RIESZ PRODUCTS

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To J. E. LITTLEWOOD *on his* 80*th birthday*

1. Introduction

A Riesz product is an infinite product

$$\prod_{\nu=1}^{\infty} (1+\alpha_\nu \cos n_\nu x), \tag{1}$$

where the positive integers n_ν satisfy a condition

$$n_{\nu+1}/n_\nu \geqslant q \geqslant 3, \tag{2}$$

and where α_ν is real and satisfies the conditions $0 < |\alpha_\nu| \leqslant 1$ for all ν. Let

$$\mu_k = \sum_1^k n_i, \quad \mu'_k = n_{k+1} - \mu_k.$$

Then $\mu_k < \mu'_k$ since, by (2),

$$\mu_k < n_k \sum_0^\infty q^{-i} = n_k q/(q-1) \leqslant n_k q(q-2)/(q-1) \leqslant n_{k+1}\left(1 - \sum_1^\infty q^{-i}\right) < \mu'_k.$$

The partial products of (1) are *non-negative* since $|\alpha_\nu| \leqslant 1$. Replacing products of cosines by linear combinations of cosines, we see that the kth partial product of (1) is a *non-negative* trigonometric polynomial

$$p_k(x) = 1 + \sum_1^{\mu_k} \gamma_\nu \cos \nu x = \prod_1^k (1+\alpha_i \cos n_i x), \tag{3}$$

with $\gamma_\nu = 0$ if ν is not of the form $n_{i_1} \pm n_{i_2} \pm \ldots$, where $k \geqslant i_1 > i_2 > \ldots$. p_{k+1} is formed by adding to p_k a polynomial

$$p_{k+1} - p_k = p_k \alpha_{k+1} \cos n_{k+1} x,$$

the lowest term of which is of rank $\mu'_k > \mu_k$. Allowing $k \to \infty$ we obtain from (3) the formal series

$$1 + \sum_1^\infty \gamma_\nu \cos \nu x. \tag{4}$$

No two terms obtained by replacing products of cosines by linear combinations of cosines are of the same rank, for suppose that an integer N can be expressed in the two forms

$$N = n_{i_1} \pm n_{i_2} \pm \ldots = n_{j_1} \pm n_{j_2} \pm \ldots \quad (i_1 > i_2 > \ldots j_1 > j_2 > \ldots), \tag{5}$$

with $j_1 < i_1$. Then $n_{i_1} = an_{i_1-1} + bn_{i_1-2} + \ldots$, where a, b, ... take only the values 0, ± 1, ± 2. But

$$an_{i_1-1} + bn_{i_1-2} + \ldots < 2n_{i_1-1}(1+3^{-1}+\ldots) = 3n_{i_1-1},$$

i.e. $n_{i_1} < 3n_{i_1-1}$, which contradicts (2). Thus $j_1 = i_1$, and we have

$$n_{i_2} \pm n_{i_3} \pm \ldots = n_{j_2} \pm n_{j_3} \pm \ldots.$$

Repetition of this argument shows that $i_2 = j_2$, $i_3 = j_3$, ..., and that corresponding signs in the two forms in (5) are the same. It follows, in particular, that

$$\gamma_{n_\nu} = \alpha_\nu. \tag{6}$$

In this paper I shall sketch briefly some of the applications of Riesz products which have been made. The background which I give to each application is minimal (or even less!). Most of these applications, like the earliest one, due to F. Riesz and mentioned in the next section, are of a negative character; the property

$$p_k(x) \geqslant 0 \tag{7}$$

provides for the satisfaction of certain conditions, and a suitable choice of the coefficients α_ν in (1) then shows that certain further conditions are not necessarily satisfied. It is, however, a notable fact that some of the most important applications of Riesz products have been in the proof of results (with a distinctly positive flavour)† concerning lacunary series and lacunary coefficients. The proofs of these results contain features not to be found in the purely negative applications of Riesz products. They are described briefly in §§ 2–4. The content of §§ 1–4 is in Zygmund's book (**1**).

