Bochner's Theorem on the Fourier Transform on $\mathbb{R}$

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1 Introduction

Typically, the Fourier transformation sends suitable functions on $\mathbb{R}$ to functions on $\mathbb{R}$. This can be defined on the space $L^1(\mathbb{R}) + L^2(\mathbb{R})$, i.e. functions which can be written as the sum of a function in $L^1(\mathbb{R})$ and a function in $L^2(\mathbb{R})$. A celebrated result (the Hausdorff–Young inequality) states that the Fourier transform takes functions in $L^p(\mathbb{R})$ to $L^q(\mathbb{R})$ for $1 \le p \le 2$, where $\frac{1}{p} + \frac{1}{q} = 1$.

However, this does not extend to the case when $p > 2$. In addition, the maps $\mathcal{F} : L^p(\mathbb{R}) \to L^q(\mathbb{R})$ are not surjective when $p < 2$. Therefore it seems natural to try to extend the Fourier transform to objects other than functions. The most complete method of doing this is extending the Fourier transform to the space of tempered distributions, i.e. the space of linear functionals on the Schwartz functions.

Instead, we will study the *Fourier–Stieltjes transform*, a slight generalisation of the Fourier transform. We now transform complex finite Borel measures rather than functions, and output a function. Bochner's Theorem answers the question of which functions φ are the Fourier–Stieltjes transform of some positive Borel measure. It states that the function is continuous and positive–definite is a necessary and sufficient condition for it to be a Fourier–Stieltjes transform.

We shall first explore the analogous situation on the torus (or the circle here, when the dimension is one). *Fourier–Stieltjes coefficients* will be examined, and are related to Fourier coefficients. There is much similarity between Fourier–Stieltjes coefficients and the Fourier–Stieltjes transform. However, the theory building up to a 'Bochner–type' result on the torus is clearer and simpler than going directly to Bochner's theorem on $\mathbb{R}$.

2 Preliminaries

Let $\mathbb{T}$ be the torus $[0, 2\pi]$ with the points 0 and 2π identified, i.e. the same point. In this paper, we shall be working with two spaces of continuous functions, both equipped with the supremum norm: $C(\mathbb{T})$, continuous functions on $\mathbb{T}$, and $C_0(\mathbb{R})$, continuous functions that vanish at infinity.

We will also consider two spaces of finite complex Borel measures on $\mathbb{T}$ and $\mathbb{R}$, namely $M(\mathbb{T})$ and $M(\mathbb{R})$. For clarity, μ will denote a measure in $M(\mathbb{T})$, while ν will denote a measure in $M(\mathbb{R})$. The norm on both spaces are given by $\|\mu\|_{M(\mathbb{T})} = |\mu|(\mathbb{T})$ and $\|\nu\|_{M(\mathbb{R})} = |\nu|(\mathbb{R})$.

Recall that if V is a Banach space, then the **dual space** V^* is the set of linear functionals $\psi : V \to \mathbb{C}$ which are continuous/bounded. The following theorem is very useful in multiple ways; the proof can be found in [3].

Theorem 1 (Riesz representation theorem)**.** *(1) Any linear functional $\psi \in (C(\mathbb{T}))^*$ can be identified with a unique measure $\mu \in M(\mathbb{T})$ such that $\psi(f) = \frac{1}{2\pi}\int_{\mathbb{T}} f(t)\, d\mu(t)$. In addition, $\|\psi\|_{C(\mathbb{T})} = \|\mu\|_{M(\mathbb{T})}$.*

(2) Any linear functional $\phi \in (C_0(\mathbb{R}))^*$ can be identified with a unique measure $\nu \in M(\mathbb{R})$ such that $\phi(g) = \int_{\mathbb{R}} g(x)\, d\nu(x)$. Furthermore, $\|\phi\|_{C(\mathbb{R})} = \|\nu\|_{M(\mathbb{R})}$.

