and that the associated eigenvectors $e_1, e_2, \cdots, e_n$ have components with exactly $0, 1, 2, \cdots, n-1$ variations in sign.

Is it necessary for a matrix to be totally positive to have this property?

Let us construct a matrix A with

$$\begin{aligned} \lambda_1 &= 3, & e_1 &= (1, 1, 1) \\ \lambda_2 &= 2, & e_2 &= (1, -1, -1); \\ \lambda_3 &= 1, & e_3 &= (1, -1, 1) \end{aligned} \qquad \text{then } E = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$$

and

$$A = EJE^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 5 & 2 & -1 \\ 1 & 4 & 1 \\ 1 & 2 & 3 \end{pmatrix}$$

which is not totally positive. This sufficient condition for the stated property is therefore not necessary.

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THE MAXIMAL THEOREMS OF HARDY AND LITTLEWOOD

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1. Introduction. In this paper we give a unified treatment of the famous maximal theorems of Hardy and Littlewood. Roughly speaking, these theorems give estimates of the maximal value of the difference quotient of an indefinite integral. They were originally proved by Hardy and Littlewood in 1930 [5]. They have had numerous applications, and in fact were proved in order to solve certain problems in complex function theory. Other applications appear in [9] (e.g., to conjugate functions, or Hilbert transforms), [1] and [8] (to ergodic theory), [7] (to harmonic functions), and [2] (to singular integrals). We are not concerned in this paper with applications, but with presenting the main facts and methods associated with maximal functions. Subsection (3.6) is of secondary interest, and can be skipped. Section 4 is not required for Section 5.

We begin by establishing notation. First, f will denote a nonnegative, extended real-valued, Lebesgue measurable function on R (the real numbers) such that $\int_F f d\lambda < \infty$ for all compact sets F; λ is Lebesgue measure for R. (Later, in sect. 5, f will be defined on R^m). Define three "maximal functions" of f as follows:

$$M_r f(x) = \sup\left\{\frac{1}{\lambda(I)}\int_I f d\lambda : I = [x, u], x < u < \infty\right\}$$

(1.1) $$M_l f(x) = \sup\left\{\frac{1}{\lambda(I)}\int_I f d\lambda : I = [u, x], -\infty < u < x\right\}$$

$$Mf(x) = \sup\left\{\frac{1}{\lambda(I)}\int_I f d\lambda : I \text{ is a closed interval containing } x\right\}.$$

Thus $M_r f(x)$, $M_l f(x)$, and $Mf(x)$ are three different maximal averages of the function f. All of the maximal theorems are inequalities giving bounds for the integral of one of the maximal functions composed with a monotonic function. As a result, the mappings M_r, M_l, and M carry certain function spaces to others. Notice that the suprema defining our maximal functions are over finite intervals of all possible lengths. By the continuity of the indefinite integrals involved, each of the maximal functions is lower semicontinuous (and hence Borel measurable). The inequalities $f \leqq M_j f \leqq Mf (j = r, l)$ are immediate, but much more is true.

(1.2) THEOREM. *The equality* $Mf = \text{Max}(M_r f, M_l f)$ *holds.*

Proof. Fix x. For $u < x < t$, let

$$A(t) = (t - x)^{-1}\int_x^t f d\lambda \quad \text{and} \quad B(u) = (x - u)^{-1}\int_u^x f d\lambda; \qquad A(x) = B(x) = 0.$$

Let $p_1(u, t) = (t-x)(t-u)^{-1}$, $p_2(u, t) = (x-u)(t-u)^{-1}$, and $C = p_1 A + p_2 B$, for $u \leqq x \leqq t$ and $u \neq t$; then

$$Mf(x) = \sup\{C(u, t) : u \leqq x \leqq t, t \neq u\}.$$

Let $(u_n, t_n)_{n=1}^{\infty}$ be such that $\lim_{n\to\infty} C(u_n, t_n) = Mf(x)$. The sequence $(p_1(u_n, t_n), p_2(u_n, t_n))_{n=1}^{\infty}$ is contained in the compact subset $[0, 1] \times [0, 1]$ of the plane, and so has a convergent subsequence. Hence there is a sequence $(u_k, t_k)_{k=1}^{\infty}$ such that

$$\lim_{k\to\infty} C(u_k, t_k) = Mf(x) \quad \text{and} \quad \lim_{k\to\infty} (p_1(u_k, t_k), p_2(u_k, t_k)) = (a, b),$$

where $a + b = 1$. For each k, the inequality

$$p_1(u_k, t_k) A(t_k) + p_2(u_k, t_k) B(u_k) \leqq p_1(u_k, t_k) M_r f(x) + p_2(u_k, t_k) M_l f(x)$$

holds; hence the inequality

$$Mf(x) \leqq a M_r f(x) + b M_l f(x)$$

holds. Since $a + b = 1$, the inequality $Mf(x) \leqq \text{Max}(M_r f(x), M_l f(x))$ follows. The reverse inequality is immediate from the definitions (1.1). ∎

2. Two fundamental lemmas. The heart of our proofs of the maximal theorems is in the following two lemmas. The lemmas themselves give precise information about the size of the maximal functions. For a positive real number t and an extended real-valued nonnegative function g on R, we let

$$E_t[g] = \{x \in R: g(x) > t\}.$$

(2.1) LEMMA. *The equalities*

$$\lambda(E_t[M_jf]) = \frac{1}{t}\int_{E_t[M_jf]} fd\lambda \qquad (j = r, l) \tag{i}$$

and the inequality

$$\lambda(E_t[Mf]) \leqq \frac{2}{t}\int_{E_t[Mf]} fd\lambda \tag{ii}$$

hold for every $t>0$.