2. Further properties of Riesz products

By (3) and (7), the partial sums of (4) have the property

$$s_{\mu_k}(x) = p_k(x) \geqslant 0,$$

and as a consequence we deduce ((**1**) I, 209)

THEOREM 1. *The series* (4) *associated with* (1) *is the Fourier–Stieltjes series of a non-decreasing continuous function F defined by*

$$F(x) - F(0) = \lim_{k\to\infty} \int_0^x p_k(t)\,dt. \tag{8}$$

In particular, if α_ν does not tend to 0 (e.g. if $\alpha_\nu = 1$, $n_\nu = 3^\nu$) then, by (6), we obtain an example of a continuous function of bounded variation whose Fourier coefficients are not $o(1/n)$. This was historically the first such example and the first use made of a Riesz product (**2**).

† Though applications of *these* are usually of a negative nature.

The next two theorems, which are due to Zygmund, give more information about the function F of (8).

THEOREM 2. *If* $\sum \alpha_\nu^2 = \infty$ *then* $F'(x) = 0$ *almost everywhere.*

Combination of Theorems 1 and 2 shows that, if $\sum \alpha_\nu^2 = \infty$, then $F(x)$ is a continuous singular function and (4) is not a Fourier series. This fact was used in a paper by M. Weiss mentioned in § 5.

THEOREM 3. *If* $\alpha_\nu \to 0$ *and* $\sum \alpha_\nu^2 = \infty$, *then both the series* (4) *and the conjugate series converge almost everywhere, the former to zero.*

Some negative consequences of this theorem, and of Weiss's theorem described in § 5, are to be found in ((**3**) 79–80).

3. Szidon's theorem on lacunary series

A trigonometric series $\sum (a_j \cos m_j x + b_j \sin m_j x)$ is said to be *lacunary* if an inequality $m_{j+1}/m_j > \lambda > 1$ is satisfied for all j. The behaviour of such series is peculiar in that, roughly speaking, what happens over any interval in $(0, 2\pi)$ (or even, in some cases, over any set of positive measure) determines what happens over the whole of the interval $(0, 2\pi)$, and virtually any regular behaviour of the series implies strong consequences in the behaviour of the coefficients a_j, b_j. The following theorem is due to Szidon (see ((**1**) I, 247–48), where the theorem is stated in a slightly stronger form).

THEOREM 4.† *If a lacunary series*

$$\sum (a_j \cos m_j x + b_j \sin m_j x) \quad (m_{j+1}/m_j > \lambda > 1)$$

is the Fourier series of a bounded function, then $\sum (|a_j| + |b_j|) < \infty$.

Let

$$\rho_j = (a_j^2 + b_j^2)^{1/2}, \quad a_j \cos m_j x + b_j \sin m_j x = \rho_j \cos(m_j x + x_j),$$

and consider

$$\prod_1^\infty \{1 + \cos(n_j x + \xi_j)\}, \tag{9}$$

where

$$n_{j+1}/n_j \geqslant q \geqslant 3.$$

(9) is a modified form of Riesz product with essentially the same properties as (1). In particular, the partial products of (9) are non-negative polynomials with constant term 1, and associated with (9) is a series

$$1 + \sum \gamma_\nu \cos(\nu x + \eta_\nu). \tag{10}$$

† The conclusion of this theorem also holds under the weaker assumption that the lacunary series is the Fourier series of a function bounded over an interval in $(0, 2\pi)$.

As in the case of the product (1), $\gamma_\nu = 0$ unless

$$\nu = n_{j_1} \pm n_{j_2} \pm \dots \quad (j_1 > j_2 > \dots), \tag{11}$$

and there is at most one such representation of any given ν, so that in particular the n_jth term of (10) is $\cos(n_j x + \xi_j)$.

