Chapter 6

Riesz Representation Theorems

6.1 Dual Spaces

DEFINITION **6.1.1.** Let V and W be vector spaces over $\mathbb{R}$. We let

$$L(V,W) = \{T : V \to W \mid T \text{ is linear}\}.$$

The space $L(V,\mathbb{R})$ is denoted by $V^\sharp$ and elements of $V^\sharp$ are called linear functionals.

EXAMPLE **6.1.2.** 1) Let $V = \mathbb{R}^n$. Then we can identify $\mathbb{R}^\sharp$ with $\mathbb{R}$ as follows:
For each $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ define $\phi_{\mathbf{a}} : \mathbb{R}^n \to \mathbb{R}$ by

$$\phi_{\mathbf{a}}((x_1, x_2, \ldots, x_n)) = \mathbf{x} \cdot \mathbf{a} = \sum_{i=1}^{n} x_i a_i.$$

2) Let (X, d) be a compact metric space. Let $x_0 \in X$. Define $\phi_{x_0} : C(X) \to \mathbb{R}$ by

$$\phi_{x_0}(f) = f(x_0).$$

Then $\phi_{x_0} \in C(X)^\sharp$.

DEFINITION **6.1.3.** Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed linear spaces. Let $T : V \to W$ be linear. We say that T is bounded is

$$\sup_{\|x\|_V \leq 1} \{\| T(x) \|_W\} < \infty.$$

In this case, we write

$$\| T \| = \sup_{\|x\|_V \leq 1} \{\| T(x) \|_W\}.$$

Otherwise, we say that T is unbounded.

The next result establishes the fundamental criterion for when a linear map between normed linear spaces is continuous. It's proof is left as an exercise.

THEOREM **6.1.4.** *Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed linear spaces. Let $T : V \to W$ be linear. Then the following are equivalent.*

1) T is continuous.

2) T is continuous at 0.

2) T is bounded.

Proof. 1) $\Rightarrow$ 2) This is immediate.

2) $\Rightarrow$ 3) Assume that T is continuous at 0. Let δ be such that if $\| x \|_V \leq \delta$, then $\| T(x) \|_W$. It follows easily that $\| T \| \leq \frac{1}{\delta}$.

3) $\Rightarrow$ 1) Note that we may assume that $\| T \| > 0$ otherwise $T = 0$ and hence is obviously continuous. Let $x_0 \in V$ and let $\epsilon > 0$. Let $\delta = \frac{\epsilon}{\|T\|}$. Then if $\| x - x_0 \|_V < \delta$, we have

$$\| T(x) - T(x_0) \|_W = \| T(x - x_0) \|_W \leq \| T \| \cdot \| x - x_0 \|_V < \epsilon.$$

∎

REMARK **6.1.5.** 1) Let $(V, \| \cdot \|_V)$ and $(W, \| \cdot \|_W)$ be normed linear spaces. Let $T : V \to W$ be linear. Then we can easily deduce from the previous theorem that . if T is bounded, then T is uniformly continuous.

2) Let

$$B(V,W) = \{T : X \to Y | T \text{ is linear and } T \text{ is bounded}\}.$$

Let T_1 and T_2 be in $B(V,W)$. Then if $x \in V$, we have

$$\begin{aligned} \| T_1 + T_2(x) \|_W &= \| T_1(x) + T_2(x) \|_W \\ &\leq \| T_1(x) \|_W + \| T_2(x) \|_W \\ &\leq \| T_1 \| \| x \|_V + \| T_1 \| \| x \|_V \\ &= (\| T_1 \| + \| T_2 \|) \| x \|_V . \end{aligned}$$

As such $T_1 + T_2 \in B(V,W)$ and in particular

$$\| T_1 + T_2 \| \leq \| T_1 \| + \| T_1 \|$$

It follows that $(B(V,W), \| \cdot \|)$ is also a normed linear space.

THEOREM **6.1.6.** *Assume that $(W, \| \cdot \|_W)$ be a Banach space. Then so is $(B(V,W), \| \cdot \|)$.*

Proof. Assume that $\{T_n\}$ is Cauchy. Let $x \in V$. Since

$$\| T_n(x) - T_m(x) \|_W \leq \| T_n - T_2 \| \| x \|_V$$

it follows easily that $\{T_n(x)\}$ is also Cauchy in W. As such we can define T_0 by

$$T_0(x) = \lim_{n \to \infty} T_n(x).$$

To see that T_0 is linear observe that

$$\begin{aligned} T_0(\alpha x + \beta y) &= \lim_{n \to \infty} T_n(\alpha x + \beta y) \\ &= \lim_{n \to \infty} \alpha T_n(x) + \beta T_n(y) \\ &= \alpha T_0(x) + \beta T_0(y) \end{aligned}$$

To see that T_0 is bounded first observe that being Cauchy, $\{T_n\}$ is bounded. Hence we can find an $M > 0$ such that $\|T_n\| \leq M$ for each $n \in \mathbb{N}$. Moreover, since $\|T_0(x)\|_W = \lim\limits_{n \to \infty} \|T_n(x)\| \leq M\|x\|_V$, we have that $\|T_0\| \leq M$.