Proof. We prove (i) with $j=r$, the case $j=l$ being almost the same. It is easy to see that $E_t[M_rf]$ is open. Let $\{]\beta_k, \gamma_k[\}_{k=1}^{\infty}$ be the unique pairwise disjoint intervals with union $E_t[M_rf]$. Consider an interval $]\beta_k, \gamma_k[$ (which may of course be infinite). For each $x \in]\beta_k, \gamma_k[$, the set

$$N_x = \left\{s: \int_x^s fd\lambda > t(s - x),\ s \in]x, \gamma_k] \cap R\right\}$$

is nonvoid. This is trivial if $\gamma_k = \infty$. If $\gamma_k < \infty$ and $N_x = \varnothing$, there would be a $w > \gamma_k$ such that $\int_x^w fd\lambda > t(w-x)$. We would have

$$\int_{\gamma_k}^{w} fd\lambda = \int_x^w fd\lambda - \int_x^{\gamma_k} fd\lambda > t(w - \gamma_k),$$

a contradiction since $\gamma_k \notin E_t[M_rf]$. Let $\gamma = \sup N_x$. If $\gamma < \gamma_k$, then the equality $\int_x^{\gamma} fd\lambda = t(\gamma - x)$ must hold; an obvious argument proves this. Therefore (since N_γ is nonvoid) there is a $y \in]\gamma, \gamma_k] \cap R$ such that $\int_\gamma^y fd\lambda > t(y-\gamma)$. It follows that $\int_x^y fd\lambda > t(y-x)$, a contradiction since $y > \gamma$. We thus have $\gamma_k = \sup N_x$, and hence the inequality $\int_x^{\gamma_k} fd\lambda \geqq t(\gamma_k - x)$ holds for all $x \in]\beta_k, \gamma_k[$; letting $x \to \beta_k$, we get

$$\int_{\beta_k}^{\gamma_k} fd\lambda \geqq t(\gamma_k - \beta_k). \tag{1}$$

If $]\beta_k, \gamma_k[$ is infinite, the equality (i) follows. If $]\beta_k, \gamma_k[$ is finite, then the reverse inequality in (1) is immediate ($\beta_k \notin E_t[M_rf]$). Hence in all cases equality holds in (1), and so (i) is proved for $j = r$.

To prove (ii), note that Theorem (1.2) implies that $E_t[Mf] = E_t[M_lf] \cup E_t[M_rf]$. Hence, using (i), we have

$$\lambda(E_t[Mf]) \leqq \lambda(E_t[M_rf]) + \lambda(E_t[M_lf]) \leqq \frac{2}{t}\int_{E_t[Mf]} fd\lambda. \ \blacksquare$$

(2.2) LEMMA. *For every $k \in]0, 1[$ and every $t > 0$, the following inequalities hold*:

(i) $$\lambda(E_t[M_j f]) \leqq \frac{1}{(1-k)t} \int_{E_{kt}[f]} f d\lambda \qquad (j = r, l);$$

(ii) $$\lambda(E_t[Mf]) \leqq \frac{2}{(1-k)t} \int_{E_{kt}[f]} f d\lambda.$$

Proof. Let

$$g(x) = \begin{cases} f(x) & \text{if } f(x) > kt \\ 0 & \text{otherwise.} \end{cases}$$

Then we have $M_r f \leqq M_r g + kt$, and $E_t[M_r f] \subset E_{(1-k)t}[M_r g]$. Applying (2.1.i), we obtain

(1) $$\lambda(E_t[M_r f]) \leqq \frac{1}{(1-k)t} \int_{E_{(1-k)t}[M_r f]} g d\lambda.$$

Since $g = 0$ on $E_{kt}[f]'$, the integral in (1) is less than $\int_{E_{kt}[f]} f d\lambda$.

It is clear how to obtain (i) for $j = l$, and also how to use (2.1.ii) to prove (ii). ∎

Our proof of (2.1.i) is a variant of one of F. Riesz [6]. Lemma (2.2) appears in [4]; the method of proof is due to Wiener [8].

3. $\mathfrak{L}_p$ maximal theorems. For a Lebesgue measurable function g (defined on R) and a positive real number p, we let $\|g\|_p = [\int_R |g|^p d\lambda]^{1/p}$. The function space $\mathfrak{L}_p$ is all g's such that $\|g\|_p < \infty$. The following lemma, an easy application of Fubini's theorem, will be useful. We use the symbol ξ for characteristic functions.

(3.1) LEMMA. *Let χ be a function on $[0, \infty[$ that is absolutely continuous on every finite interval, and satisfies $\chi(0) = 0$. For every nonnegative Lebesgue measurable function g on R and every Lebesgue measurable set E, the equality*

$$\int_E (\chi \circ g) d\lambda = \int_0^\infty \chi'(t) \lambda(E \cap E_t[g]) dt$$

holds.