The sum in (11) is contained between

$$n_{j_1}(1 - q^{-1} - q^{-2} - \dots) = n_{j_1}(q-2)/(q-1)$$

and

$$n_{j_1}(1 + q^{-1} + q^{-2} + \dots) = n_{j_1}q/(q-1),$$

so if q is large enough, say $q > q_0(\lambda)$, the ν with $\gamma_\nu \neq 0$ are confined to the intervals $(n_j/\lambda, n_j\lambda)$. We split $\{m_j\}$ into r subsequences

$$\{m_{jr+s}\}_{j=0,1,\dots} \quad (s = 1, 2, \dots, r),$$

and take r so large that $q = \lambda^r > \max(3, q_0(\lambda))$. Fixing s, we write

$$R^s(x) = \prod_{j=0}^{k} \{1 + \cos(m_{jr+s}x + x_{jr+s})\}.$$

Suppose, now, that the given lacunary series is the Fourier series of a function $f(x)$, where $|f(x)| \leqslant M$. Since the ranks of the non-zero terms of R^s are in the intervals $(m_{jr+s}/\lambda, m_{jr+s}\lambda)$, $j = 0, 1, \dots, k$, and the only non-zero term of the lacunary series which has its rank in such an interval has rank m_{jr+s}, it follows that

$$\frac{1}{\pi}\int_0^{2\pi} f(x)R^s(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} f(x)\sum_{j=0}^{k}\cos(m_{jr+s}x + x_{jr+s})\,dx = \sum_{j=0}^{k}\rho_{jr+s}.$$

But, since $R^s \geqslant 0$, we have

$$\left|\frac{1}{\pi}\int_0^{2\pi} f(x)R^s(x)\,dx\right| \leqslant M\frac{1}{\pi}\int_0^{2\pi} R^s(x)\,dx = 2M,$$

so $\sum_{j=0}^{k}\rho_{jr+s} \leqslant 2M$. Allowing $k \to \infty$ we obtain $\sum_{j=0}^{\infty}\rho_{jr+s} \leqslant 2M$, and summing over s, $\sum \rho_j \leqslant 2Mr$, which implies that $\sum(|a_j|+|b_j|) \leqslant 2Mr$.

4. Banach's theorem on the prescription of lacunary coefficients

Suppose that

$$\{a_n, b_n\} \tag{12}$$

is a sequence of pairs of real numbers. By the Riemann–Lebesgue theorem, if the sequence (12) is that of the Fourier coefficients of an integrable function, then $|a_n|+|b_n| \to 0$. By Parseval's theorem, if the sequence is that of the Fourier coefficients of a continuous function, then $\sum(a_n{}^2 + b_n{}^2)$ is convergent. Trivially, if the sequence is that of the Fourier–Stieltjes coefficients of a function of bounded variation, then

$|a_n|+|b_n| = O(1)$. The converse of each of these three statements is false, but a converse does hold if only certain subsequences of (12) are prescribed.

THEOREM 5.† *Let $\{n_k\}$ be a sequence of positive integers such that*

$$n_{k+1}/n_k > \lambda > 1$$

for all k, and let $\{x_k, y_k\}$ be a sequence of pairs of real numbers.

(i) *If $\sum (x_k^2+y_k^2) < \infty$, then there is a continuous function with Fourier coefficients satisfying*

$$a_{n_k} = x_k, \quad b_{n_k} = y_k. \tag{13}$$

(ii) *If $|x_k|+|y_k| \to 0$, then there is an integrable function with Fourier coefficients satisfying* (13).

(iii) *If x_k, y_k are bounded, then there is a continuous non-decreasing function with Fourier–Stieltjes coefficients satisfying* (13).

We sketch the proof of (iii) only. (ii) then follows from (iii) by an argument not relevant to Riesz products, and the proof of (i) is on similar lines. As in the last section, we write $\rho_k = (x_k^2+y_k^2)^{1/2}$, and we write $A_{n_k}(x) = x_k \cos n_k x + y_k \sin n_k x$. There is no loss of generality in assuming that $\rho_k \leqslant 1$ for all k. We suppose first that $\lambda \geqslant 3$, and consider the modified Riesz product

$$\prod_{k=1}^{\infty} \{1+A_{n_k}(x)\}.$$

Since $\rho_k \leqslant 1$, $\lambda \geqslant 3$, expansion of this product yields a formal trigonometric series analogous to (10), and the conclusion of (iii) follows by a correspondingly modified form of Theorem 1.

