ELENA NARDI

MATHEMATICS UNDERGRADUATES' RESPONSES TO SEMANTIC ABBREVIATIONS, 'GEOMETRIC' IMAGES AND MULTI-LEVEL ABSTRACTIONS IN GROUP THEORY

ABSTRACT. The study, parts of which are reported in this paper, aimed at the identification and exploration of the difficulties in the novice mathematician's encounter with mathematical abstraction. For this purpose twenty first-year mathematics undergraduates were observed in their weekly tutorials in four Oxford Colleges during the first two terms of Year 1. Tutorials were tape-recorded and fieldnotes kept during observation. The students were also interviewed at the end of each term of observation. During data analysis, sets of Episodes were extracted from the transcripts of the tutorials and the interviews within five topics in pure mathematics. This topical analysis was followed by a cross-topical synthesis of themes that were found to characterise the novices' cognition. Here, in three characteristic Episodes, I discuss issues related to the learning of one mathematical topic, Group Theory, and in particular: 1. The static and operational duality within the concept of order of an element as well as the semantic abbreviation contained in $|g|$ 2. The often problematic use of 'times' and 'powers of' in association with the group operation. 3. The ambivalent use of geometric images as part of meaning bestowing processes with regard to the notion of coset. 4. The problematic conceptualisation of the multi-level abstractions embedded in the concept of isomorphism.

1. On the genesis of group-theoretical concepts

Research interest in learning difficulties with respect to Abstract Algebra, and in particular Group Theory, is relatively recent. First insights into the cognition of Abstract Algebra are now available. In this paper I intend to extend or refine some of these insights.

The concept of group is an example of a new mental object the construction of which causes fundamental difficulties in the transition from school to university mathematics. As Robert and Schwarzenberger (1991) point out, one root of this difficulty is historical and epistemological: "the problems from which these concepts arose in an essential manner are not accessible to students who are beginning to study (and expected to understand) the concepts today". This historically decontextualised presentation deprives the learners of a potentially meaningful construction of these concepts. As Burn (1996) notes "historically, the fundamental concepts of Group Theory were those of permutation and symmetry".

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For example, the concept of quotient group (Nicholson, 1993) appears implicitly in the works of many 19th century mathematicians, including Galois and Jordan, but only as a context-embedded tool, namely not formally defined. Despite Dedekind's early understanding of the potential power of formalising the concept since the 1850s, it is not until the end of the century that Hölder systematically and explicitly defines the quotient group. It is noteworthy that Dedekind's lectures were lukewarmly received (members of the mathematically strong audience admitted they understood little); that is, it seems that from a phylogenetic (Pinxten, 1987) point of view the concept had not completed its cycle of fermentation. As a result, professional mathematicians of the time resisted its significance and its acceptance through formalisation.

Given these historical origins and recent debates on the relevance of these origins in investigations on the learning of Group Theory (Burn's (1996) critique on Dubinsky et al. (1994) and their response (1997)), it is necessary to stress that the research on students' difficulties reported here espouses a perspective on Group Theory where an understanding of set-theoretic logic (see, for example, Selden and Selden's (1987) list of 'errors and misconceptions' regarding theorem proving in Abstract Algebra), of the concept of set (see, for example, Dubinsky et al.'s (1994) account of the psychogenesis of the notion of a group from a 'primitive' notion of set to one that incorporates the notion of a binary operation) and of an abstract definition of function (see, for example, Harel and Kaput (1991)) are psychological prerequisites to an understanding of fundamental concepts of Group Theory.