Proof. Write

$$\begin{aligned} \int_E \chi \circ g \, d\lambda &= \int_R \xi_E(x) \int_0^{g(x)} \chi'(t) dt dx \\ &= \int_0^\infty \int_R \xi_{E \cap E_t[g]}(x) \chi'(t) dx dt = \int_0^\infty \chi'(t) \lambda(E \cap E_t[g]) dt; \end{aligned}$$

the use of Fubini's theorem is justified since the set $\{(x, t): x \in E \text{ and } g(x) > t\}$ is product measurable. ∎

(3.2) THEOREM. *For $p > 1$ and $f \in \mathfrak{L}_p$, the inequalities*

(i)
$$\|M_j f\|_p \leqq \frac{p}{p-1} \|f\|_p \qquad (j = r, l)$$

and

(ii)
$$\|Mf\|_p \leqq \frac{2^{1/p} p}{p-1} \|f\|_p$$

hold.

Proof. The following calculation (using (3.1), (2.2.i), and Fubini's theorem) shows that $M_j f \in \mathfrak{L}_p$:

$$\int_R (M_j f)^p d\lambda = p \int_0^\infty \lambda(E_t[M_j f]) t^{p-1} dt \leqq \frac{p}{1-k} \int_0^\infty t^{p-2} \int_{E_{kt}[f]} f d\lambda dt$$
$$= \frac{p}{1-k} \int_R f(x) \int_0^\infty \xi_{E_{kt}[f]}(x) t^{p-2} dt dx$$
$$= \frac{p}{(p-1)(1-k)k^{p-1}} \int_R f(x)^p dx.$$

The constant k satisfies $0 < k < 1$. The application of Fubini's theorem is justified because the set $\{(x, t): f(x) > kt\}$ is product measurable. To obtain (i), we use (2.1.i), Fubini's theorem, and Hölder's inequality to calculate as follows:

$$\int_R (M_j f)^p d\lambda = p \int_0^\infty t^{p-2} \int_{E_t[M_j f]} f(x) dx dt = p \int_R \int_0^\infty \xi_{E_t[M_j f]}(x) f(x) t^{p-2} dt dx$$
$$= p \int_R f(x) \frac{(M_j f(x))^{p-1}}{p-1} dx$$
$$\leqq \frac{p}{p-1} \left[\int_R f^p d\lambda\right]^{1/p} \left[\int_R (M_j f)^{p'(p-1)} d\lambda\right]^{1/p'}.$$

Since $p'(p-1) = p$ and $1 - 1/p' = 1/p$, the inequality (i) follows. The use of Fubini's theorem is justified because the set $\{(x, t): M_j f(x) > t\}$ is product measurable.

To prove (ii), let $A = \{x: M_l f(x) \geqq M_r f(x)\}$. Then we have, by (1.2),

$$\int_R (Mf)^p d\lambda = \int_A (M_l f)^p d\lambda + \int_{A'} (M_r f)^p d\lambda \leqq 2\left(\frac{p}{p-1}\right)^p \int_R f^p d\lambda;$$

the inequality (ii) follows. ∎

(3.3) REMARKS. (a) The calculation giving (3.2.i) in the above proof is a modification of one given by Flett [4].

(b) The inequalities of Theorem (3.2) are false if $p=1$. In fact, if $f\in\mathfrak{L}_1(R)$ and $0<b<x$, then the inequality $M_l f(x)\geqq x^{-1}\int_0^b fd\lambda$ holds. Thus, except for trivial f's, $M_l f$ is not in $\mathfrak{L}_1(R)$. The fact that we allow arbitrary intervals in our definition of the maximal functions is not, however, the only difficulty in the $\mathfrak{L}_1$ case. If $f(x)=\xi_{[0,1/2]}(x)/[x(\log x)^2]$, then $f\in\mathfrak{L}_1(R)$; but $(M_l f)\xi_{[0,1/2]}\notin\mathfrak{L}_1(R)$. In fact, we have $M_l f(x)\geqq(x|\log x|)^{-1}$ if $x\in]0,\frac{1}{2}[$. (This example appears in [9], p. 33, but with the equality $M_l f(x)=(x|\log x|)^{-1}$. Strict inequality holds, however, for $x\in]1/e, 1/2[$.) In the case $p=1$, there are two replacement results for Theorem (3.2). We state and prove them below.

(3.4) THEOREM. *Let E be a Lebesgue measurable set, and suppose $k\in]0, 1[$. The inequalities*

$$\text{(i)}\qquad \int_E (M_j f)d\lambda \leqq \frac{1}{k}\lambda(E)+\frac{1}{1-k}\int_R f(x)\log^+ f(x)dx \qquad (j=r, l)$$

and

$$\text{(ii)}\qquad \int_E (Mf)d\lambda \leqq \frac{1}{k}\lambda(E)+\frac{2}{1-k}\int_R f(x)\log^+ f(x)dx$$

hold. For $0<p<1$, the inequalities

$$\text{(iii)}\qquad \int_E (M_j f)^p d\lambda \leqq \frac{\lambda(E)^{1-p}}{1-p}\left(\int_R fd\lambda\right)^p \qquad (j=r, l)$$

and

$$\text{(iv)}\qquad \int_E (Mf)^p d\lambda \leqq \frac{2^p\lambda(E)^{1-p}}{1-p}\left(\int_R fd\lambda\right)^p$$

hold.

