Covering Theorems

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Contents

1 Introduction

We are going to present three types of covering theorems. First we discuss Vitali covering Theorem. The theorem states that it is possible to cover, up to a lebesgue-negligible set, a given subset E of $\mathbb{R}^d$ by a disjoint family extracted from a Vitali covering of E.
In third section we define a Besicovitch covering, that is a cover of a subset E of the Euclidean space $\mathbb{R}^d$ by closed balls such that each point of E is the center of some ball in the cover. Then we prove geometric Besicovitch Theorem that asserts there exists a collection $\{B_n\}$ of Besicovitch covering of E that covers E and a constant c_N depending only on the Dimension d such that the balls $\{B_n\}$ can be organized into at most c_N subcollections, in such a way that the balls in each subcollection are disjoint.
In the last section we introduce another type of Besicovitch covering and with the help of geometric Besicovitch theorem we prove measure theoretical Besicovitch theorem for any Radon measure and outer measure associated with it.

2 Vitali coverings

Let $\{X, \mathcal{A}, \mu\}$ be $\mathbb{R}^N$ endowed with Lebesgure measure, and let $\mathcal{F}$ denote a family of closed, nontrivial cubes in $\mathbb{R}^N$.

Definition. *We say that $\mathcal{F}$ is a **fine Vitali covering** for a set $E \subset \mathbb{R}^N$ if for every $x \in E$ and every $\epsilon > 0$, there exists a cube $Q \in \mathcal{F}$ such that $x \in Q$ and* $\operatorname{diam} Q < \epsilon$.

Example. *The collection of N-dimensional closed diadic cubes of diameter not exceeding some given positive number is a fine Vitali covering for any set $E \subset \mathbb{R}^N$.*

Theorem (Vitali)**.** *Let E be a bounded, Lebesgue-measurable set in $\mathbb{R}^N$, and let $\mathcal{F}$ be a fine Vitali covering for E. There exists a countable collection $\{Q_n\}$ of cubes $Q_n \in \mathcal{F}$ with pairwise-disjoint interior such that*

$$\mu \left(E - \bigcup Q_n \right) = 0. \tag{1}$$

Proof. Without loss of generality, we may assume that E and the cubes making up the family $\mathcal{F}$ are all included in some larger cube Q. Label by $\mathcal{F}_0$ the family $\mathcal{F}$, and out of $\mathcal{F}_0$ select a cube Q_0. If Q_0 covers E, then the theorem is proven. Otherwise, introduce the family of cubes

$$\mathcal{F}_1 \equiv \left\{ Q \in \mathcal{F}_0 : \mathring{Q} \bigcap \mathring{Q}_0 = \emptyset \right\}.$$

If Q_0 does not cover E, such a family is nonempty. (That follows from closeness of Q_0), also introduce the number

$$d_1 = \sup\{\operatorname{diam} Q : Q \in \mathcal{F}_1\}.$$

Then out of $\mathcal{F}_1$ select a cube Q_1 whose diameter is larger than $\frac{1}{2}d_1$. If $Q_0 \cup Q_1$ covers E, then the theorem is proven. Otherwise, introduce the family of cubes

$$\mathcal{F}_2 \equiv \left\{ Q \in \mathcal{F}_1 : \mathring{Q} \cap \mathring{Q}_1 = \emptyset \right\},$$

and the number

$$d_2 = \sup\{\operatorname{diam} Q : Q \in \mathcal{F}_2\}.$$

Then out of $\mathcal{F}_2$ select a cub Q_2 whose diameter is larger than $\frac{1}{2}d_1$. Proceeding in this fashion, we inductively define families $\{\mathcal{F}_n\}$, positive numbers $\{d_n\}$, and cubes $\{Q_n\}$ by the recursive procedure
$\mathcal{F}_n \equiv \left\{ Q \in \mathcal{F}_{n-1} : \mathring{Q} \bigcap \mathring{Q}_{n-1} = \emptyset \right\}$,
$d_n = \sup\{\operatorname{diam} Q : Q \in \mathcal{F}_n\}$,
$Q_n=$ A cub select out of $\mathcal{F}_n$ such that $\operatorname{diam} Q_n > \frac{1}{2}d_n$.
The cubes $\{Q_n\}$ have pairwise-disjoint interior, and they are all included in some larger cube Q. Therefore,

$$\sum_{n=1}^{\infty} \left(\frac{\operatorname{diam} Q_n}{\sqrt{N}} \right)^N = \sum_{n=1}^{\infty} \mu(Q_n) \leq \mu(Q) < \infty. \tag{2}$$