Now let $\epsilon > 0$ and choose an $N \in \mathbb{N}$ so that if $n, m \geq N$, then

$$\|T_n - T_m\| < \epsilon.$$

Let $x \in V$ with $\|x\|_V \leq 1$. Then since $\|T_n(x) - T_m(x)\|_W < \epsilon$ for each $m \geq N$, we have

$$\|T_n(x) - T_0(x)\|_W = \lim_{m\to\infty} \|T_n(x) - T_m(x)\|_W \leq \epsilon.$$

In particular

$$T_0 = \lim_{n\to\infty} T_n$$

in $B(X, Y)$.

∎

DEFINITION **6.1.7.** Let $(V, \|\cdot\|)$ be a normed linear space. The space $B(V, \mathbb{R})$ is called the dual space of V and is denoted by V^*.

EXAMPLE **6.1.8.** 1) Let $V = \mathbb{R}^n$ with the usual norm $\|\cdot\|_2$. For each $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ we defined $\phi_{\mathbf{a}} : \mathbb{R}^n \to \mathbb{R}$ by

$$\phi_{\mathbf{a}}((x_1, x_2, \ldots, x_n)) = \mathbf{x} \cdot \mathbf{a} = \sum_{i=1}^{n} x_i a_i.$$

Then in fact $\phi_{\mathbf{a}} \in \mathbb{R}^{n^*}$ and

$$\|\phi_{\mathbf{a}}\| = \|\mathbf{a}\|_2.$$

2) Let (X, d) be a compact metric space. Again, if $x_0 \in X$ and we define $\phi_{x_0} : (C(X), \|\cdot\|_\infty) \to \mathbb{R}$ by

$$\phi_{x_0}(f) = f(x_0),$$

then $\phi_{x_0} \in C(X)^*$. In this case $\|\phi_{x_0}\|$.

3) Let $(X, \mathcal{A}, \mu)$ be a measure space and let $1 \leq p \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Hölder's Inequality allows us to define for each $g \in L_q(X, \mathcal{A}, \mu)$ and element $\phi_g \in L_p(X, \mathcal{A}, \mu)^\sharp$ by

$$\phi_g(f) = \int fg\, d\mu.$$

Moreover, Hölder's Inequality also shows that $\phi_g \in L_p(X, \mathcal{A}, \mu)^*$ with

$$\|\phi_g\| \leq \|g\|_p.$$

Note that ϕ_g has the additional property that if $g \geq 0$ μ-a.e., then $\phi_g(f) \geq 0$ whenever $f \in L_p(X, \mathcal{A}, \mu)$ and $f \geq 0$ μ-a.e.

4) Let (X, d) be a compact measure space and let μ be a finite regular signed measure on $\mathcal{B}(X)$. Define $\phi_\mu \in C(X)^\sharp$ by

$$\phi_\mu(f) = \int f\, d\mu.$$

Since

$$|\phi_\mu(f)| \leq \int |f|\, d|\mu| \leq \|f\|_\infty \|\mu\|_{meas}$$

we see that in fact $\phi_\mu \in C(X)^*$ and $\|\phi_\mu\| \leq \|\mu\|_{meas}$.

We note again that ϕ_μ has the additional property that if μ is a positive measure on $\mathcal{B}(X)$, then $\phi_\mu(f) \geq 0$ whenever $f \in C(X)$ and $f \geq 0$. Furthermore, in this case since

$$\phi_\mu(1) = \int 1\, d\mu = \mu(X) = \|\mu\|_{meas},$$

if μ is a positve measure we have

$$\|\phi_\mu\| = \|\mu\|_{meas}.$$

PROBLEM **6.1.9.** *In Examples 3) and 4) above we have shown respectively that every element in $L_q(X, \mathcal{A}, \mu)$ determines a continuous functional on $L_p(X, \mathcal{A}, \mu)$ and that if (X, d) is a compact metric space, then every finite regular signed measure on $\mathcal{B}(X)$ determines a continuous linear functional on $C(X)$. It is natural to ask:*

Do all continuous linear functionals on $L_p(X, \mathcal{A}, \mu)$ and $C(X)$ arise in this fashion?