Our proof of (i) and (ii) (below) is essentially as in [4]. Our proof of (iii) and (iv), based also on (2.2), uses a technique appearing in [3] and [2] (but not in [4]). The definition of $\log^+$ is $\log^+ t=\operatorname{Max}\{\log t, 0\}$. If g is Lebesgue measurable and $\lambda(E)=0$, then $\int_E gd\lambda=0$, even if $\lambda(\{x: g(x)=\pm\infty\})>0$.

Proof. For $j=r, l$, we have

$$\int_E (M_j f)d\lambda=\int_0^\infty \lambda(E_t[M_j f]\cap E)dt=\int_0^{1/k}+\int_{1/k}^\infty$$
$$\leqq \frac{1}{k}\lambda(E)+\frac{1}{1-k}\int_{1/k}^\infty \frac{1}{t}\int_{E_{kt}[f]} f(x)dxdt$$

$$= \frac{\lambda(E)}{k} + \frac{1}{1-k}\int_R f(x)\left\{\int_{1/k}^{\infty} \xi_{E_{kt}[f]}(x)\,\frac{1}{t}\,dt\right\}dx$$

$$= \frac{\lambda(E)}{k} + \frac{1}{1-k}\int_R f(x)\log^+ f(x)\,dx;$$

and so (i) is established. Clearly (ii) can be proved similarly, using (2.2.ii).

To prove (iii), suppose that $\lambda(E)>0$ and $\int_R f d\lambda<\infty$. Let α be any positive real number. We write

(1)
$$\int_E (M_{\jmath}f)^p d\lambda = p\int_0^{\infty} t^{p-1}\lambda(E_t[M_{\jmath}f]\cap E)\,dt = p\left(\int_0^{\alpha/k} + \int_{\alpha/k}^{\infty}\right)$$
$$\leqq \frac{\alpha^p}{k^p}\lambda(E) + \frac{p}{1-k}\int_R f(x)\left\{\int_{\alpha/k}^{\infty} t^{p-2}\xi_{E_{kt}[f]}(x)\,dt\right\}dx.$$

The inner integral above is $[f(x)^{p-1}-\alpha^{p-1}]/k^{p-1}(p-1)$ if $f(x)>\alpha$ and is 0 otherwise; it therefore does not exceed $\alpha^{p-1}/k^{p-1}(1-p)$. Thus we get the estimate

(2)
$$\int_E (M_{\jmath}f)^p d\lambda \leqq \frac{\lambda(E)}{k^p}\alpha^p + \frac{p\alpha^{p-1}}{(1-p)k^{p-1}(1-k)}\int_R f d\lambda.$$

Taking $\alpha=(k/(1-k))(\lambda(E))^{-1}\int_R f d\lambda$ (this value minimizes the right side of (2)), we obtain

$$\int_E (M_{\jmath}f)^p d\lambda \leqq \frac{1}{(1-k)^p}\,\frac{\lambda(E)^{1-p}}{1-p}\left[\int_R f d\lambda\right]^p.$$

Letting $k\to 0$, we obtain (iii). Obvious modifications in the proof yield (iv). ∎

(3.5) Remark. All of the inequalities in (3.4) are, of course, interesting only if $\lambda(E)<\infty$. Consider the function $\xi=\xi_{[-1,1]}$. We have $M\xi(x)=2/(1+|x|)$ if $|x|\geqq 1$ (see 3.6). Thus ξ and $\xi\log^+\xi$ are in $\mathfrak{L}_1(R)$, but $M\xi$ is not in $\mathfrak{L}_1(R)\cup\mathfrak{L}_p(R)$, $p<1$. This example shows that M does not carry $\mathfrak{L}_1\log^+\mathfrak{L}_1$ into $\mathfrak{L}_1$ or $\mathfrak{L}_1$ into $\mathfrak{L}_p$, $p<1$; and so we cannot expect replacement results for (3.4) when $\lambda(E)=\infty$.

(3.6) Best Constants. In various places in the preceding sections, the constants obtained can be shown to be the best possible. First, let $\xi_\delta=\xi_{[-\delta,\delta]}$, $0<\delta<1$. An elementary calculation shows that

$$M\xi_\delta(x) = \begin{cases} 1 & \text{if } |x|\leqq\delta \\ 2\delta/(\delta+|x|) & \text{if } |x|>\delta. \end{cases}$$

We have $\lambda(E_t[M\xi_\delta])=2$ and $t^{-1}\int_{-1}^{1}\xi_\delta d\lambda=\delta+1$, if $t=2\delta(\delta+1)^{-1}$; therefore, 2 in (2.1.ii) is best possible.