Recall that the Fourier series on $\mathbb{T}$ of an integrable function f is given by $\hat{f}(n) = \int_{\mathbb{T}} f(t)e^{-int}\, dt$. *Trigonometric polynomials* are functions of the form $P(t) = \sum_{n=-N}^{N} a_n e^{int}$. It can be easily verified that $P(t) = \sum_{n=-N}^{N} \hat{P}(n)e^{int}$. A basic result in Fourier analysis is that the partial sums $\sum_{n=-N}^{N} \hat{f}(n)e^{int}$ do not necessarily converge to the function itself. Nevertheless, we have convergence in a related series. A proof of this statement can be found in [2].

Lemma 2 (Fejér)**.** *For positive integer N, define the **Fejér kernel** by*

$$F_N(x) = \frac{1}{N+1}\frac{\sin^2[(N+1)x/2]}{\sin^2[x/2]}.$$

Then for a continuous function f on $\mathbb{T}$, we have the following convolution:

$$(F_N * f)(t) = \sum_{n=-N}^{N}\left(1 - \frac{|n|}{N+1}\right)\hat{f}(n)e^{int}.$$

Furthermore, $(F_N * f)(t)$ converges uniformly to f as $N \to \infty$.

As the convergents in the previous lemma are trigonometric polynomials, any function can be approximated by trigonometric polynomials. Hence trigonometric polynomials are dense in $C(\mathbb{T})$.

3 Fourier–Stieltjes Coefficients

Fourier–Stieltjes coefficients are an extension of Fourier coefficients, defined with measures in $M(\mathbb{T})$:

Definition 3. The **Fourier–Stieltjes coefficients** of a measure $\mu \in M(\mathbb{T})$ is a function on $\mathbb{Z}$ given by the expression

$$\hat{\mu}(n) = \frac{1}{2\pi}\int_{\mathbb{T}} e^{-int}\, d\mu(t).$$

Note that if we have an integrable function f, we can identify it with a measure $d\mu = f(t)\,dt$ (dt represents the usual Lebesgue measure). Computing the Fourier–Stieltjes coefficients gives $\hat{\mu}(n) = \frac{1}{2\pi}\int_{\mathbb{T}} e^{-int}\,d\mu = \frac{1}{2\pi}\int_{\mathbb{T}} e^{-int} f(t)\,dt = \hat{f}(n)$, so we really do have an extension of Fourier coefficients.

For Fourier transforms of functions in $L^2(\mathbb{R})$, an important result is Parseval's formula: $\int_{\mathbb{R}} f(x)\overline{g(x)}\,dx = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\xi)\overline{\hat{g}(\xi)}\,d\xi$. An analogue holds in the context of Fourier–Stieltjes coefficients in the following sense:

Proposition 4 (Parseval's Formula). *If $\mu \in M(\mathbb{T})$ and $f \in C(\mathbb{T})$, then*

$$\frac{1}{2\pi}\int_{\mathbb{T}} f(t)\overline{d\mu(t)} = \lim_{N\to\infty}\sum_{n=-N}^{N}\left(1 - \frac{|n|}{N+1}\right)\hat{f}(n)\overline{\hat{\mu}(n)}.$$

Proof. Suppose f is any continuous function. Then by using lemma 2, we can approximate f uniformly as $\lim\limits_{N\to\infty}\left(1 - \frac{|n|}{N+1}\right)\hat{f}(n)e^{int}$. As μ is a finite measure, we can exchange the limit and integral to yield

$$\begin{aligned}
\int_{\mathbb{T}} f(t)\overline{d\mu(t)} &= \lim_{N\to\infty}\int_{\mathbb{T}}\sum_{n=-N}^{N}\left[\left(1 - \frac{|n|}{N+1}\right)\hat{f}(n)e^{int}\right]\overline{d\mu(t)} \\
&= \lim_{N\to\infty}\sum_{n=-N}^{N}\left(1 - \frac{|n|}{N+1}\right)\hat{f}(n)\int_{\mathbb{T}} e^{int}\,\overline{d\mu(t)} \\
&= \lim_{N\to\infty}\sum_{n=-N}^{N}\left(1 - \frac{|n|}{N+1}\right)\hat{f}(n)\overline{\hat{\mu}(n)}.
\end{aligned}$$

□

We remark that without the $\left(1 - \frac{|n|}{N+1}\right)$ factor, the right hand side may not necessarily converge.