6.2 Riesz Representation Theorem for $L^p(X, \mathcal{A}, \mu)$

In this section we will focus on the following problem:

PROBLEM **6.2.1.** *What is $L^p(X, \mathcal{A}, \mu)^*$?*

We have already established most of the following result:

LEMMA **6.2.2.** *If $(X, \mathcal{A}, \mu)$ is a measure space and if $1 \leq p \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, then for every $g \in L^q(X, \mu)$ the map $\Gamma_g : L^p(X, \mu) \to \mathbb{R}$ defined by $\Gamma_g(f) = \int_X f\, g\, d\mu$ is a continuous linear functional on $L^p(X, \mu)$. Further, $\|\Gamma_g\| \leq \|g\|_q$ and if $1 < p \leq \infty$ then $\|\Gamma_g\| = \|g\|_q$.*

Proof. Assignment. ∎

If (X, μ) is σ-finite, then equality holds for $p = 1$ as well.

LEMMA **6.2.3.** *Let $(X, \mathcal{A}, \mu)$ be a finite measure space and if $1 \leq p < \infty$. Let g be an integrable function such that there exists a constant M with $|\int g\varphi\, d\mu \leq M\|\varphi\|_p$ for all simple functions φ. Then $g \in L^q(X, \mu)$, where $\frac{1}{p} + \frac{1}{q} = 1$.*

Proof. Assume that $p > 1$. Let ψ_n be a sequence of simple functions with $\psi_n \nearrow |g|^q$. Let $\varphi_n = (\psi_n)^{\frac{1}{p}}\mathrm{sgn}(g)$. Then φ_n is also simple and $\|\varphi_n\|_p = (\int \psi_n d\mu)^{\frac{1}{p}}$. Since $|\varphi_n g| \geq |\varphi_n||\psi_n|^{\frac{1}{q}} = |\psi_n|$, we have

$$\int \psi_n\, d\mu \leq \int \varphi_n g\, d\mu \leq M\|\varphi_n\|_p = M\left(\int \psi_n d\mu\right)^{\frac{1}{p}}$$

Therefore $\int \psi_n\, d\mu \leq M^q$. By the Monotone Convergence Theorem we get that $\|g\|_q \leq M$, so $g \in L^q(X, \mu)$. If $p = 1$, then we need to show that g is bounded almost everywhere. Let $E = \{x \in X | \, |g(x)| > M\}$. Let $f = \frac{1}{\mu(E)}\chi_E \mathrm{sgn}(g)$. Then f is a simple function and $\|f\|_1 = 1$. This is a contradiction. ∎

LEMMA **6.2.4.** *Let $1 \leq p < \infty$. Let $\{E_n\}$ be a sequence of disjoint sets. Let $\{f_n\} \subseteq L^p(X, \mu)$ be such that $f_n(x) = 0$ if $x \notin E_n$ for each $n \geq 1$. Let $f = \sum_{n=1}^\infty f_n$. Then $f \in L^p(X, \mu)$ if and only if $\sum_{n=1}^\infty \|f\|_p^p < \infty$. In this case, $\|f\|_p^p = \sum_{n=1}^\infty \|f\|_p^p$.*

Proof. Exercise. ∎

THEOREM **6.2.5** [RIESZ REPRESENTATION THEOREM, I]. *Let $\Gamma \in L^p(X,\mu)^*$, where $1 \le p < \infty$ and μ is σ-finite. Then if $\frac{1}{p} + \frac{1}{q} = 1$, there exists a unique $g \in L^q(X,\mu)^*$ such that*

$$\Gamma(f) = \int_X fg\,d\mu = \phi_g(f)$$

Moreover, $\|\Gamma\| = \|g\|_q$.

Proof. Assume that μ is finite. Then every bounded measurable function is in $L^p(X,\mu)$. Define $\lambda : \mathcal{A} \to \mathbb{R} : E \mapsto \Gamma(\chi_E)$. Let $\{E_n\} \subseteq \mathcal{A}$ be a sequence of disjoint sets, and let $E = \bigcup_{n=1}^{\infty} E_n$. Let $\alpha_n = \text{sgn}\Gamma(\chi_{E_n})$ and $f = \sum_{n=1}^{\infty} \alpha_n \chi_{E_n}$. Then $f \in L^p(X,\mu)$ and $\Gamma(f) = \sum_{n=1}^{\infty} |\lambda(E_n)| < \infty$ and so $\sum_{n=1}^{\infty} |\lambda(E_n)| = \Gamma(\chi_E) = \lambda(E)$. Therefore λ is a finite signed measure. Clearly, if $\mu(E) = 0$ then $\chi_E = 0$ almost everywhere, so $\lambda(E) - \Gamma(0) = 0$. Therefore $|\lambda| \ll \mu$. By the Radon-Nikodym Theorem, there is an integrable function g such that $\lambda(E) = \int_E g\,d\mu$ for all $E \in \mathcal{A}$. If φ is simple, then $\Gamma(\varphi) = \int \varphi g\,d\mu$ by linearity of the integral. But $|\Gamma(\varphi)| \le \|\Gamma\|\|\varphi\|_p$ for all simple functions φ, so $g \in L^q(X,\mu)$ by the lemma above. Now $\Gamma - \phi_g \in L^p(X,\mu)^*$ and $\Gamma - \phi_g = 0$ on the space of simple functions. Since the simple functions are dense in $L^p(X,\mu)$, $\Gamma - \phi_g = 0$ on $L^P(X,\mu)$, so $\Gamma = \phi_g$. We have that $\|\Gamma\| = \|\phi_g\| = \|g\|_q$ as before.