Let $p>1$ and $0<\alpha<1$, and put $f_\alpha(x)=x^{-\alpha/p}\xi_{[0,1]}(x)$. The equality $\lim_{\alpha\to 1}\|M_{\imath}f_\alpha\|_p\|f_\alpha\|_p^{-1}=p/(p-1)$ shows that the constant in (3.2.i) is the best possible. The example is due to Flett, [4]. Let $0<p<1$, $0<\alpha<1$, and $f_\alpha(x)=x^{-\alpha}\xi_{[0,1]}(x)$. The inequality

$$\left[\int_0^1 (M_l f_\alpha)^p d\lambda\right]\left[\int_R f_\alpha d\lambda\right]^{-p} \geqq (1-\alpha p)^{-1}$$

shows that the constant in (3.4.iii) is the best possible.

Finally, the constant $(1-k)^{-1}$ in (3.4.i) is the best possible in the following sense. If B_1 and B_2 are constants independent of f and E such that

$$\int_E (M_l f)d\lambda \leqq B_2\lambda(E) + B_1\int_R f(x)\log^+ f(x)dx,$$

then we have $B_1 \geqq 1$. The following example is due to Flett [4]. Let $E=[0, b]$ and let $f(x)=x^{-1}[\log 1/x]^{-(2+\epsilon)}\xi_E(x)$. Suppose that b is small enough so that f is decreasing on $]0, b]$ and so that $0<x\leqq b$ implies that $\log(\log 1/x)\geqq 0$. An elementary calculation shows that

$$M_l f(x) = \frac{1}{1+\epsilon}\,\frac{1}{x}\left[\log\frac{1}{x}\right]^{-(1+\epsilon)}, \qquad 0<x\leqq b.$$

Hence for any constant B_2 we have

$$\begin{aligned}\left[\int_E M_l f\,d\lambda - B_2\lambda(E)\right]\left[\int_R f\log^+ f\right]^{-1} &\geqq (1+\epsilon)^{-1}\left[\int_0^b \left(\frac{1}{x}\log\frac{1}{x}\right)^{-(1+\epsilon)} dx - B_2 b\right]\\ &\quad\times\left[\int_0^b \frac{1}{x}\left(\log\frac{1}{x}\right)^{-(1+\epsilon)} dx\right]^{-1}.\end{aligned}$$

Letting $b\to 0$, we obtain $(1+\epsilon)^{-1}$ on the right.

4. The Hardy-Littlewood maximal theorem. Versions of the inequalities (3.2.i), (3.2.ii), and (3.4.i) appear in Hardy and Littlewood's original paper [5]. Sections 2 and 3 combine some of the refinements made in subsequent years, and some related results. As is apparent from the previous references, important contributions have been made by several authors: F. Riesz, Wiener, Flett, and Edwards and Hewitt, to mention those who bear the most influence on our treatment. Hardy and Littlewood based their proofs on a more general maximal theorem. This theorem, which we now prove by the methods of Section 3, is usually called the Hardy-Littlewood maximal theorem.

Notation is as in Sections 1 and 2. We also let f^* denote a decreasing function on $]0, \infty[$ that is equimeasurable with f; that is, $\lambda(E_t[f^*])=\lambda(E_t[f])$ for all $t>0$. (Such functions exist; e.g., f^* can be taken as the inverse of the decreasing function $y\to\lambda(E_y[f])$, with suitable modifications when this function is not strictly decreasing.) The equality

$$\int_R f d\lambda = \int_0^\infty f^* d\lambda = \int_0^\infty \lambda(E_t[f])dt \tag{4.1}$$

is a result of (3.1).

(4.2) LEMMA. *The inequality*

$$\int_B f d\lambda \leqq \int_0^{\lambda(B)} f^* d\lambda$$

holds for every Lebesgue measurable set B.

Proof. Clearly we have $\lambda(E_0[f\xi_B]) \leqq \lambda(B)$. Thus $(f\xi_B)^* = 0$ on $]\lambda(B), \infty[$, and we have

$$\int_B f d\lambda = \int_0^{\lambda(B)} (f\xi_B)^* d\lambda \leqq \int_0^{\lambda(B)} f^* d\lambda.$$

We will prove the inequality $(f\xi_B)^* \leqq f^*$(a.e.) used here. In fact, if $g \leqq f$ and $g^*(t_0) > f^*(t_0)$ for some point t_0 of continuity of g^*, then $g^* > f^*$ on some interval $[t_0, t_1]$. Thus we have

$$\lambda(E_{f^*(t_0)}[f^*]) \leqq t_0 < t_1 \leqq \lambda(E_{f^*(t_0)}[g^*]),$$

a contradiction since the extremes are equal to $\lambda(E_{f^*(t_0)}[f])$ and $\lambda(E_{f^*(t_0)}[g])$. Since g^* is continuous a.e., we have $g^* \leqq f^*$ a.e. ∎

The above lemma is well known; it appears, e.g., in [**9**], p. 31.

We extend f^* by letting $f^*(t) = 0$ if $t \leqq 0$.