For $1 < \lambda < 3$, we choose an integer r, as in the proof of Szidon's theorem, such that $\lambda^r \geqslant 3$, and we split $\{n_k\}$ into r sequences:

$$n_1, n_{r+1}, n_{2r+1}, \ldots; n_2, n_{r+2}, n_{2r+2}, \ldots; \ldots; n_r, n_{2r}, n_{3r}, \ldots.$$

The sum

$$\prod_{s=0}^{\infty} (1+A_{n_{sr+1}}) + \prod_{s=0}^{\infty} (1+A_{n_{sr+2}}) + \ldots + \prod_{s=0}^{\infty} (1+A_{n_{sr+r}}) \tag{14}$$

is then the Fourier–Stieltjes series of a continuous non-decreasing function. Again as in the proof of Szidon's theorem, if r is large enough, the r series originating from the products in (14) have no terms in common, and the conclusion of (iii) again follows.

In the remainder of this paper I describe some of the negative results which have been obtained by using Riesz products.

† See ((1) II, 131–32), where appropriate references are given.

5. M. Weiss's theorem

If the partial sums $s_n(x)$ of a trigonometric series

$$\sum_1^\infty (a_n \cos nx + b_n \sin nx) \tag{15}$$

satisfy the condition

$$\int_0^{2\pi} |s_n(x)|\,dx = O(1), \tag{16}$$

then (15) is a Fourier–Stieltjes series. Steinhaus conjectured that, under the hypothesis (16), $|a_n|+|b_n| \to 0$. This was shown to be true by Helson.† Littlewood raised the question whether (16) implies that (15) is a Fourier series. Mary Weiss has provided a negative answer to this question (**4**).

THEOREM 6. *If* $\sum \alpha_n^2 = \infty$ *and*

$$A_k = |\alpha_1|+|\alpha_2|+\ldots+|\alpha_n|, \quad \alpha_n A_n = O(1) \tag{17}$$

(*e.g. if* $\alpha_n = n^{-1/2}$), *then the series* (4) *associated with* (1) *is the Fourier–Stieltjes series of a continuous singular function, and its partial sums satisfy the condition* (16).

Under the hypothesis that $\sum \alpha_n^2 = \infty$, the first part of the conclusion is contained in Theorems 1 and 2. The proof of the second part of the conclusion under the additional hypothesis (17) is ingenious, but difficult to summarize in a paper of this nature. The question whether (15) is a Fourier series under the stronger hypothesis $s_n(x) \geqslant 0$ for all x and n seems to remain open.

6. A problem on rearrangements of Fourier coefficients

Suppose that $c_0, c_1, c_{-1}, c_2, c_{-2}, \ldots$ is a sequence of complex numbers tending to 0, and let $c_0^* \geqslant c_1^* \geqslant c_{-1}^* \geqslant c_2^* \geqslant c_{-2}^* \geqslant \ldots$ be the sequence $|c_0|, |c_1|, |c_{-1}|, \ldots$ rearranged in descending order of magnitude (with corresponding repetitions in the c_n^* if several of the $|c_n|$ are equal). For a measurable function $f(x)$ defined in $(0, 2\pi)$, we write

$$J(f) = \frac{1}{2\pi}\int_0^{2\pi} |f(x)|\,dx.$$

Hardy and Littlewood (**5**) drew attention to the problem of determining whether, if $\sum c_n e^{n\theta i}$ is the Fourier series of a function $f(x)$, $\sum c_n^* e^{n\theta i}$ is then the Fourier series of a function $f^*(x)$ and

$$J(f^*) \leqslant AJ(f), \tag{18}$$

where A is an absolute constant. The question was raised, in particular,

† See, for example ((**1**) I, 286).

whether (18) is true for $A = 1$. Lehmer (**6**) found a trigonometric polynomial which demonstrated that (18) is false for $A = 1$. Subsequently I showed (**7**), using a Riesz product, that (18) is false for any A, and even for trigonometric polynomials. The main features of the argument relating to Riesz products were use of the fact that the polynomial $p_k(x)$ of (3) is non-negative (in the case $\alpha_\nu = 1$, $n_\nu = 3^\nu$), and a calculation of the number of cosines in $p_k(x)$ which have the same coefficient, the result of which is contained in

THEOREM 7. *In the polynomial*

$$q_m(x) = \prod_0^{m-1} (1+\cos 3^k x)$$

there are exactly $2^s\binom{m}{s+1}$ *cosines with coefficient equal to*

$$2^{-s}, \quad s = 0, 1, 2, \ldots, m-1.$$

In the same paper the stronger result was proved that if $\sum c_n e^{n\theta i}$ is a Fourier series then $\sum c_n{}^* e^{n\theta i}$ need not be.