The convergence of this series implies that $\lim \operatorname{diam} Q_n = 0$.
To prove (1), we argue by contradiction. Assume that

$$\mu \left(E - \bigcup Q_n \right) \geq 2\epsilon \qquad for\ some\ \epsilon > 0. \tag{3}$$

First, for each Q_n, we construct a larger cube $Q_n^{'}$ of diameter

$$\operatorname{diam} Q_n^{'} = (4\sqrt{N} + 1) \operatorname{diam} Q_n, \tag{4}$$

with the same center as Q_n and faces parallel to the faces of Q_n. By the convergence of the series in (2), there exists some $n_\epsilon \in \mathbb{N}$, such that

$$\mu \left(\bigcup_{n=n_\epsilon+1}^{\infty} Q_n^{'} \right) \leq \sum_{n=n_\epsilon+1}^{\infty} \mu(Q_n^{'}) \leq \epsilon. \tag{5}$$

Using this inequality and (3), we estimate

$$\mu \left(\left(E - \bigcup_{n=1}^{n_\epsilon} Q_n \right) - \bigcup_{n=n_\epsilon+1}^{\infty} Q_n^{'} \right) \geq \mu \left(E - \bigcup_{n=1}^{n_\epsilon} Q_n \right) - \mu \left(\bigcup_{n=n_\epsilon+1}^{\infty} Q_n^{'} \right) \geq \epsilon. \tag{6}$$

This implies that there exists an element

$$x \in \left(E - \bigcup_{n=1}^{n_\epsilon} Q_n \right) - \bigcup_{n=n_\epsilon+1}^{\infty} Q_n^{'}, \tag{7}$$

such an element must have positive distance 2σ from the union of the first n_ϵ cubes. Indeed, such a finite union is closed and x does not belong to any of the cubes Q_n, $n = 1, 2, \cdots, n$.
By the definition of a Vitali covering, given such a σ, there exists a cube $Q_\delta \in \mathcal{F}$ of positive diameter $0 < \delta \leq \sigma$ that covers x. By construction, Q_δ does not intersect the interior of any of the first n cubes Q_n ;

$$Q_\delta \cap \mathring{Q}_n = \emptyset \qquad n = 1, 2, \cdots, n_\epsilon.$$

It follows that Q_δ belongs to the family $\mathcal{F}_{n_\epsilon+1}$. Next we claim that

$$Q_\delta \cap \mathring{Q}_n \neq \emptyset \qquad for\ some \quad n \in \{n_\epsilon + 1, n_\epsilon + 2, \cdots\}.$$

Indeed, if Q_δ did not intersect the interior of any such cubes, it would belong to all the families $\mathcal{F}_n$. This, however, would imply that

$$0 < \delta = \operatorname{diam} Q_\delta \leq d_n \to 0 \qquad as\ n \to \infty.$$

Let $m \geq (n_\epsilon + 1)$ be the smallest positive integer for which $Q_\delta \cap \mathring{Q}_m \neq \emptyset$.
Then

$$Q_\delta \notin \mathcal{F}_{m+1}, \qquad Q_\delta \in \mathcal{F}_m, \qquad \delta \leq d_m.$$

By the selection (7), the element x does not belong to $Q_m^{'}$. Therefore, the intersection $Q_\delta \cap \mathring{Q}_m$ can be nonempty only if the diameter of Q_δ is larger than the difference of edges of $Q_m^{'}$ and Q_m, by Pythagoras that is equal to, $\frac{1}{2\sqrt{N}}(\operatorname{diam} Q_m^{'} - \operatorname{diam} Q_m)$. Hence

$$\delta = \operatorname{diam} Q_\delta > \frac{1}{2\sqrt{N}}(\operatorname{diam} Q_m^{'} - \operatorname{diam} Q_m).$$

From this and (4), we find the contradiction $d_m \geq \delta > d_m$. □

Remark. *The theorem does not claims that $\bigcup Q_n$ covers E. Rather, $\bigcup Q_n$ covers E in a measure-theoretical sense. However, the proof shows that $E \subset \bigcup Q_n^{'}$ where $Q_n^{'}$ are the cubes congruent to Q_n and with the double edge. Because in each step $\operatorname{diam} Q_n > \frac{1}{2} d_n$ and d_n goes to zero, as n goes to infinity. Therfore when we double the Edge of Q_n we can cover each $x \in E$.*