Bochner's theorem is about trying to determine which sequences are the Fourier–Stieltjes coefficients of a measure. As a first step, we give a necessary and sufficient condition for a sequence to the Fourier–Stieltjes coefficients of a measure μ:

Proposition 5. *Let $\{a_n\}_{n=-\infty}^{\infty}$ be a sequence of complex numbers. Then the following are equivalent:*

(a) There exists $\mu \in M(\mathbb{T})$ with $\|\mu\| \le C$ and $\hat{\mu}(n) = a_n$ for all n.

(b) For all trigonometric polynomials P, $\left|\sum \hat{P}(n)\overline{a_n}\right| \le C\sup\limits_{t\in\mathbb{T}} P(t)$.

Proof. (a) $\Longrightarrow$ (b): Given any trigonometric polynomial P, use Parseval's formula to get

$$\left|\sum_{n=-N}^{N}\hat{P}(n)\overline{a_n}\right| = \left|\int_{\mathbb{T}} P(t)\,\overline{d\mu}\right| \le \|\mu\|\sup_{t\in\mathbb{T}} P(t)$$

where we used the Riesz representation theorem to show that $|\mu(f)| \le \|\mu\| \cdot \|f\|_{C(\mathbb{T})} = \|\mu\|\sup_{t\in\mathbb{T}} f(x)$.

(b) $\implies$ (a): The linear map $P \mapsto \sum \hat{P}(n)\overline{a_n}$ defines a linear functional on the space of trigonometric polynomials. As trigonometric polynomials are dense in $C(\mathbb{T})$, we can extend our linear functional to $C(\mathbb{T})$.

By the Riesz representation theorem our linear functional is of the form $f \mapsto \frac{1}{2\pi}\int_{\mathbb{T}} f\,\overline{d\mu}$ for a suitable measure $\mu \in M(\mathbb{T})$ of norm $\leq C$. Substituting $f = e^{int}$ into both $P \mapsto \sum \hat{P}(n)\overline{a_n}$ and $f \mapsto \frac{1}{2\pi}\int_{\mathbb{T}} f\,\overline{d\mu}$ immediately gives $\overline{\hat{\mu}(n)} = \overline{a_n}$ or $\hat{\mu}(n) = a_n$. □

Consider a sequence $\{a_n\}_{n\in\mathbb{Z}}$ which eventually vanishes, i.e. $a_n = 0$ when $|n| > K$. Then it can be checked that the measure given by $d\mu_N := \sum_{n=-K}^{K} a_n e^{int}\,dt$ satisfies $\widehat{\mu_N}(n) = a_n$ for all n. This allows us to make sense of the following corollary. The proof involves algebraic manipulations and Parseval's formula only, and hence will not be presented here.

Corollary 6. *A sequence $\{a_n\}_{n\in\mathbb{Z}}$ is the Fourier–Stieltjes coefficients of some μ with $\|\mu\| \leq C$, if, and only if, $\|\mu_N\|_{M(\mathbb{T})} \leq C$ for all N. Here, μ_N is the measure such that $\widehat{\mu_N}(n) = (1 - \frac{|n|}{N+1})a_n$ whenever $|n| \leq N$, and zero when $|n| > N$.*

4 Hergoltz's Theorem

Hergoltz's theorem is the analogue of Bochner's theorem on the torus, as in it gives necessary and sufficient conditions for a sequence to be the Fourier–Stieltjes coefficients of a positive measure. To prove this, we first need the following lemma:

Lemma 7. *A sequence $\{a_n\}_{n\in\mathbb{Z}}$ is the Fourier–Stieltjes series of a positive measure if, and only if, for all N and $t \in \mathbb{T}$,*

$$\sigma_N(t) := \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) a_n e^{int} \geq 0.$$