Now asume that μ is σ-finite. We can write $X = \bigcup_{n=1}^{\infty} X_n$, where $\mu(X_n) < \infty$ and $X_n \subseteq X_{n+1}$ for all $n \ge 1$. For each $n \ge 1$, the proof above gives us $g_n \in L^q(X,\mu)$, vanishing outside X_n, such that $\Gamma(f) = \int fg\,d\mu$ for all $f \in L^p(X,\mu)$ vanishing off of X_n. Moreover, $\|g_n\|_q \le \|\Gamma\|$. By the uniqueness of the g_n's, we can assume that $g_{n+1} = g_n$ on X_n. Let $g(x) = \lim_{n\to\infty} g_n(x)$. We have that $|g_n| \nearrow |g|$. By the Monotone Convergence Theorem

$$\int |g|^q\,d\mu = \lim_{n\to\infty} \int |g_n|^q\,d\mu \le \|\Gamma\|q$$

Hence $g \in L^q(X,\mu)$. Let $f \in L^p(X,\mu)$ and $f_n = f\chi_{X_n}$. Then $f_n \to f$ pointwise and $f_n \in L^p(X,\mu)$ for all $n \ge 1$. Since $|fg| \in L^1(X,\mu)$ and $f_n g| \le |fg|$, the Lebesgue Dominated Convergence Theorem shows

$$\int fg\,d\mu = \lim_{n\to\infty} \int f_n g\,d\mu = \lim_{n\to\infty} \int f_n g_n\,d\mu = \lim_{n\to\infty} \Gamma(f_n) = \Gamma(f)$$

∎

If $p = 1$, then we cannot drop the assumption of σ-finiteness.

THEOREM **6.2.6** [RIESZ REPRESENTATION THEOREM, II]. *Let $\Gamma \in L^p(X,\mu)^*$, where $1 < p < \infty$. Then if $\frac{1}{p} + \frac{1}{q} = 1$, there exists a unique $g \in L^q(X,\mu)$ such that*

$$\Gamma(f) = \int fg\,d\mu$$

for all $f \in L^p(X,\mu)$. Moreover, $\|\Gamma\| = \|g\|_q$.

Proof. Let $E \subseteq X$ be σ-finite. then there exists a unique $g_E \in L^q(X,\mu)$, vanishing outside of E, such that $\Gamma(f) = \int fg_E\,d\mu$ for all $g \in L^p(X,\mu)$ vanishing outside of E. Moreover, if $A \subseteq E$, then $g_A = g_E$ almost everywhere on A. For each σ-finite set E let $\lambda(E) = \int |g_E|^q d\mu$. If $A \subseteq E$, then $\lambda(A) \le \lambda(E) \le \|\Gamma\|^q$. Let $M = \sup\{\lambda(E) | E \text{ is } \sigma\text{-finite}\}$. Let $\{E_n\}$ be a sequence of σ-finite sets such that $\lim_{n\to\infty} \lambda(E_n) = M$. If $H = \bigcup_{n=1}^{\infty} E_n$ then H is σ-finite and $\lambda(H) = M$. If E is σ-finite with $H \subseteq E$, then $g_E = g_H$ almost everywhere on H. But

$$\int |g_E|^q d\mu = \lambda(E) \le \lambda(H) = \int |g_H|^q d\mu$$

so $g_E = 0$ almost everywhere on $E \backslash H$. Let $g = g_{H\chi H}$. Then $g \in L^q(X,\mu)$ and if E is σ-finite with $H \subseteq E$ then $g_E = g$ almost everywhere. If $f \in L^p(X,\mu)$, then let $E = \{x \in X | f(x) \neq 0\}$. E is σ-finite and hence $E_1 = E \cup H$ is σ-finite. Hence

$$\Gamma(f) = \int f g_{E_1}\, d\mu = \int fg\, d\mu = \phi_g(f)$$

Therefore $\Gamma = \phi_g$ and as before $\|\Gamma\| = \|g\|_q$. ∎

We have shown that if $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then for any measure space $(X, \mathcal{A}, \mu)$, $L^p(X,\mu)^* \cong L^q(X,\mu)$. If μ is σ-finite, then $L^1(X,\mu)^* \cong L^\infty(X,\mu)$. What happens when $p = \infty$? $L^1(X,\mu) \hookrightarrow L^\infty(X,\mu)^*$, but this embedding is not usually surjective. There exists a compact Hausdorff space Ω such that $L^\infty(X,\mu) \cong C(\Omega)$. What is $C(\Omega)$?