The data discussed here originate in a doctoral study on the learning of mathematics at university level, supported by the Economic and Social Research Council in the UK ((Nardi, 1996) – see Abstract and Note 1 for a brief description). The study set out to explore the tensions of the novice mathematician's encounter with mathematical abstraction, an activity that ranks among the least accessible mental activities and, at least in the UK, a crucial step of the transition from informal school mathematics to the formalism of university mathematics. A combination of techniques from Data Grounded Theory (e.g. Glaser and Strauss, 1967) and Discourse Analysis (Dijk, 1985) was used for the extraction of Episodes (paradigmatic cases, representatives of a class with regard to an analytical theme) on the students' explicit or implicit articulations of their difficulties within the mathematical areas of Foundational Analysis, Calculus, Topology, Linear Algebra and Group Theory. The analytical thinking of the study lies primarily in the Piagetian concept of *Reflective Abstraction* (Dubinsky and Lewin, 1986) and in dominant theories in the field of the Psychology of Advanced Mathematical Thinking (Tall, 1991) such as *Concept Image and Concept Definition* (e.g. Tall and Vinner, 1981) and theories related to *the encapsulation of mathematical processes into conceptual entities* (e.g. Sfard, 1991; Gray and Tall, 1994). In addition to this prevalently developmental perspective, a perspective on advanced mathematical learning as an enculturation process (e.g. Sierpinska, 1994) was employed where the new culture is Advanced Mathematics introduced to the novice by an expert mathematician, the tutor.

2. ON THE LEARNING OF THE CONCEPTS OF COSET, ORDER OF AN ELEMENT AND ISOMORPHISM

In the following I present a factual and interpretive account of three Episodes from the second term of data collection each one focusing on the concepts of order of an element, coset and the First Isomorphism Theorem for Groups. Subsequently I synthesise these findings and refer briefly to a few implications on the teaching of Group Theory.

Episode 1: Linguistic and conceptual difficulties with order of an element, generating $<g>$ and the group operation

Constructing new meaning often involves resorting to familiarity and, in the context of the students' understanding of the group operation, this becomes evident in their use of expressions that seem to 'borrow properties' from $\Re$ (Hazzan, 1994). Hazzan comments on the various degrees of legitimacy in using expressions such as 'product' for a*b; in this Episode, I offer an analogous commentary on 'times' and 'powers of'.

Background to the episode. In the beginning of the tutorial student Connie asks the tutor about the meaning of 'generator' in Group Theory. She also asks the tutor to explain his proof (previously discussed in a whole class session) for the following Question from that week's problem sheet:

> *If p is a prime and H and K are distinct subgroups of a group G, each of order p, show that $H \cap K = \{e\}$ (use Lagrange). If G is finite, deduce that the number of elements of order p in G is a multiple of $(p - 1)$.*

The tutor claims that, on the basis of his previous experiences with this student, she lacks clarity in her conception of 'order' either of a group or of an element. He then defines order of an element as follows:

> *n is the order of an element G in a group G if it is the least positive integer for which $g^n = e$, where e is the identity element of the group.*

If G is a finite group, he adds, then n exists because "all powers of g would have to start repeating themselves at some point"; also: $<g>\leq G$, the inverse of an element g^t in $<g>$ is g^{n-t} and there are exactly n elements in $<g>$.

The Episode. Ostensibly about the Question, the following are mostly about $<g>$, $|g|$ and the group operation.

C1: Is this ($<g>$) a cyclic group?
T1: Yes, a cyclic subgroup.
C2: Were we talking about them there *[in the Question]?*

The tutor explains that they haven't mentioned it yet in relation to the Question and that he is still "trying to clarify" the meaning of order of an element or of a group for her. Another thing "she needs to know" is, he says, Lagrange's Theorem (if $H<G$ then $|H| \mid |G|$) because the cosets of H in G are "distinct chunks of G of equal size". Also, he adds, $|g| \mid |G|$ and if $|H| = p$, were p is a prime number, then the $p-1$ elements of H, except e, have order p. He then turns to the Question and explains that if two subgroups of order p overlap only at the identity, they together contain 2($p-1$) elements of order p.

C3: Well, I thought that H had just p elements because when you've got these groups when you take the order of the group you count the . . . identity?

The tutor repeats that each of the elements in H, except e, is of order p.

C4: I don't understand how an element can have an order.