Remark. *The proof relies on the structure of the Lebesgue measure in $\mathbb{R}^N$ and would not hold for a general Radon measure (A Borel measure that is finite on compact subsets) in $\mathbb{R}^N$.*

Corollary. *Let E be a bounded , Lebesgue-measurable set in $\mathbb{R}^N$, and let $\mathcal{F}$ be a fine Vitali covering for E. For every $\epsilon > 0$, there exists a finite collection of cubes*

$$\mathcal{F}_\epsilon \equiv \{Q_1, Q_2, \cdots, Q_{n_\epsilon}\} \quad (Q_n \in \mathcal{F}),$$

with pairwise-disjoint interior such that

$$\sum \mu(Q_n) - \epsilon \leq \mu(E) \leq \mu\left(\bigcup_{n=1}^{n_\epsilon} E \bigcap Q_n\right) + \epsilon. \tag{8}$$

Proof. Having fixed $\epsilon > 0$, let $E_{0,\epsilon}$ be an open set containig E and satisfying $\mu(E_{0,\epsilon}) \le \mu(E) + \epsilon$. Introduce the subfamily

$$\mathcal{F}_\epsilon \equiv \{\textit{the collection of the cubes out of } \mathcal{F} \textit{ that are contained in } E_{0,\epsilon}\},$$

and out of $\mathcal{F}_\epsilon$ select a countable collection of closed cubes $\{Q_n\}$ with pairwise-disjoint interior satisfying (1). By construction

$$\sum_{n\in\mathbb{N}} \mu(Q_n) \le \mu(E_{0,\epsilon}) \le \mu(E) + \epsilon. \tag{9}$$

This, in turn, implies that there exists a positive integer n_ϵ such that

$$\sum_{n_\epsilon+1}^{\infty} \mu(Q_n) \le \epsilon.$$

From this and (1),

$$\mu(E) = \mu\left(\bigcup_{n\in\mathbb{N}} (E \cap Q_n)\right) \le \mu\left(\bigcup_{n=1}^{n_\epsilon} E \bigcap Q_n\right) + \epsilon \tag{10}$$

The corollary follows from (9),(10). □

3 The geometric Besicovitch covering theorem

Definition. *Let E be a subset of $\mathbb{R}^N$. A collection $\mathcal{F}$ of nontrivial closed balls in $\mathbb{R}^N$ is a **Besicovitch covering for** E , if each $x \in E$ is the center of a nontrivial ball $B(x)$ belonging to $\mathcal{F}$.*

Theorem (Besicovitch)**.** *Let E be a bounded subset of $\mathbb{R}^N$ and let F be a Besicovitch covering for E. There exist a countable collection $\{x_n\}$ of points in E and a corresponding collection of balls $\{B_n\}$ in $\mathcal{F}$,*

$$B_n = B_{\rho_n}(x_n) \quad \textit{balls centered at } x_n \textit{ and radius } \rho_n, \tag{11}$$

such that $E \subset \bigcup B_n$. Moreover, there exists a positive c_N depending only upon the dimension N and independent of E and the covering $\mathcal{F}$ such that the balls $\{B_n\}$ can be organized into at most c_N subcollections, in such a way that the balls $\{B_{n_j}\}$ of each subcollection $\mathcal{B}_j$ are disjoint.

Remark. *The theorem continues to hold, if the balls making up the Besicovitch covering $\mathcal{F}$ are replaced by cubes with faces parallel to the coordinate planes.*

Proof. Since E is bounded, we may assume that E and the balls making up the family $\mathcal{F}$ are all included in some large ball B_0 centered at the origin. Set $E_1 = E$ and

$$\mathcal{F}_1 = \{\textit{the collection of balls } B(x) \in \mathcal{F} \textit{ whose center is in } E_1\},$$

$$r_1 = \sup\{radius\ of\ the\ balls\ in\ \mathcal{F}_1\}.$$

Select $x_1 \in E_1$ and a ball

$$B_1 = B_{\rho_1}(x_1) \in \mathcal{F}_1 \qquad of\ radius\ \rho_1 > \frac{3}{4}r_1.$$

If $E_1 \subset B_1$, the process terminates. Otherwise, set $E_2 = E_1 - B_1$ and

$\mathcal{F}_2 = \{the\ collection\ of\ balls\ B(x) \in \mathcal{F}\ whose\ center\ is\ in\ E_2\}$,
$r_2 = \sup\{radius\ of\ the\ balls\ in\ \mathcal{F}_2\}$.