(4.3) HARDY-LITTLEWOOD MAXIMAL THEOREM. *Let χ be an increasing nonnegative function on $[0, \infty[$, and suppose $f \in \mathfrak{L}_1(R)$. The following inequalities hold:*

(i) $$\int_R \chi \circ (M_j f) d\lambda \leqq \int_R \chi \circ (M_l f^*) d\lambda \qquad (j = r, l);$$

(ii) $$\int_R \chi \circ (Mf) d\lambda \leqq 2 \int_R \chi \circ (M_l f^*) d\lambda.$$

Proof. It is easy to see that $M_l f^*(x) = (1/x)\int_0^x f^* d\lambda$, since f^* is decreasing. It follows from this equality that $M_l f^*$ is decreasing. By (2.1.i) and (4.2), we have

(1) $$\lambda(E_t[M_j f]) \leqq \frac{1}{t} \int_0^{\lambda(E_t[M_j f])} f^* d\lambda \qquad (t > 0);$$

and this inequality implies that $M_l f^*(\lambda(E_t[M_j f])) \geqq t$, if $0 < \lambda(E_t[M_j f]) < \infty$. Since $M_l f^*$ is decreasing, we infer that

(2) $$\lambda(E_t[M_j f]) \leqq \lambda(E_t[M_l f^*]),$$

if $\lambda(E_t[M_j f]) < \infty$. The equality $\lambda(E_t[M_j f]) = \infty$ would imply, by (1), that $f^*, f \notin \mathfrak{L}_1(R)$; hence, (2) holds for all $t > 0$. Let $F_t[g] = \{x : g(x) \geqq t\}$ $(t > 0,\ g \geqq 0)$. An application of Fatou's lemma, using (2), shows that

(3) $$\lambda(F_t[M_j f]) \leqq \lambda(F_t[M_l f^*]).$$

Next, let $S=\{x: 0 \leqq x_1 < x < x_2 \Rightarrow \chi(x_1) < \chi(x) < \chi(x_2)\}$. Then χ is strictly increasing on S. On S, we define α as the inverse of $\chi: \chi(\alpha(t))=t$ if $t \in \chi(S)$. If $t \in \chi([0, \infty[)'$, let $\beta = \sup[\{x: \chi(x) < t\} \cup \{0\}]$ and $\gamma = \inf[\{x: \chi(x) > t\} \cup \{\infty\}]$. Then we have $]\beta, \gamma[\subset \chi([0, \infty[)'$, $t \in [\beta, \gamma]$, and β or $\gamma \in \chi([0, \infty[)$. If $\gamma \in \chi([0, \infty[)$, let $\alpha(t) = \inf\{x: \chi(x) = \gamma\}$; if $\beta \in \chi([0, \infty[)$ and $\gamma \notin \chi([0, \infty[)$, let $\alpha(t) = \sup\{x: \chi(x) = \beta\}$. Then $\chi(\alpha(t))$ is either β or γ, and is unequal to t. The function α is thus defined a.e. on $[0, \infty[$ (it is not defined on $\chi(S')$). Let $A = \{t: \chi(\alpha(t)) \leqq t\}$ and $B = \{t: \chi(\alpha(t)) > t\}$. Then the equalities

$$\begin{aligned} E_t[\chi \circ g] &= E_{\alpha(t)}[g], \qquad t \in A \\ E_t[\chi \circ g] &= F_{\alpha(t)}[g], \qquad t \in B \end{aligned} \tag{4}$$

hold for any nonnegative function g. Using (2), (3), (4), and our previous results, we calculate as follows:

$$\begin{aligned} \int_R \chi \circ (M_j f) d\lambda &= \int_0^\infty \lambda(E_t[\chi \circ M_j f]) dt \\ &= \int_A \lambda(E_{\alpha(t)}[M_j f]) dt + \int_B \lambda(F_{\alpha(t)}[M_j f]) dt \\ &\leqq \int_0^\infty \lambda(E_t[\chi \circ M_l f^*]) dt = \int_R \chi \circ (M_l f^*) d\lambda. \end{aligned}$$

To prove (ii), we write

$$\begin{aligned} \int_R \chi \circ (Mf) d\lambda &= \int_R \mathrm{Max}\{\chi \circ (M_r f), \chi \circ (M_l f)\} d\lambda \\ &\leqq \int_R \{\chi \circ (M_r f) + \chi \circ (M_l f)\} d\lambda \leqq 2 \int_R \chi \circ (M_l f^*) d\lambda. \quad \blacksquare \end{aligned}$$

The constant 2 in (4.3.ii) is the best possible; for, an equimeasurable function for ξ_1 (see Section 3.6) is $\xi_1^* = \xi_{[0,2]}$, and we have

$$\lim_{\substack{p \to 1 \\ p > 1}} \left[\int_R (M\xi_1)^p d\lambda \right] \left[\int_R (M_l \xi_1^*)^p d\lambda \right]^{-1} = 2.$$

5. Generalizations for R^m. The maximal function Mf defined in Section 1 can be written as

$$Mf(x) = \sup \left\{ \lambda(I)^{-1} \int_{x+I} f d\lambda : I \text{ is a finite interval containing } 0 \right\} \cdot$$

We see that the mapping M depends on the measure λ and the collection of finite intervals containing 0; averages are taken over intervals containing x. Given a measure space $(X, \mathcal{A}, \mu)$, what is needed to define formally an operator