7. A property of bounded starlike functions

Suppose that $f(z) = \sum_1^\infty a_n z^n$ is regular and univalent for $|z| < 1$, and let

$$g(z) = zf'(z)/f(z).$$

$f(z)$ is said to be *starlike* (with respect to the origin) if the image of $|z| < 1$ under the transformation $w = f(z)$ is a domain D with the property that, for any w in D, all of the points tw, $0 \leqslant t \leqslant 1$, are contained in D. It is well known that a necessary and sufficient condition for $f(z)$ to be starlike is that

$$\mathscr{R}g(z) > 0 \quad (|z| < 1).$$

Clunie and Keogh (**8**) showed that, if $f(z)$ is starlike and bounded, then

$$a_n = O(1/n).$$

We then showed that this result is best possible in the sense that the inequality

$$|a_n| > \frac{k}{n}$$

can occur for a constant $k > 0$ and a sequence of values of n. The function $f(z)$ which was constructed for this purpose was one for which

$$g(z) = 1 + \sum \gamma_n z^n,$$

where $1 + \sum_1^\infty \gamma_n \cos nx$ is again of the form (4) with $\alpha_\nu = 1$, $n_\nu = 4^\nu$ in (1). The only property of Riesz products which we used was (7).

8. A strengthened form of a theorem of Wiener

Let $\{s_n\}$, $n = 0, 1, 2, \ldots$, be a sequence of complex numbers. $\{s_n\}$ is said to be almost convergent to s if, as $p \to \infty$,

$$(p+1)^{-1} \sum_{n}^{n+p} s_m$$

tends to s uniformly with respect to n.

$\{s_n\}$ is said to be almost periodic if, corresponding to any positive number ε, there are numbers N, l such that in every interval $(k, k+l)$ $(k \geqslant 0)$ there is an integer p for which $|s_{n+p} - s_n| < \varepsilon$ for all $n \geqslant N$.

Let $F(x)$ be a real function of bounded variation over $(0, 2\pi)$, and let

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} e^{n\theta i}\, dF(\theta).$$

Suppose that

$$F(\theta) = G(\theta) + H(\theta),$$

where $G(\theta) = \sum_{\lambda_m \leqslant \theta} \mu_m$ is a function with discontinuities μ_m at λ_m, $m = 1, 2, \ldots$, and $H(\theta)$ is a continuous function. In (**9**), Keogh and Petersen, strengthening a theorem of Wiener, proved that

(i) $\{|c_n|^2\}$ *is almost convergent to* $(4\pi)^{-2} \sum_{1}^{\infty} \mu_m{}^2$*;* (ii) *for* $F(\theta)$ *to be continuous it is necessary that* $\{|c_n|^2\}$ *(or* $\{|c_n|\}$*) should be almost convergent to zero, and sufficient that* $\{|c_n|^2\}$ *(or* $\{|c_n|\}$*) should be summable to zero by some summation method which contains almost convergence.*

In the course of the proof of this result we showed that if $H(\theta)$ has no singular part, then $\{c_n\}$, $\{|c_n|\}$, and $\{|c_n|^2\}$ are almost periodic sequences. We pointed out that this is not true, in general, if $H(\theta)$ has a singular part, and gave the counter-example

$$F(x) = \lim_{m \to \infty} \int_0^x \prod_{p=1}^{m} (1 + \cos 4^p t)\, dt.$$

For this function, $c_n = 1$ if n is of the form 4^p, but otherwise $c_n \leqslant \frac{1}{2}$, so none of the above-mentioned sequences is almost periodic.

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