Then Select $x_2 \in E_2$ and a ball

$$B_2 = B_{\rho_2}(x_2) \in \mathcal{F}_2 \qquad of\ radius\ \rho_2 > \frac{3}{4}r_2.$$

Proceeding recursively, define countable collections of sets E_n balls B_n, families $\mathcal{F}_n$ and positive numbers r_n by

$E_n = E - \bigcup_{j=1}^{n-1} B_j, \qquad x_n \in E_n,$
$\mathcal{F}_n = \{the\ collection\ of\ balls\ B(x) \in \mathcal{F}\ whose\ center\ is\ in\ E_n\}$,
$r_n = \sup\{radius\ of\ the\ balls\ in\ \mathcal{F}_n\}$,
$B_n = B_{\rho_n}(x_n) \in \mathcal{F}_n \qquad of\ radius\ \rho_n > \frac{3}{4}r_n.$

By construction, if $m > n$

$$\rho_n > \frac{3}{4}r_n \geq \frac{3}{4}r_m \geq \frac{3}{4}\rho_m. \tag{12}$$

This implies the balls $B_{\frac{1}{3}\rho_n}(x_n)$ are disjoint. Indeed, since $x_m \notin B_n$,

$$|x_n - x_m| > \rho_n = \frac{1}{3}\rho_n + \frac{2}{3}\rho_n \geq \frac{1}{3}\rho_n + \frac{1}{3}\rho_m. \tag{13}$$

The balls $B_{\frac{1}{3}\rho_n}(x_n)$ are contained in B_0 and are disjoint. Therefore, $\rho_n \to 0$ as $n \to \infty$. The union of the balls $\{B_n\}$ covers E. If not, select $x \in E - \bigcup B_n$ and a nontrivial ball $B_\rho(x)$ centered at x and radius $\rho > 0$. Such a ball exists since F is a Besicovitch covering. By construction, $B_\rho(x)$ must belong to all the families $\mathcal{F}_n$. Therefore, $0 < \rho \leq r_n \longrightarrow 0$. The contadiction implies that $E \subset \bigcup B_n$.

The proof of last statement based on the following geometrical fact.

Proposition. *There exists a positive integer c_N depending only on N such that for every index k, at most C_N balls out of $\{B_1, B_2, \cdots, B_{k-1}, B_k\}$ intersect B_k.*

The collection $\mathcal{B}_j$ are constructed by regarding them initially as empty boxes to be filled with disjoint balls taken out of $\{B_n\}$. Each element of $\{B_n\}$ is allocated to some of the boxes $\mathcal{B}_j$ as follows:

First, for $j = 1, 2, \cdots, c_N$, put B_j into $\mathcal{B}_j$. Next, consider the ball B_{c_n+1}. By Proposition, at least one of the first c_N balls does not intersect B_{c_N+1}, say, for example, B_1.
Then allocate B_{c_N+1} to $\mathcal{B}_1$.
Consider the subsequent ball B_{c_N+2}. At least two of the first $(c_N + 1)$ balls do not intersect B_{c_N+1}. If one of the $B_j, (j = 2, \cdots, c_N)$ say, for example, B_2, does not intersect B_{c_N+2}, allocate B_{c_N+2} to $\mathcal{B}_2$. If all the balls $B_j, j = 2, \cdots, c_N$ intersect B_{c_N+2}, then B_1 and B_{c_N+1} do not intersect B_{c_N+2} since at least two of the first $(c_N + 1)$ balls do not intersect B_{c_N+2}. Then allocate B_{c_N+2} to $\mathcal{B}_1$, which now would contain three disjoint balls.
proceeding recursively, assume that all the balls

$$B_1, \cdots, B_{c_N}, \cdots, B_{c_N+n-1} \qquad for\ some\ n \in \mathbb{N},$$

have been allocated so that at the $(n-1)th$ step of the process, each of the $\mathcal{B}_j$ contains at most n disjoint balls. To allocate B_{c_N+n} observe that by Proposition, at least n of the first (c_N+n-1) balls must be disjoint from B_{c_N+n}. This implies that the element of at least one of the boxes $\mathcal{B}_j, (j = 1, 2, \cdots, c_N)$, are all disjoint from B_{c_N+n}. Allocate B_{c_N+n} to one such a box and proceed inductively. □

proof of Proposition. Fix some positive integer k, consider those balls B_j for, $j = 1, 2, \cdots, k$, that intersect $B_k = B_{\rho_k}(x_k)$ and divide them into two sets:

$\mathcal{G}_1 = \left\{B_j = B_{\rho_j}(x_j) : j = 1, \cdots, k \text{ that intersect } B_k \text{ and } \rho_j \le \frac{3}{4}M\rho_k\right\},$
$\mathcal{G}_2 = \left\{B_j = B_{\rho_j}(x_j) : j = 1, \cdots, k \text{ that intersect } B_k \text{ and } \rho_j > \frac{3}{4}M\rho_k\right\}.$
where $M > 3$ is a positive integer to be chosen.

Lemma. *The number of balls in $\mathcal{G}_1$ does not exceed $4^N(M+1)^N$.*

Proof. Let $\{B_{\rho_j}(x_j)\}$ be the collection of balls in $\mathcal{G}_1$ and let $\#\{\mathcal{G}_1\}$ denote their number. The balls $\{B_{\frac{1}{3}\rho_j}(x_j)\}$ are disjoint and are contained in $B_{(M+1)\rho_k}(x_k)$. Indeed,
since $B_j \cap B_k \neq \emptyset$,

$$|x_j - x_k| \le \rho_j + \rho_k \le \left(\frac{3}{4}M + 1\right)\rho_k.$$

Morever for any $x \in B_{\frac{1}{3}\rho_j}(x_j)$,

$$|x - x_k| \le |x - x_j| + |x_j - x_k|,$$
$$\le \frac{1}{3}\rho_j + \left(\frac{3}{4}M + 1\right)\rho_k \le (M+1)\rho_k.$$

From this, denoting by κ_N the volume of the unit ball in $\mathbb{R}^N$,

$$\sum_{j:B_j\in\mathcal{G}_1} \kappa_N \left(\frac{1}{3}\rho_j\right)^N \le \kappa_N (M+1)^N {\rho_k}^N.$$

Since $j < k$, it follows from (12) that $\frac{1}{3}\rho_j > \frac{1}{4}\rho_k$. Therefore,

$$\#\{\mathcal{G}_1\}\kappa_N \left(\frac{1}{4}\rho_k\right)^N \leq \kappa_N (M+1)^N \rho_k^N.$$

□

An upper estimate of the number of balls in $\mathcal{G}_2$ is drivied by counting the number of rays originating from the center x_k of B_k to each of the centers x_j of $B_j \in \mathcal{G}_2$. We first establish that the angle between any two such a rays is not less than an absolute angle θ_0. Then we estimate the number of rays originating from x_k and mutually forming an angle of at least θ_0.
Let $B_{\rho_n}(x_n)$ and $B_{\rho_m}(x_m)$ be any two balls in $\mathcal{G}_2$ and set:

$$\theta\text{=angle between the rays from } x_k \text{ to } x_n \text{ and } x_m.$$

Lemma. *The number M can chosen so that, $\theta > \theta_0 = \arccos \frac{5}{6}$.*

Proof. Assume $n < m < k$. By construction, $x_m \notin B_{\rho_n}(x_n)$; that means:

$$|x_n - x_m| > \rho_n. \tag{14}$$

Also, $x_k \notin B_{\rho_n}(x_n) \bigcup B_{\rho_m}(x_m)$,

$$\rho_n < |x_n - x_k| \qquad \textit{and} \qquad \rho_m < |x_m - x_k|.$$

Since both $B_{\rho_n}(x_n)$ and $B_{\rho_m}(x_m)$ intersect B_k and are in $\mathcal{G}_2$,

$$\begin{aligned} \frac{3}{4}M\rho_k < \rho_n \leq |x_n - x_k| \leq \rho_n + \rho_k, \\ \frac{3}{4}M\rho_k < \rho_m \leq |x_m - x_k| \leq \rho_m + \rho_k. \end{aligned} \tag{15}$$