M on the positive measurable functions is a family $\mathcal{B}$ of subsets of X over which to take averages. Maximal functions have been successfully defined and heavily used by several mathematicians by obtaining suitable families $\mathcal{B}$ on certain measure spaces. Examples are: Calderon [1], to ergodic theorems on locally compact groups; Calderon and Zygmund [2], to singular integrals in R^m; Edwards and Hewitt [3], to pointwise convergence of convolutions on locally compact groups; Smith [7], to harmonic functions in R^m. For the presentation of the maximal theorems in sections 2, 3, and 4, the crucial result is really Lemma (2.1). The other results are obtained from it with no significant use of specific properties of the underlying space R. In this section we prove an analogue of (2.1) for Euclidean m-space R^m, using open spheres in place of intervals. The precise arguments of the proof of (2.1) are not possible if $m \geqq 2$, for the open sets in R^m are not unique disjoint countable unions of spheres. Instead, we need the following covering theorem. All writers dealing with generalized maximal theorems use covering arguments of roughly the same type. The one given below is patterned after Theorem (2.2) of [3]. For $r>0$, let $S_r=\{x\in R^m: \|x\|<r\}$; and for $n\in Z$ (the integers), let $B_n=S_{2^{-n}}$. Notice that $B_n+B_n=B_{n-1}$. Let $\mathcal{B}=\{x+B_n: x\in R^m, n\in Z\}$.

(5.1) A Covering Theorem. *Let* $\mathcal{B}^\dagger\subset\mathcal{B}$ *and suppose* $E\subset R^m$ *satisfies*

(i) $\lambda(E+B_n)<\infty$ *for all* $n\in Z$;

(ii) *for each* $x\in E$, *there is some* n *such that* $x+B_n\in\mathcal{B}^\dagger$;

(iii) $\{n: x+B_n\in\mathcal{B}^\dagger$ *for some* $x\in E\}$ *is bounded below. Then there are (possibly finite) sequences* $(x_k)_{k=1}^\infty$ *and* $(n_k)_{k=1}^\infty$ *such that* $x_k\in E$, $(x_k+B_{n_k})_{k=1}^\infty$ *is a pairwise disjoint sequence in* $B^\dagger$, *and* $\lambda(E)\leqq 2\sum_{k=1}^\infty\lambda(B_{n_k})$.

Proof. Let $n_1=\mathrm{Min}\{n: x+B_n\in\mathcal{B}^\dagger$ for some $x\in E\}$, and choose $x_1\in E$ such that $x_1+B_{n_1}\in\mathcal{B}^\dagger$. Sequences (x_k) and (n_k) satisfying

(1) $\{x_k+B_{n_k}\}_{k=1}^p$ is a pairwise disjoint sequence in $\mathcal{B}^\dagger$ for all p;

(2) $n_k=\mathrm{Min}\{n: x+B_n\in\mathcal{B}^\dagger$ and $(x+B_n)\subset[\bigcup_{i=1}^{k-1}(x_i+B_{n_i})]'$, some $x\in E\}$;

(3) $x_{p+1}\in[\bigcup_{i=1}^p(x_k+B_{n_k-1})]'$ for all p

will be defined by induction. Thus suppose that $(x_k)_{k=1}^p$ and $(n_k)_{k=1}^p$ satisfy (1), (2), (3). If $E\subset\bigcup_{k=1}^p[x_k+B_{n_k-1}]$, we obtain a finite sequence satisfying (1) and (2) and $E\subset\bigcup_{k=1}^p(x_k+B_{n_k-1})$. Otherwise, let $x\in E\cap[\bigcup_{k=1}^p(x_k+B_{n_k-1})]'$ and let j be the smallest integer such that $x+B_j\in\mathcal{B}^\dagger$. We will show that $x+B_j\subset[\bigcup_{k=1}^p(x_k+B_{n_k})]'$. If $(x+B_j)\cap(x_1+B_{n_1})\neq\varnothing$, we have $x\in x_1+(B_{n_1}+B_j)\subset x_1+B_{n_1-1}$; a contradiction. If there are k's such that $(x+B_j)\cap(x_k+B_{n_k})\neq\varnothing$, let q be the least among them $(1<q\leqq p)$. Then the inclusion $x+B_j\subset[\bigcup_{i=1}^{q-1}(x_i+B_{n_i})]'$ holds. Therefore, the inequality $j\geqq q$ holds, and $x\in x_q+B_{n_q-1}$; a contradiction. We can thus define $n_{p+1}=\mathrm{Min}\{j: x+B_j\in\mathcal{B}^\dagger$ and $(x+B_j)\subset[\bigcup_{i=1}^p(x_i+B_{n_i})]'$, some $x\in E\}$, and select x_{p+1} such that $x_{p+1}+B_{n_{p+1}}\in\mathcal{B}^\dagger$.