Proof. First suppose $\hat{\mu}(n) = a_n$ for some positive measure $\mu \in M(\mathbb{T})$. Let $f \in C(\mathbb{T})$ be an arbitrary non-negative function. Then

$$\begin{aligned}
\frac{1}{2\pi}\int_{\mathbb{T}} f(t)\overline{\sigma_N(t)}\,dt &= \frac{1}{2\pi}\int_{\mathbb{T}} f(t)\overline{\left[\sum_{n=-N}^{N}\left(1 - \frac{|n|}{N+1}\right) a_n e^{int}\right]}\,dt \\
&= \sum_{n=-N}^{N}\left(1 - \frac{|n|}{N+1}\right)\hat{f}(n)\overline{\hat{\mu}(n)} \\
&= \sum_{n=-N}^{N}\left(1 - \frac{|n|}{N+1}\right)\hat{f}(n)\overline{\int_{\mathbb{T}} e^{int}\,d\mu(t)} \\
&= \int_{\mathbb{T}}\left(1 - \frac{|n|}{N+1}\right)\hat{f}(n)e^{int}\,\overline{d\mu} \\
&= \int_{\mathbb{T}}(F_N * f)(t)\,\overline{d\mu} \qquad \text{(from lemma 2).}
\end{aligned}$$

Now note that the Fejér kernel is non-negative. As f is non-negative, the convolution $(F_N * f)$ is also non-negative. From the positivity of the measure μ, our quantity is non-negative. As this is true for any non-negative function f, we get that $\sigma_N(t) \geq 0$.

Now suppose $\sum_{n=-N}^{N}\left(1-\frac{|n|}{N+1}\right)a_n e^{int} \geq 0$. As before, denote μ_N as the measure with Fourier–Stieltjes coefficients $\left(1-\frac{|n|}{N+1}\right)a_n$. Then we want to compute $\|\mu_N\|_{M(\mathbb{T})}$. By the Riesz representation theorem we just need to compute the norm of the functional $f \mapsto \frac{1}{2\pi}\int_{\mathbb{T}} f\,\overline{d\mu_N}$. However, from earlier discussion, $\overline{d\mu_N} = \left(1-\frac{|n|}{N+1}\right)a_n\,dt$. Therefore

$$\|\mu_N\|_{M(\mathbb{T})} = \sup_{\|f\|_{C(\mathbb{T})}=1} \frac{1}{2\pi}\int_{\mathbb{T}} f(t)\,\overline{d\mu_N} = \sup_{\|f\|=1}\frac{1}{2\pi}\int_{\mathbb{T}} f(t)\,\overline{\sigma_N(t)}\,dt.$$

As $\sigma_N(t)$ is positive, μ_N is a positive measure and the above quantity is maximised when $f(t) = 1$. This means that

$$\|\mu_N\|_{M(\mathbb{T})} = \frac{1}{2\pi}\int_{\mathbb{T}}\sum_{n=-N}^{N}\left(1-\frac{|n|}{N+1}\right)a_n e^{int}\,dt = a_0.$$

This must be true for all N. Therefore, $\|\mu_N\|_{M(\mathbb{T})}$ are uniformly bounded by a_0. By corollary 6, we can find a $\mu \in M(\mathbb{T})$ such that $\hat{\mu}(n) = a_n$ for all n. To show that μ is a positive measure, take an arbitrary non-negative function f. Then

$$\int_{\mathbb{T}} f\,d\mu = \lim_{N\to\infty}\int_{\mathbb{T}} f\,d\mu_N \geq 0$$

as this convergence comes from taking the limit in Parseval's formula. Thus μ is positive. $\square$

We remark in the above proof that one can replace the condition $\sigma_N(t) \geq 0$ for all N with $\sigma_N(t) \geq 0$ for infinitely many N. The proof remains the same, except that we only take a subsequence N_j where $\sigma_{N_j}(t) \geq 0$.

Definition 8. A sequence $\{a_n\}_{n\in\mathbb{Z}}$ is **positive definite** if for all sequences of complex numbers $\{z_n\}_{n\in\mathbb{Z}}$ which have all but a finite number of terms zero, we have

$$\sum_{n,m\in\mathbb{Z}} a_{n-m} z_n \overline{z_m} \geq 0.$$

The work from the last few pages can now be combined to prove Herglotz's theorem:

Theorem 9 (Herglotz). *A sequence $\{a_n\}_{n\in\mathbb{Z}}$ is the Fourier–Stieltjes transform of a positive measure $\mu \in M(\mathbb{T})$ if, and only if, the sequence is positive definite.*