Let $\varphi : [a,b] \to \mathbb{R}$ be defined by $\varphi(f) = f(x_0)$. Then $\varphi \in C[a,b]^*$, and $\|\varphi\| = 1$. Let μ_{x_0} be the measure on $[a,b]$ of the point mass x_0. If $g \in L^1([a,b], m)$, then $\varphi_g(f) = \int_a^b fg\, dm$ is a linear functional on $C[a,b]$, and $\|\varphi_g\| \le \|g\|_1$. g is the Radon-Nikodym derivative of an absolutely continuous measure μ on $[a,b]$, and $\varphi_g(f) = \int f\, d\mu$. If $\mu \in \text{Meas}[a,b]$, then $\varphi_n(f) = \int f\, d\mu$ is a bounded linear functional on $C[a,b]$, with $\|\varphi_\mu\| \le \|\mu\|_{\text{Meas}}$.

6.3 Riesz Representation Theorem for C([a,b])

Theorem **6.3.1.** *[Jordan Decomposition Theorem]*
Let $\Gamma \in C([a,b])^*$. Then there exist positive linear functionals $\Gamma^+, \Gamma^- \in C([a,b])^*$ such that

$$\Gamma = \Gamma^+ - \Gamma^-$$

and

$$\| \Gamma \| = \Gamma^+(1) + \Gamma^-(1).$$

Proof. Assume that $f \ge 0$. Define

$$\Gamma^+(f) = \sup_{0 \le \phi \le f} \Gamma(\varphi).$$

Then $\Gamma^+(f) \ge 0$ and $\Gamma^+(f) \ge \Gamma(f)$. It is also easy to see that if $c \ge 0$, then $\Gamma^+(cf) = c\Gamma^+(f)$.
Let $f, g \ge 0$. If $0 \le \phi \le f$ and $0 \le \psi \le g$, then $0 \le \phi + \psi \le f + g$ so

$$\Gamma(\phi) + \Gamma(\psi) \le \Gamma^+(f+g)$$

and hence,

$$\Gamma^+(f) + \Gamma^+(g) \le \Gamma^+(f+g).$$

If $0 \le \psi \le f + g$, then let $\varphi = inf\{f, \psi\}$ and $\xi = \psi - \varphi$. Then $0 \le \varphi \le f$ and $0 \le \xi \le g$. It follows that

$$\Gamma(\psi) = \Gamma(\varphi) + \Gamma(\xi) \le \Gamma^+(f) + \Gamma^+(g).$$

This shows that

$$\Gamma^+(f+g) \le \Gamma^+(f) + \Gamma^+(g)$$

Therefore,

$$\Gamma^+(f+g) = \Gamma^+(f) + \Gamma^+(g)$$

Let $f \in C[a,b]$. Let α, β be such that $f + \alpha 1 \ge 0$ and $f + \beta 1 \ge 0$. Then

$$\begin{aligned} \Gamma^+(f + \alpha 1 + \beta 1) &= \Gamma^+(f + \alpha 1) + \Gamma^+(\beta 1) \\ &= \Gamma^+(f + \beta 1) + \Gamma^+(\alpha 1) \end{aligned}$$

This shows that

$$\Gamma^+(f + \alpha 1) - \Gamma^+(\alpha 1) = \Gamma^+(f + \beta 1) - \Gamma^+(\beta 1)$$

As such , if we let

$$\Gamma^+(f) = \Gamma^+(f + \alpha 1) - \Gamma^+(\alpha 1),$$

then Γ^+ is well defined.

Let $f, g \in C[a,b]$. Let α, β be chosen so that $f + \alpha 1 \geq 0$ and $g + \beta 1 \geq 0$. Then $f + g + (\alpha+\beta)1 \geq 0$ so

$$\begin{aligned}
\Gamma^+(f+g) &= \Gamma^+(f+g+(\alpha+\beta)1) - \Gamma^+((\alpha+\beta)1) \\
&= \Gamma^+(f+\alpha 1) + \Gamma^+(g+\beta 1) - \Gamma^+((\alpha+\beta)1) \\
&= \Gamma^+(f+\alpha 1) - \Gamma^+(\alpha 1) + \Gamma^+(g+\beta 1) - \Gamma^+(\beta)1) \\
&= \Gamma^+(f) + \Gamma^+(g).
\end{aligned}$$

That is Γ^+ is additive.

It is also clear that $\Gamma^+(cf) = c\Gamma^+(f)$ when $c \geq 0$. But since $\Gamma^+(-f) + \Gamma^+(f) = \Gamma^+(0) = 0$, we get that

$$\Gamma^+(-f) = -\Gamma^+(f)$$

so Γ^+ is linear.

