\ &= \inf_{(\lambda_i)_{i=1}^m} \frac{1}{F_g((\lambda_i)_{i=1}^m)} = \frac{1}{\sup_{(\lambda_i)_{i=1}^m} F_g((\lambda_i)_{i=1}^m)} = \frac{1}{F_g}\ . \end{aligned}$$

To prove (12), we simply compute the four factors on the left-hand side. As before, let $A = \sum_{i=1}^m c_i\lambda_i v_i \otimes v_i$ and let $\|x\| = \sqrt{\langle Ax, x\rangle}$. The factor at the upper right we have computed before:

$$\int_{\mathbb{R}^n} \prod_{i=1}^m f_i(\langle x, v_i\rangle)^{c_i}\, dx = \int_{\mathbb{R}^n} e^{-\langle Ax, x\rangle}\, dx = \sqrt{\frac{\pi^n}{\det A}}$$

by (10). The factors at the bottom left and bottom right are similar:

$$\prod_{i=1}^{m}\left(\int_{\mathbb{R}}\tilde{f}_i\right)^{c_i} = \sqrt{\frac{\pi^n}{\prod_{i=1}^m \lambda_i^{-c_i}}} \quad \text{and} \quad \prod_{i=1}^{m}\left(\int_{\mathbb{R}} f_i\right)^{c_i} = \sqrt{\frac{\pi^n}{\prod_{i=1}^m \lambda_i^{c_i}}} \ .$$

For the factor at the top left, note that $\prod_{i=1}^m \tilde{f}_i(\theta_i)^{c_i} = e^{-\sum_{i=1}^m c_i\theta_i^2/\lambda_i}$, and so

$$\sup\left\{\prod_{i=1}^m \tilde{f}_i(\theta_i)^{c_i} \mid y = \sum_{i=1}^m c_i\theta_i v_i\right\} = e^{-\inf\,\{\sum_i c_i\theta_i^2/\lambda_i \,\mid\, y=\sum_i c_i\theta_i v_i\}} = e^{-\|y\|_*^2} \ ,$$

using one of the expressions in (11). Thus the integrand at the top left is measurable, as claimed, and moreover, the integral is

$$\int_{\mathbb{R}^n} e^{-\|y\|_*^2}\, dy = \int_{\mathbb{R}^n} e^{-\langle A^{-1}y,y\rangle}\, dy = \sqrt{\frac{\pi^n}{\det(A^{-1})}} \ ,$$

again by (10). Multiplying the four factors together establishes (12).

2.3 Third part: $E \geq DF$

The supremum in the definition of F can be considered to arise from an optimization problem: find functions $(f_i)_{i=1}^m$ minimizing a certain objective function, namely, the ratio considered in that supremum. Similarly for E. We will consider candidate solutions for these two optimization problems, use optimal transport methods to transport one to the other, and compute how the objective functions behave under that transportation.

Accordingly, let $(f_i)_{i=1}^m$ and $(g_i)_{i=1}^m$ be nonnegative integrable $\mathbb{R}\to\mathbb{R}$ functions. Suppose $h\colon \mathbb{R}\to\mathbb{R}$ is measurable and satisfies (3). We wish to show that

$$\frac{\int_{\mathbb{R}^n} h(y)\, dy}{\prod_{i=1}^m \left(\int_{\mathbb{R}} f_i\right)^{c_i}} \geq D\,\frac{\int_{\mathbb{R}^n}\prod_{i=1}^m g_i(\langle x, v_i\rangle)^{c_i}\, dx}{\prod_{i=1}^m \left(\int_{\mathbb{R}} g_i\right)^{c_i}} \ .$$

By an approximation argument, we may assume the f_i and g_i are continuous and strictly positive everywhere. (We will see later why this is a desirable assumption.) By homogeneity, we may also assume $\int f_i = \int g_i = 1$ for all i.

Let $T_i\colon \mathbb{R}\to\mathbb{R}$ push the measure with density f_i forward to the measure with density g_i, that is,

$$\int_{-\infty}^{T_i(t)} f_i = \int_{-\infty}^{t} g_i \qquad \text{for all } t\in\mathbb{R}.$$

(Formally: let $\Phi_i(s) = \int_{-\infty}^s f_i$; since f_i is strictly increasing, Φ_i is invertible; define $T_i(t) = \Phi_i^{-1}(\int_{-\infty}^t g_i)$.) Differentiating yields

$$f_i(T_i(t))T_i'(t) = g_i(t) \qquad \text{for all } t\in\mathbb{R}. \tag{13}$$

Note that, since the f_i and g_i are strictly positive everywhere, T_i' is also strictly positive everywhere.