The Carnot formula applied to the triangle of vertices x_k, x_n, x_m yields:

$$\cos(\theta) = \frac{|x_n - x_k|^2 + |x_m - x_k|^2 - |x_n - x_m|^2}{2|x_n - x_k||x_m - x_k|}.$$

Assuming $\cos(\theta) > 0$ and using $14, 15$, estimate:

$$\begin{aligned} \cos\theta &\leq \frac{(\rho_n + \rho_k)^2 + (\rho_m + \rho_k)^2 - \rho_n^2}{2\rho n \rho m} \\ &\leq \frac{\rho_m^2 + 2\rho_k^2 + 2\rho_k(\rho_n + \rho_m)}{2\rho_n\rho_m} \\ &\leq \frac{1}{2}\frac{\rho_m}{\rho_n} + \frac{\rho_k}{\rho_n}\frac{\rho_k}{\rho_m} + \frac{\rho_k}{\rho_m} + \frac{\rho_k}{\rho_n} \\ &\leq \frac{1}{2}\frac{\rho_m}{\rho_n} + \left(\frac{4}{3}\right)^2 \frac{1}{M^2} + 2\frac{4}{3}\frac{1}{M}. \end{aligned}$$

Since $m > n$, from (12) , it follows that $\rho_n > \frac{3}{4}\rho_m$. Therefore,

$$\cos\theta \leq \frac{2}{3} + \frac{4}{3}\frac{1}{M}\left(\frac{4}{3}\frac{1}{M} + 2\right).$$

Now choose M so large, that the $\cos\theta \leq \frac{5}{6}$. □

If $N = 2$, the number of rays originating from the origin and mutually forming an angle $\theta > \theta_0$ is at most $\frac{2\pi}{\theta_0}$.
If $N \geq 3$, let $\mathcal{C}(\theta_0)$ be a circular cone in $\mathbb{R}^N$ with vertex at the origin whose axial cross-section with a two-dimensional hyperplane forms an angle $\frac{1}{2}\theta_0$. Denote by $\sigma_N(\theta_0)$ the solid angle corresponding to $\mathcal{C}(\theta_0)$[1]. Then the number of rays originating from the origin and mutually forming an angle $\theta > \theta_0$ is at most $\frac{\omega_N}{\sigma_N(\theta_0)}$.
The number c_N claimed by Proposition is estimated by:

$$c_N = \#\{\mathcal{G}_1\} + \{\mathcal{G}_2\} \leq 4^N(M+1)^N + \frac{\omega_N}{\sigma_N(\theta_0)}.$$

□

4 Besicovitch measure-theoretical covering theorem

Definition. *Let $\mathcal{F}$ denote a family of nontrivial closed balls in$\mathbb{R}^N$. We say that $\mathcal{F}$ is a **fine Besicovitch covering** for a set $E \subset \mathbb{R}^N$ if for every $x \in E$ and every $\epsilon > 0$, there exists a ball $B_\rho(x) \in \mathcal{F}$ centered at x and of radius $\rho < \epsilon$.*

A fine Besicovitch covering of a set $E \subset \mathbb{R}^N$ differs from a fine Vitali covering in that each $x \in E$ is required to be a center of a ball of arbitrary small radius.

Theorem (Besicovitch measure-theoretical)**.** *Let E be a bounded set in $\mathbb{R}^N$ and let $\mathcal{F}$ be a fine besicovitch covering for E. Let μ be a Radon measure [2] in $\mathbb{R}^N$ and let μ_e be the outer measure associated with it.*
There exists a countable collection $\{B_n\}$ of disjoint balls $B_n \in \mathcal{F}$ such that

$$\mu_e\left(E - \bigcup B_n\right) = 0. \tag{16}$$

Remark. *The set E is not required to be μ-measurable.*

Remark. *It is not claimed here that $E \subset \bigcup B_n$. The collection $\{B_n\}$ forms a measure-theoretical covering of E in the sense of* (16)*.*

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[1]That is, the area of the intersection of $\mathcal{C}(\theta_0)$ with the unit sphere in $\mathbb{R}^N$. The area of the unit sphere in $\mathbb{R}^N$ is denoted by ω_N. Accordingly, the solid angle of the unit sphere is ω_N