Let

$$\Gamma^- = \Gamma^+ - \Gamma$$

Since it is clear that $\Gamma^+(f) \geq \Gamma(f)$ if $f \geq 0$, Γ^- is also positive.

We know that

$$\| \Gamma \| \leq \| \Gamma^+ \| + \| \Gamma^- = \Gamma^+(1) + \Gamma^-(1)$$

Let $0 \leq \psi \leq 1$. Then $\| 2\psi - 1 \|_\infty \leq 1$. As such

$$\| \Gamma \| \geq \Gamma(2\psi - 1) = 2\Gamma(\psi) - \Gamma(1)$$

and therefore

$$\begin{aligned}
\| \Gamma \| &\geq 2\Gamma^+(1) - \Gamma(1) \\
&= \Gamma^+(1) + \Gamma^-(1)
\end{aligned}$$

Hence

$$\| \Gamma \| = \Gamma^+(1) + \Gamma^-(1).$$

∎

THEOREM **6.3.2.** *[Riesz Representation Theorem for $C([a,b])$]*

Let $\Gamma \in C([a,b])^*$. Then there exists a unique finite signed measure μ on the Borel subsets of $[a,b]$ such that

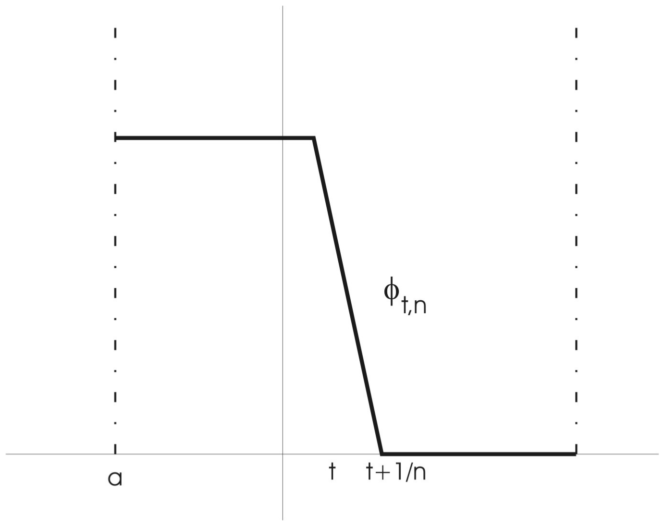
$$\Gamma(f) = \int_{[a,b]} f \, d\mu$$

for each $f \in C([a,b])$. Moreover, $\| \Gamma \| = | \mu | ([a,b])$.

Proof. First, we will assume that Γ is positive.

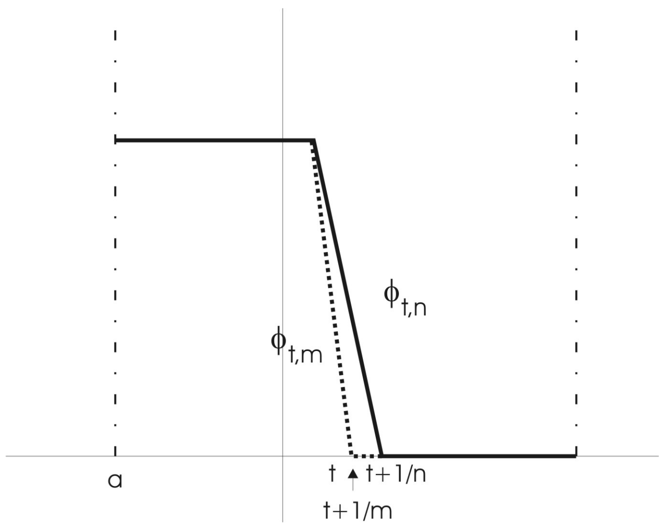
For $a \leq t < b$ and for n large enough so that $t + \frac{1}{n} \leq b$, let

$$\varphi_{t,n}(x) = \begin{cases} 1 & \text{if } x \in [a,t] \\ 1 - n(x-t) & \text{if } x \in (t, t+\frac{1}{n}] \\ 0 & \text{if } x \in (t+\frac{1}{n}, b] \end{cases}$$



Note that if $n \leq m$, then

$$0 \leq \varphi_{t,m} \leq \varphi_{t,n} \leq 1$$

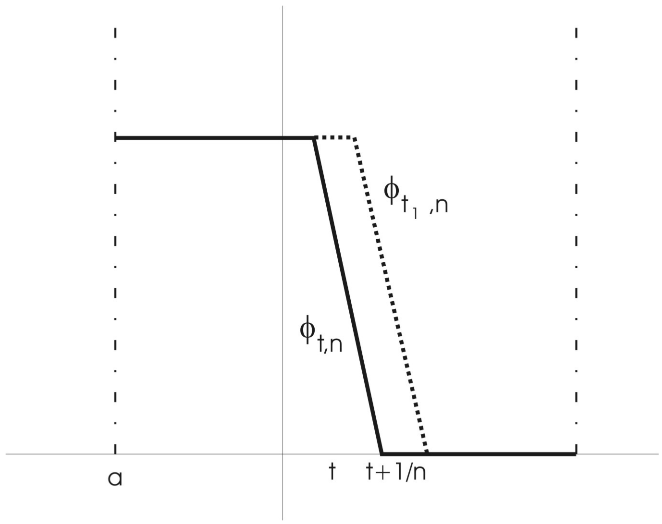


It follows that $\{\Gamma(\varphi_{t,n})\}$ is decreasing and bounded below by 0. Therefore, we can define