(We have constructed the T_i "by hand"; we could instead have constructed them using our optimal transportation machinery — as Barthe indeed does for the multivariable versions of these inequalities. In this single-variable setting, it is more convenient to proceed by hand for a technical reason, as will be seen.)

Now we compute:

$$\int_{\mathbb{R}^n} \prod_{i=1}^m g_i(\langle x, v_i \rangle)^{c_i}\, dx$$

$$= \int_{\mathbb{R}^n} \prod_{i=1}^m f_i(T_i(\langle x, v_i \rangle))^{c_i} \prod_{i=1}^m T_i'(\langle x, v_i \rangle)^{c_i}\, dx \qquad \text{(by (13))}$$

(Now we take $\theta_i = T_i(\langle x, v_i \rangle)$ in (3), and $\lambda_i = T_i'(\langle x, v_i \rangle)$ in (6).)

$$\le \frac{1}{D} \int_{\mathbb{R}^n} h\underbrace{\Big(\sum_{i=1}^m c_i T_i(\langle x, v_i \rangle) v_i \Big)}_{=:B(x)} \underbrace{\det \Big(\sum_{i=1}^m c_i T_i'(\langle x, v_i \rangle) v_i \otimes v_i \Big)}_{\text{Jacobian of } B(x)}\, dx$$

$$= \frac{1}{D} \int_{B(\mathbb{R}^n)} h(y)\, dy$$

$$\le \frac{1}{D} \int_{\mathbb{R}^n} h(y)\, dy$$

as desired. It remains only to check that the change of variable $y = B(x)$ is valid, that is, that B is injective. First note that, since $T_i'(t) > 0$ for all $t \in \mathbb{R}$, the Jacobian matrix of $B(x)$ is positive definite: for any nonzero $z \in \mathbb{R}^n$,

$$\langle B'(x) z, z \rangle = \sum_{i=1}^m c_i T_i'(\langle x, v_i \rangle) \langle v_i, z \rangle^2 > 0 \ .$$

Now, let $x_1, x_2 \in \mathbb{R}^n$ with $x_1 \neq x_2$ and define $\Psi(s) = \langle B(x_1 + s(x_2 - x_1)), x_2 - x_1 \rangle$. Then

$$\Psi'(s) = \langle B'(x_1 + s(x_2 - x_1))(x_2 - x_1), x_2 - x_1 \rangle > 0 \ ,$$

whence $\langle B(x_1) - B(x_2), x_2 - x_1 \rangle = \Psi(1) - \Psi(0) > 0$. In particular, B is injective.

(It is in this last argument that it is convenient to have constructed the T_i "by hand"; otherwise we would have to verify the validity of the change of variable by methods more advanced than those of elementary calculus.)

3 Conditions for $D > 0$

As noted in section 1, in order for the inequalities (BL) and (RBL) to be non-trivial we require that $E > 0$ and $F < \infty$, which in turn requires that $D > 0$. Barthe proves that $D > 0$ if and only if c (that is, the sequence $(c_i)_{i=1}^m$ considered as a vector in $\mathbb{R}^m$) is in the convex hull of the indicator functions $\mathbb{1}_I$ of those subsets $I \subseteq \{1, \dots, m\}$ of cardinality n such that $(v_i \mid i \in I)$ is a basis for $\mathbb{R}^n$.

In applications in convex geometry, the v_i are often unit vectors and satisfy

$$\sum_{i=1}^{m} c_i v_i \otimes v_i = I_n \ , \tag{14}$$

where I_n is the identity map on $\mathbb{R}^n$. We will show that in this situation, $D = 1$.

First, two remarks:

1. The condition (14) should be viewed as a generalization of the v_i forming an orthonormal basis. Indeed, applying both sides of (14) to some vector $x \in \mathbb{R}^n$ yields
$$\sum_{i=1}^{m} c_i \langle x, v_i \rangle v_i = x \ . \tag{15}$$
If the v_i are an orthonormal basis and all $c_i = 1$, then (15) simply expresses x in coordinate form with respect to the v_i. Next, applying $\langle \cdot, x\rangle$ to both sides of (15) yields
$$\sum_{i=1}^{m} c_i \langle x, v_i \rangle^2 = |x|^2 \ , \tag{16}$$
which again is a familiar formula when the v_i are an orthonormal basis and all $c_i = 1$. (In fact all collections of unit vectors satisfying (14) arise by orthogonally projecting an orthonormal basis of $\mathbb{R}^m$ onto an n-dimensional subspace, then renormalizing (and adjusting the weights c_i accordingly).)

2. If the v_i are unit vectors and (14) holds then our necessary conditions (1) and (2) hold. Indeed, taking traces in (14) yields $\sum_{i=1}^{m} c_i = n$, and (15) implies that the v_i span $\mathbb{R}^n$.