Proof. If μ is a positive measure with $\hat{\mu}(n) = a_n$, then

$$\sum_{n,m\in\mathbb{Z}} a_{n-m} z_n \overline{z_m} = \sum_{n,m\in\mathbb{Z}} \int e^{-int} e^{imt} z_n \overline{z_m}\,d\mu = \int \left|\sum_{n\in\mathbb{Z}} z_n e^{-int}\right|^2 d\mu \geq 0.$$

On the other hand, suppose $\{a_n\}$ is positive definite. Fix values t, N. Define numbers $z_n = \begin{cases} e^{int}, & |n| \le N \\ 0, & \text{else} \end{cases}$. By direct substitution, one can verify that $\sum_{n,m} a_{n-m} z_n \overline{z_m} = \sum_j C_{j,N} a_j e^{ijx}$, where $C_{j,N} = \max(0, 2N+1-|j|)$. Therefore by positive–definiteness,

$$\begin{aligned}\sigma_{2N}(t) &= \sum_{j=-2N}^{2N} \left(1 - \frac{|j|}{2N+1}\right) a_j e^{ijx} = \frac{1}{2N+1} \sum_{j=-2N}^{2N} C_{j,N} a_j e^{ijx} \\ &= \frac{1}{2N+1} \sum_{n,m\in\mathbb{Z}} a_{n-m} z_n \overline{z_m} \ge 0.\end{aligned}$$

We proved that $\sigma_N(t) \ge 0$ for even N. The result then follows from lemma 7. □

5 The Fourier–Stieltjes Transform

We define the main object of interest here, the Fourier–Stieltjes transform.

Definition 10. The **Fourier–Stieltjes** transform of a measure $\nu \in M(\mathbb{R})$ is a function on $\mathbb{R}$ given by the expression

$$\hat{\nu}(\xi) = \int_{\mathbb{R}} e^{-i\xi x}\, d\nu(x).$$

Like Fourier–Stieltjes coefficients, there is consistency between the definition here, and the Fourier transform of L^1 functions. If g is an integrable function, identify it with a measure $d\nu = g\,dx$. Hence $\hat{\nu}(\xi) = \int_{\mathbb{R}} e^{-i\xi x} g(x)\,dx = \hat{g}(\xi)$, which shows consistency in the transforms.

The Riemann–Lebesgue lemma on the ordinary Fourier transform states that if $g \in L^1(\mathbb{R})$, then $\hat{g}(\xi)$ is uniformly continuous and goes to 0 as $|\xi| \to \infty$. The uniform continuity still holds for the Fourier–Stieltjes transform, but $\hat{\nu}(\xi)$ does not necessarily go to 0. A simple example is $\nu = \delta_0$, the measure with mass at 0. It can be easily verified that $\hat{\nu}(\xi) = 1$ for all ξ.

We can deduce another version of Parseval's formula in this new context:

Proposition 11 (Parseval's Formula)**.** *If $\nu \in M(\mathbb{R})$ and both $g, \hat{g}$ are in $L^1(\mathbb{R})$, then*

$$\int_{\mathbb{R}} g(x) d\nu(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{g}(\xi) \hat{\nu}(-\xi)\, d\xi.$$

Proof. If both g and $\hat{g}$ are integrable, then the Fourier inversion formula holds: $g(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{g}(\xi) e^{i\xi x}\, d\xi$. Therefore,

$$\int_{\mathbb{R}} g(x) d\nu(x) = \frac{1}{2\pi} \iint_{\mathbb{R}} \hat{g}(\xi) e^{i\xi x}\, d\nu(x)\, d\xi = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{g}(\xi) \hat{\nu}(-\xi),$$

where we used the integrability of $\hat{g}(\xi)$ to justify the usage of Fubini's theorem. □

The following proposition gives a necessary and sufficient statement for a function to be the Fourier–Stieltjes transform of a measure, and is the first step to Bochner's theorem on $\mathbb{R}$.

Proposition 12. *If φ is a continuous function defined on $\mathbb{R}$, then it is the Fourier–Stieltjes transform of a measure if, and only if, there exists a constant C such that*

$$\left| \frac{1}{2\pi} \int_{\mathbb{R}} \hat{g}(\xi)\varphi(-\xi)\, d\xi \right| \leq C \sup_{x\in\mathbb{R}} |g(x)|$$

for every continuous function $g \in L^1(\mathbb{R})$ such that $\hat{g}$ has compact support.