$$g(t) = \begin{cases} 0 & \text{if } t < a \\ \lim_{n\to\infty} \Gamma(\varphi_{t,n}) & \text{if } t \in [a, b) \\ \Gamma(1) & \text{if } t \geq b \end{cases}$$

Moreover, if $t_1 > t$, we have

$$\varphi_{t,m} \leq \varphi_{t_1,n}.$$



Since Γ is positive, $g(t)$ is monotonically increasing.

It is clear that $g(t)$ is right continuous if $t < a$ or if $t \geq b$. Assume that $t \in [a, b)$. Let $\epsilon > 0$ and choose n large enough so that

$$n > \max(2, \frac{\| \Gamma \|}{\epsilon})$$

and

$$g(t) \leq \Gamma(\varphi_{t,n}) \leq g(t) + \epsilon.$$

Let

$$\psi_n(x) = \begin{cases} 1 & \text{if } x \in [a, t + \frac{1}{n^2}] \\ 1 - \frac{n^2}{n-2}(x - t - \frac{1}{n^2}) & \text{if } x \in (t + \frac{1}{n^2}, t + \frac{1}{n} - \frac{1}{n^2}] \\ 0 & \text{if } x \in (t + \frac{1}{n} - \frac{1}{n^2}, b] \end{cases}$$

Then

$$\| \psi_n - \varphi_{t,n} \|_\infty \leq \frac{1}{n}$$

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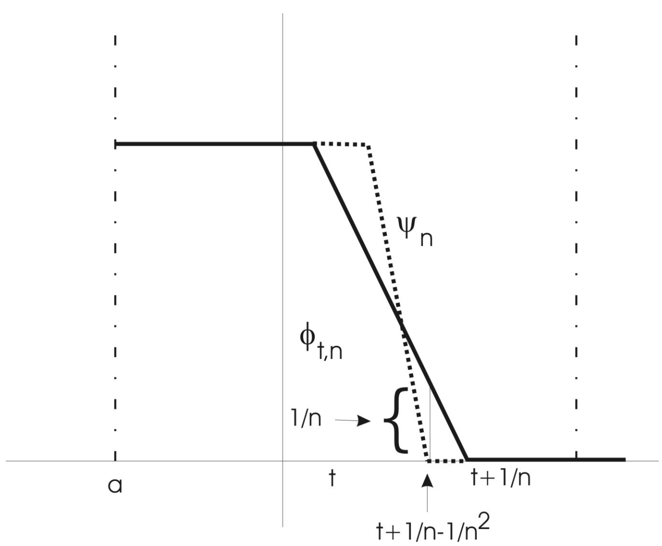

Therefore,

$$\Gamma(\psi_n) \leq \Gamma(\varphi_{t,n}) + \frac{1}{n} \| \Gamma \| \leq g(t) + 2\epsilon.$$

This means that

$$g(t) \leq g(t + \frac{1}{n^2}) \leq g(t) + 2\epsilon.$$

However, as $g(t)$ is increasing, this is sufficient to show that $g(t)$ is right continuous.

The Hahn Extension Theorem gives a Borel measure μ such that $\mu((\alpha, \beta]) = g(\beta) - g(\alpha)$. In particular, if $a \leq c \leq b$, then

$$\mu([a, c]) = \mu((a - 1, c]) = g(c).$$

Let $f \in C([a, b])$ and let $\epsilon > 0$. Let δ be such that if $| x - y |< \delta$ and $x, y \in [a, b]$, then

$$| f(x) - f(y) |< \epsilon.$$

Let $P = \{a = t_0, t_1, \ldots, t_m = b\}$ be a partition with $\sup(t_k - t_{k-1}) < \frac{\delta}{2}$. Then choose n large enough so that $\frac{2}{n} < \inf(t_k - t_{k-1})$ and

$$(*) \qquad g(t_k) \leq \Gamma(\varphi_{t,n}) \leq g(t_k) + \tfrac{\epsilon}{m\|f\|_\infty}.$$