First we prove that if the v_i are unit vectors and (14) holds then for any $n \times n$ matrix A,

$$\det A \le \prod_{i=1}^{m} |Av_i|^{c_i} \ . \tag{17}$$

Note that if the v_i are the standard basis for $\mathbb{R}^n$ and all $c_i = 1$, then (17) is Hadamard's inequality.

Replacing A with QA, where Q is an orthogonal matrix, does not alter the inequality (17), so by polar decomposition, we may assume A is symmetric and positive definite, say,

$$A = \sum_{j=1}^{n} \alpha_j e_j \otimes e_j \ ,$$

where $(e_j)_{j=1}^n$ is an orthonormal basis of eigenvectors of A and $(\alpha_j)_{j=1}^n$ are the associated (positive) eigenvalues. Then

$$|Av_i|^2 = \Big| \sum_{j=1}^{n} \alpha_j \langle v_i, e_j \rangle e_j \Big|^2 = \sum_{j=1}^{n} \alpha_j^2 \langle v_i, e_j \rangle^2 \ge \prod_{j=1}^{n} \alpha_j^{2\langle v_i, e_j\rangle^2} \ ,$$

where the last step invokes the AM/GM inequality with weights $\langle v_i, e_j\rangle^2$. (Note that $\sum_{j=1}^n \langle v_i, e_j\rangle^2 = |v_i|^2 = 1$, as required.) Thus

$$\prod_{i=1}^m |Av_i|^{c_i} \geq \prod_{i=1}^m \prod_{j=1}^n \alpha_j^{c_i\langle v_i, e_j\rangle^2} = \prod_{j=1}^n \alpha_j^{\sum_{i=1}^m c_i\langle v_i, e_j\rangle^2} = \prod_{j=1}^n \alpha_j^{|e_j|^2} = \det A \ ,$$

using (16). This completes the proof of (17).

Now we show that if the v_i are unit vectors and (14) holds then $D = 1$. To show $D \leq 1$, simply take all $\lambda_i = 1$ in the definition (6). To show $D \geq 1$, we must show that, for any positive real numbers $(\lambda_i)_{i=1}^m$,

$$\det\Big(\sum_{i=1}^m c_i\lambda_i v_i \otimes v_i\Big) \geq \prod_{i=1}^m \lambda_i^{c_i} \ . \tag{18}$$

Let $B = \sum_{i=1}^m c_i\lambda_i v_i \otimes v_i$. Note that B is symmetric and positive definite. We now make a magical computation:

$$\begin{aligned}
1 &= \frac{1}{n}\operatorname{tr}(B^{-1}B) = \frac{1}{n}\operatorname{tr}\Big(\sum_{i=1}^m c_i\lambda_i v_i \otimes B^{-1}v_i\Big) = \frac{1}{n}\sum_{i=1}^m c_i\lambda_i\langle v_i, B^{-1}v_i\rangle \\
&= \frac{1}{n}\sum_{i=1}^m c_i\lambda_i\langle B^{-1/2}v_i, B^{-1/2}v_i\rangle = \frac{1}{n}\sum_{i=1}^m c_i\lambda_i|B^{-1/2}v_i|^2 \\
&\geq \prod_{i=1}^m (\lambda_i|B^{-1/2}v_i|^2)^{c_i/n} = \Big(\prod_{i=1}^m \lambda_i^{c_i/n}\Big)\Big(\prod_{i=1}^m |B^{-1/2}v_i|^{c_i}\Big)^{2/n} \\
&\geq \Big(\prod_{i=1}^m \lambda_i^{c_i/n}\Big)(\det B^{-1/2})^{2/n}
\end{aligned}$$

(The inequalities are the AM/GM inequality with weights $\frac{c_i}{n}$ and (17) applied to $B^{-1/2}$.) Thus $\det(B)^{1/n} \geq \prod_{i=1}^m \lambda_i^{c_i/n}$, which proves (18).

4 References

Barthe's proof for the single-variable case is in [2], and for the multivariable case in [3]. We follow the organization of [3]. The exact condition for $D > 0$ mentioned but not proved in section 3 is Proposition 3 of [3]. For the proof that $D = 1$ under condition (14), we follow Ball [1]; Barthe gives a different proof in Proposition 9 of [3].

For further references and history, see §15 and §16 of Gardner's survey [4].

[1] Keith Ball. Convex geometry and functional analysis. In William B. Johnson and Joram Lindenstrauss, editors, *Handbook of the Geometry of Banach Spaces*, volume 1, pages 161–194. North-Holland, Amsterdam, 2001.

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