Proof. Firstly, suppose that $\varphi = \hat{\nu}$. Then the statement follows directly from Parseval's formula by setting $C = \|\nu\|_{M(\mathbb{R})}$.

On the other hand, suppose our inequality is valid. Then the linear functional ψ which maps g to $\frac{1}{2\pi}\int_{\mathbb{R}} \hat{g}(\xi)\varphi(-\xi)\, d\xi$ is a bounded, continuous linear functional on the set of continuous functions g such that $\hat{g}$ is compactly supported. This is a dense subset of $C_0(\mathbb{R})$. Therefore we can extend ψ to a bounded functional on $C_0(\mathbb{R})$.

By the Riesz representation theorem, ψ can be represented by a measure $\nu \in M(\mathbb{R})$, where $\|\nu\|_{M(\mathbb{R})} \leq C$. Therefore ψ maps g to $\int_{\mathbb{R}} \hat{g}(x)\, d\nu(x)$. Using Parseval's formula here yields that $\frac{1}{2\pi}\int_{\mathbb{R}} \hat{g}(\xi)\varphi(-\xi)\, d\xi = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{g}(\xi)\hat{\nu}(-\xi)\, d\xi$. This must hold for all g, so $\varphi = \hat{\nu}$. □

6 Connection to Fourier–Stieltjes Coefficients

We have a natural covering map $p : \mathbb{R} \to \mathbb{T}$ sending x to $x \bmod 2\pi$. This can be used to pushforward a measure $\nu \in M(\mathbb{R})$ to a measure $\mu \in M(\mathbb{T})$. If we extend a continuous function f on $\mathbb{T}$ to a 2π-periodic function g on $\mathbb{R}$ (which do not necessarily go to zero), then we get that

$$\int_{\mathbb{R}} g(x)\, d\nu(x) = \int_{\mathbb{T}} f(t)\, d\mu(t).$$

An immediate consequence is that $\hat{\nu}(n) = \hat{\mu}(n)$ for all integers n. This allows us to relate the original problem on $\mathbb{R}$ to the problem on $\mathbb{T}$. The following theorem is a key step in doing this:

Theorem 13. *If φ is a continuous function defined on $\mathbb{R}$, then it is the Fourier–Stieltjes transform if, and only if, there exists a constant $C > 0$ such that for any choice of $\lambda > 0$, $\{\varphi(\lambda n)\}_{n=-\infty}^{\infty}$ are the Fourier–Stieltjes coefficients of a measure of norm $\leq C$ on $\mathbb{T}$.*

Proof. First suppose $\varphi = \hat{\nu}$. Then $\varphi(n) = \hat{\nu}(n) = \hat{\mu}(n)$, where μ is the pushforward measure of ν with respect to the covering. Note that $\|\mu\|_{M(\mathbb{T})} \leq \|\nu\|_{M(\mathbb{R})}$. Denote $\nu(x/\lambda)$ the measure on $\mathbb{R}$ which satisfies the following equation for all g:

$$\int_{\mathbb{R}} g(x) d\nu(\tfrac{x}{\lambda}) = \int_{\mathbb{R}} g(\lambda x)\, d\nu(x).$$

We get that $\|\nu(x/\lambda)\|_{M(\mathbb{R})} = \|\nu\|_{M(\mathbb{R})}$ and $\widehat{\nu(x/\lambda)}(\xi) = \hat{\nu}(\xi\lambda)$. Setting $\xi = n$ yields that $\varphi(\lambda n) = \mu(x/\lambda)(n)$, so $\{\varphi(\lambda n)\}_{n\in\mathbb{Z}}$ form the Fourier–Stieltjes coefficients of norm at most $\|\nu\|_{M(\mathbb{R})}$.