[2]a Borel measure, that is finite on compact subsets of $\mathbb{R}^N$

Proof. We may assume that $\mu_e(E) > 0\dot{\text{O}}$therwise,t he statement is trivial. Since E is bounded, we may assume that both E and all the balls making up the covering $\mathcal{F}$ are contained in some larger ball B_0.
Let $B_j, j = 1, 2, \cdots, c_N$ be the subcollections of disjoint balls claimed by The Geometric Besicovitch Theorem. Since

$$E \subset \bigcup_{j=1}^{c_N} \bigcup_{n_j=1}^{\infty} B_{n_j}$$

it holds that

$$\mu_e \left(E \bigcap \bigcup_{j=1}^{c_N} \bigcup_{n_j=1}^{\infty} B_{n_j} \right) = \mu_e(E) > 0.$$

Therefore, there exists some index $j \in \{1, 2, \cdots, c_N\}$ for which

$$\mu_e \left(E \bigcap \bigcup_{n_j=1}^{\infty} B_{n_j} \right) \geq \frac{1}{c_N} \mu_e(E).$$

Since all the balls B_{n_j} are disjoint and are all included in B_0

$$\mu_e \left(E \bigcap \bigcup_{n_j=1}^{\infty} B_{n_j} \right) \leq \sum_{n_j=1}^{\infty} \mu(B_{n_j}) \leq \mu(B_0) < \infty.$$

Therefore, there exists some index m_1 such that

$$\mu_e \left(E \bigcap \bigcup_{n_j=1}^{m_1} B_{n_j} \right) \geq \frac{1}{2c_N} \mu_e(E). \tag{17}$$

The finite union of balls is μ-measurable. Therefore, by the Caratheodory criterion of measurability and the lower estimate in (17),

$$\begin{aligned} \mu_e(E) &= \mu_e \left(E \bigcap \bigcup_{n_j=1}^{m_1} B_{n_j} \right) + \mu_e \left(E - \bigcup_{n_j=1}^{m_1} B_{n_j} \right) \\ &\geq \frac{1}{2c_N} \mu_e(E) + \mu_e \left(E - \bigcup_{n_j=1}^{m_1} B_{n_j} \right). \end{aligned}$$

Therefore

$$\mu_e \left(E - \bigcup_{n_j=1}^{m_1} B_{n_j} \right) \leq \eta \mu_e(E) \qquad \eta = 1 - \frac{1}{2c_N} \in (0, 1) \tag{18}$$

Now set

$$E_1 = E - \bigcup_{n_j=1}^{m_1} B_{n_j}.$$

If $\mu_e(E_1) = 0$, the process terminates and the theorem is proven. Otherwise, let $\mathcal{F}_1$ denote the collection of balls in $\mathcal{F}$ that do not intersect any of the balls B_{n_j} for $n_j = 1, 2, \cdots, m_1$. Since $\mathcal{F}$ is a fine Besicovitch covering for E,the family $\mathcal{F}_1$ is nonempty, and it is a fine Besicovitch covering for E_1.
Repeating the previous selection process for the set E_1 and the Besicovitch covering $\mathcal{F}_1$ yields a finite number m_2 of closed disjoint balls B_{n_l} in $\mathcal{F}_1$ such that

$$\mu_e\left(E_1 - \bigcup_{n_l=1}^{m_2} B_{n_l}\right) \leq \eta\mu_e(E_1) \leq \eta\mu_e\left(E - \bigcup_{n_j=1}^{m_1} B_{n_j}\right) \leq \eta^2\mu_e(E).$$

Relabelling the balls B_{n_j} and B_{n_l} yields a finite number s_2 of closed, disjoint balls B_n in $\mathcal{F}$ such that

$$\mu_e\left(E - \bigcup_{n=1}^{s_2} B_n\right) \leq \eta^2\mu_e(E). \tag{19}$$

Repeating the process k times gives a collection of s_k closed disjoint balls in $\mathcal{F}$ such that

$$\mu_e\left(E - \bigcup_{n=1}^{s_k} B_n\right) \leq \eta^k\mu_e(E). \tag{20}$$

If for some $k \in \mathbb{N}$

$$\mu_e\left(E - \bigcup_{n=1}^{s_k} B_n\right) = 0.$$

the process terminated and the theorem is proven. Otherwise (18) holds for all $k \in \mathbb{N}$. Letting $k \to \infty$ proves (16). □

References

[RE] Emmanuele DiBenedetto,Real Analysis:Foundations and Applications,Birkhäuser,2002.