We claim that $E\subset\bigcup_{k=1}^\infty(x_k+B_{n_k-1})$ (this is already established if the sequences are finite). We have $\lim_{k\to\infty}n_k=\infty$; proof:

$$\sum_{k=1}^{\infty} \lambda(B_{n_k}) = \sum_{k=1}^{\infty} \lambda(x_k + B_{n_k}) < \lambda(E + B_{n_1}) < \infty.$$

Hence, for given $x \in E$ and $p \in Z$, we must have $(x+B_p) \cap (x+B_{n_k}) \neq \emptyset$ for some k; if not, x and p would have been selected for x_j and n_j as soon as n_j exceeded p. If q is the least integer such that $(x+B_p) \cap (x_q+B_{n_q}) \neq \emptyset$, then we argue as before to obtain $x \in x_q + B_{n_q - 1}$.

Finally, we have

$$\lambda(E) \leqq \sum_{k=1}^{\infty} \lambda(B_{n_k - 1}) \leqq 2 \sum_{k=1}^{\infty} \lambda(B_{n_k}). \quad \blacksquare$$

Let λ denote m-dimensional Lebesgue measure.

(5.2) THEOREM. *Let $p \geqq 1$, and suppose $f \in \mathfrak{L}_p(R^m)$. Define*

$$Mf(x) = \sup_{r>0} \lambda(S_r)^{-1} \int_{x+S_r} f d\lambda.$$

There is a constant α such that $\lambda(E_t[Mf]) \leqq (2/t) \int_{E_{\alpha t}[Mf]} f d\lambda$ for all $t > 0$.

Proof. With notation as above, let

$$M'f(x) = \sup_{n \in Z} \lambda(B_n)^{-1} \int_{x+B_n} f d\lambda.$$

Small changes in x yield small changes in $\int_{x+S_r} f d\lambda$ for fixed r; it follows easily from this fact that Mf and $M'f$ are Borel measurable, in fact lower semi-continuous. Define $\mathfrak{B}^{\dagger} = \{x+B_n : \lambda(B_n)^{-1} \int_{x+B_n} f d\lambda > t\}$. Then $E_t[M'f]$ satisfies (5.1.ii) for this $\mathfrak{B}^{\dagger}$. Condition (5.1.iii) is also satisfied, for there cannot be sequences $(x_i)_{i=1}^{\infty}$ and $(n_i)_{i=1}^{\infty}$ such that $\lim_{i\to\infty} n_i = -\infty$ and $\int_{x+B_{n_i}} f > t\lambda(B_{n_i})$. This last assertion is obvious if $p=1$ and is an easy application of Hölder's inequality if $p>1$. Condition (5.1.i) is satisfied for each set $E_j = B_j \cap E_t[M'f]$, $j \in Z$; and clearly (ii) and (iii) are also satisfied for the E_j's. For fixed j, then, let $(x_k + B_{n_k})_{k=1}^{\infty}$ be the sequence guaranteed by (5.1). The inequality

$$\lambda(E_j) \leqq 2t^{-1} \int_U f d\lambda \left(U = \bigcup_{k=1}^{\infty} (x_k + B_{n_k}) \right)$$

thus holds. If $x \in U$ and $x \in x_k + B_{n_k}$, then we have

$$\lambda(B_{n_k - 1})^{-1} \int_{x+B_{n_k - 1}} f d\lambda = 2^{-m} \lambda(B_{n_k})^{-1} \int_{x+B_{n_k - 1}} f d\lambda > 2^{-m} t;$$

therefore, $U \subset E_{2^{-m}t}[M'f]$. We conclude, letting $j \to -\infty$, that

$$\lambda(E_t[M'f] \leqq \frac{2}{t} \int_{E_{\beta t}[M'f]} f d\lambda \quad (\beta = 2^{-m}).$$

For a given $r>0$, let n be the unique integer such that $B_n \subseteq S_r \subset B_{n-1}$. Then $\lambda(S_r)^{-1}\int_{x+S_r} f d\lambda \leqq 2^m \lambda(B_{n-1})^{-1}\int_{x+B_{n-1}} f d\lambda$ for any $x \in R^m$. Hence we have $Mf \leqq 2^m M'f$ and $E_t[Mf] \subset E_{t2^{-m}}[M'f]$. The inequality

$$\lambda(E_t[Mf]) \leqq \frac{2}{t}\int_{E_{\alpha t}[Mf]} f d\lambda, \quad \alpha = 2^{-2m}$$

follows. ∎

The proofs of the R^m analogues of (2.2), (3.2), (3.4), and (4.3) are almost identical to those given in Sections 2, 3, 4 for $m=1$; (5.2) is used in place of (2.1). Of course, the constants change. Lemmas (3.1) and (4.1) offer no difficulty in R^m.

The family $\{x+S_r : x \in R^m, r>0\}$ over which the averages are taken in (5.2) can be replaced by other families. For example, m-dimensional cubes symmetric about the origin would do. In fact, m-dimensional rectangles symmetric about O and satisfying the condition that the ratio of the maximal side length to the minimal side length does not exceed a fixed constant could replace the S_r's. Sufficient conditions (on the sets over which averages are taken) to yield maximal theorems are given in each of [2], [3], and [7].

Hardy and Littlewood devoted the main part of their proofs to the discrete analogues of the theorems in Sections 3 and 4. It is instructive to formulate these analogues (for Z, e.g.). The facts in these cases are corollaries of the results in Sections 2, 3, and 4. Similarly, there are results for sums of multiple sequences (maximal functions for Z^m). These follow from the R^m theorems.

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