To show the converse, we shall use theorem 12. Let g be continuous such that $\hat{g}$ is continuous and compactly supported. Then by those conditions the integral $\frac{1}{2\pi}\int_{\mathbb{R}} \hat{g}(\xi)\varphi(-\xi)\,d\xi$ can be approximated by its Riemann sums (where the width of each rectangle is λ). For arbitrary ϵ, choose sufficiently small λ to obtain

$$\left|\frac{1}{2\pi}\int_{\mathbb{R}} \hat{g}(\xi)\varphi(-\xi)\,d\xi\right| < \left|\frac{\lambda}{2\pi}\sum_n \hat{g}(\lambda n)\varphi(-\lambda n)\right| + \epsilon.$$

Note that $\frac{\lambda}{2\pi}\hat{g}(\lambda n)$ is the Fourier coefficient for the function $G_\lambda(t) = \sum\limits_{m\in\mathbb{Z}} g((t+2\pi m)/\lambda)$ on $\mathbb{T}$. If λ is sufficiently small, then by the decay of g, we have

$$\sup_{t\in\mathbb{T}} |G_\lambda(t)| \le \sup_{x\in\mathbb{R}} |g(x)| + \epsilon.$$

By assumption, $\varphi(\lambda n) = \widehat{\mu_\lambda}(n)$ for some $\mu_\lambda \in M(\mathbb{T})$, with $\|\mu_\lambda\|_{M(\mathbb{T})} \le C$. Then Parseval's formula for $\mathbb{T}$ gives

$$\left|\frac{\lambda}{2\pi}\sum_n \hat{g}(\lambda n)\varphi(-\lambda n)\right| = \left|\sum_n \hat{G_\lambda}(n)\hat{\mu_\lambda}(-n)\right| \le C\sup_{t\in\mathbb{T}}|G_\lambda(t)|.$$

Combining all inequalities gives

$$\left|\frac{\lambda}{2\pi}\sum_n \hat{g}(\lambda n)\varphi(-\lambda n)\right| < C\sup_{x\in\mathbb{R}}|g(x)| + (C+1)\epsilon.$$

As $\epsilon > 0$ was arbitrary, the condition for proposition 12 is satisfied. □

Now here is a necessary and sufficient condition for a function φ to be a Fourier–Stieltjes transform of a positive measure. The proof is very similar to theorem 13, so it will not be presented here.

Proposition 14. *If φ is a continuous function defined on $\mathbb{R}$, then it is the Fourier–Stieltjes transform of a positive measure if, and only if, there exists a constant $C > 0$ such that for any choice of $\lambda > 0$, $\{\varphi(\lambda n)\}_{n=-\infty}^{\infty}$ are the Fourier–Stieltjes coefficients of a positive measure on $\mathbb{T}$.*

7 Bochner's Theorem

Definition 15. A function φ defined on $\mathbb{R}$ is **positive definite** if for all $\xi_1,\dots,\xi_N \in \mathbb{R}$ and $z_1,\dots,z_N \in \mathbb{C}$, we have

$$\sum_{j,k=1}^{N} \varphi(\xi_j - \xi_k) z_j \overline{z_k} \ge 0.$$

Finally, we have the machinery to prove the main subject of this paper.

Theorem 16. *A function φ defined on $\mathbb{R}$ is the Fourier–Stieltjes transform of a positive measure if, and only if, it is continuous and positive definite.*

Proof. First assume that $\varphi = \hat{\nu}$ for a positive measure $\nu \in M(\mathbb{R})$. Given numbers $\xi_1, \dots \xi_N \in \mathbb{R}$ and $z_1, \dots z_N \in \mathbb{C}$, we get

$$\begin{aligned}\sum_{j,k=1}^{N} \varphi(\xi_j - \xi_k) z_j \overline{z_k} &= \int_{\mathbb{R}} \sum_{j,k=1}^{N} e^{-i\xi_j x} z_j e^{-i\xi_k x} \overline{z_k}\, d\nu(x) \\ &= \int_{\mathbb{R}} \left| \sum_{j=1}^{N} z_j e^{-i\xi_j x} \right|^2 d\nu(x) \geq 0.\end{aligned}$$

For the other direction, if φ is positive definite, then by definition, $\{\varphi(\lambda n)\}$ is a positive definite sequence for all λ. Herglotz's theorem guarantees a measure μ_λ, such that $\widehat{\mu_\lambda}(n) = \varphi(\lambda n)$. By proposition 14, there must exist ν such that $\hat{\nu} = \varphi$. $\square$

References

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