Next, we let

$$f_1(x) = f(t_1)\varphi_{t_1,n} + \sum_{k=2}^{m} f(t_k)(\varphi_{t_k,n} - \varphi_{t_{k-1},n})$$

and

$$f_2(x) = f(t_1)\chi_{[t_0,t_1]} + \sum_{k=2}^{m} f(t_k)\chi_{[t_{k-1},t_k]})$$

Note that f_1 is continuous and piecewise linear. f_2 is a step function. It is also true that both f_1 and f_2 agree with $f(x)$ at each point t_k for $k \geq 1$. Moreover, the function f_1 takes on values between $f(t_{k-1}$ and $f(t_k)$ on the interval $[t_{k-1}, t_k]$. As such

$$\| f_1 - f \|_\infty \leq \epsilon$$

and

$$sup\{| f_2(x) - f(x) || x \in [a, b]\} \leq \epsilon.$$

From this we conclude that

$$| \Gamma(f) - \Gamma(f_1) | \leq \epsilon \| \Gamma \| .$$

We use (*) to see that for $2 \leq k \leq m$

$$| \Gamma(\varphi_{t_k,n} - \varphi_{t_{k-1},n}) - (g(t_k) - g(t_{k-1})) | \leq \frac{\epsilon}{m \| f \|_\infty}$$

Next, we apply Γ to f_1 and integrate f_2 with respect to μ to get

$$| \Gamma(f_1) - \int_{[a,b]} f_2 \, d\mu | \leq \epsilon$$

We also have that

$$\int_{[a,b]} f_2 \, d\mu - \int_{[a,b]} f \, d\mu | \leq \epsilon\mu([a, b]).$$

Therefore,

$$| \Gamma(f) - \int_{[a,b]} f \, d\mu | \leq \epsilon(2 \| \Gamma \| + \mu([a, b]).$$

Since ϵ is arbitrary,

$$\Gamma(f) = \int_{[a,b]} f \, d\mu$$

for each $f \in C[a, b]$. Moreover, $\| \Gamma \| = \Gamma(1) = | \mu | ([a, b])$.

The general result follows from the previous theorem.

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6.4 Riesz Representation Theorem for $C(\Omega)$

In this section we will briefly discuss how to extend the Riesz Representation to $C(\Omega)$ when (Ω, d) is a compact metric space. In fact we can state this extension in greater generality:

THEOREM **6.4.1.** *[Riesz Representation Theorem for $C(\Omega)$] Let (Ω, τ) be a compact Hausdorff space. Let $\Gamma \in C(\Omega)^*$. Then there exists a unique finite regular signed measure μ on the Borel subsets of Ω such that*

$$\Gamma(f) = \int_\Omega f \, d\mu$$

for each $f \in C(\Omega)$. Moreover, $\| \Gamma \| = | \mu | (\Omega)$.

REMARK **6.4.2.** Let $\mu \in Meas(\Omega, \mathcal{B}(\Omega))$. If Γ_μ is defined by

$$\Gamma_\mu(f) = \int_\Omega f \, d\mu \qquad (*)$$

for each $f \in C(\Omega)$, then $\Gamma_\mu \in C(\Omega)^*$ and

$$\| \Gamma_\mu \| = | \mu | (\Omega) = \|\mu\|_{meas}.$$

PROBLEM **6.4.3.** *For the converse how do we construct the measure μ?*

Sketch: We will sketch a solution in the special case where (Ω, d) is a compact metric space.
By the Jordan Decomposition Theorem, we may again assume that Γ is positive.
Key Observation: Let $K \subseteq \Omega$ be compact. Assume that $\{\varphi_n\}$ is a sequence of continuous functions such that

$$0 \leq \varphi_{n+1}(t) \leq \varphi_n(t) \leq 1$$

for every $t \in \Omega$ with

$$\lim_{n\to\infty} \varphi_n = \chi_K$$

pointwise. Then

$$\lim_{n\to\infty} \Gamma(\varphi_n)$$

exists. Moreover, if μ is a measure satisfying $(*)$, then the Lebesgue Dominated Convergence Theorem shows that

$$\mu(K) = \int_\Omega \chi_K \, d\mu = \lim_{n\to\infty} \int_\Omega \varphi_n \, d\mu = \lim_{n\to\infty} \Gamma(\varphi_n).$$

From here, let K be compact. For each $n \in \mathbb{N}$ let

$$U_n = \bigcup_{x \in K} B(x, \frac{1}{n})$$

and let $F_n = \Omega \setminus U_n$. Then define

$$\varphi_n(x) = \frac{dist(x, F_n)}{dist(x, F_n) + dist(x, K)}$$

where $dist(x, A) = inf\{d(x, y) \,|\, y \in A\}$.Then $\varphi_n(x) = 1$ if $x \in K$ and $\varphi_n(x) = 0$ if $x \in F_n$. Hence $\varphi_n \to \chi_K$ pointwise.

Moreover since $\{dist(x, F_n)\}$ is decreasing, we get

$$0 \leq \varphi_{n+1}(t) \leq \varphi_n(t) \leq 1.$$