As the tutor notes Connie lacks a clear understanding of the notion of order of an element – but, as C3 suggests, has a clearer understanding of the notion of order of a group. He then chooses to re-introduce her to the notion, by repeating the definition given in the lectures, and explain that $|g| = |<g>|$. In C1 and C2 Connie seems to be slightly surprised with the connection between $|g|$ and $<g>$. Her confusion is not surprising: the definition given in the *Background* appears as standard in most textbooks and was also used in these lectures. $|g| = |<g>|$ is then a theorem which, however obvious to the tutor, apparently needs to be proved for Connie. This theorem allows, in a sense, the linguistic abbreviation of the term *order of the group generated by an element g* down to the term *order of an element g*. C4 illustrates how Connie is in trouble with understanding 'how an element can have an order' as *order*, so far, has been identified as a property of groups: the order of a group is the number of its elements. Connie's confusion can be accounted for as an effect of her not realising the tacit theorem mentioned above – and the resulting linguistic abbreviation.

Finding however the order of a cyclic group by counting the number of its elements obscures the fact that if $<g>$ is of order *p*, that means that $<g>$ has *p* elements *because* the powers of g start 'repeating themselves' after *p* steps. In this sense *order of an element* is a concept that contains both a static characteristic of $<g>$ (the number of its elements) AND information about the process of obtaining these elements (how many times it is necessary to take the powers of g in order to cover $<g>$). In Connie's words, most strikingly C4, this duality, resulting from the definition and the theorem mentioned above, is missing.

In response to C4 the tutor repeats the definition of the order of an element and adds that "the reason it's called order is because it's the order of the least subgroup which contains the element".

C5: Does this mean if you take this little g times $p - 1$ you would have the same element?
T2: That's right. If you take $g^2, \ldots, (g^2)^{p-1}$ you would take all these elements in some different order. Is that what you are saying?
C6: No, I was just saying if you take g^2 times g^{p-1} . . .

The tutor explains that $g^2 g^{p-1} = g$ and stresses that every other element in H will be of order p. So, he concludes, if G has r subgroups like H, only coinciding at e, and H contains $p-1$ elements of order p then G contains $r(p-1)$ elements of order p. He stresses that the notion of the order of an element is "subtle but confusing".

C7: So you don't do g times $p - 1$ but g to the $2p - 1$ . . .
T3: I'd do this if I wanted to find the order of g^2. I would square it and cube it and so on.
C8: Ah! And you would time these together?
T4: Yes, like this is g^2. . . this is g^6. . .
C9: Are these separate things?
T5: Yes, I am looking at g^2, g^2 all squared . . . and you don't hit the identity until you get p again.
C10: OK . . .

In C5 to C9, it turns out that behind Connie's unease with the concept of *order* lies the even more fundamental unease with the notion of generating a cyclic group from the powers of an element g in *G*. Even further, her troubled notion of generating can be largely attributed to her muddled expressions regarding the operation in a group.

In C5-C8 'times' and 'to the power of' are used vaguely interchangeably; it is also not clear at all whether Connie refers to the 'multiples' or 'powers' of g or g^2. This can be due to the tutor's effort to convey the idea that, in a subgroup of order *p*, where *p* is a prime number, every element, other than *e*, is of order *p* too. Therefore, whether we consider the powers of g or g^2, eventually the generated set will be H. In this sense the tutor and Connie are far from communicating fruitfully: Connie is still struggling with her exploration of the notion of generating a group from an element; she is still vague about how this process takes place (operational stage). The tutor on the other hand assumes the clarity of the process of generating a group from g and a group from g^2 and, arguably, proceeds with an attempt at a demonstration of how these two groups coincide.

It seems fair to say that the exchange between the two interlocutors here takes place past each other. Perhaps most striking is the exchange of words in C5 to C8 with regard to 'times' and 'to the [power of]': in C5 Connie explicitly uses 'times' and in T2 the tutor responds with taking powers. In C6 and C7 she insists on 'times' and the tutor shifts from 'multiplying' elements that are powers of g to manipulating the powers to which these elements are taken. While doing so he seems to assume the clarity of these operations. C9 is evidence of how Connie, far into the discussion, is still struggling with clarifying the objects on which the operations are applied. Since the conversation is completed with C10 – it seems that the tutor has been expecting a verbal signal of understanding from Connie so that he can move on to other topics – there is no evidence of whether Connie's perceptions of the group operation, the generating process and ultimately cyclic groups and their order have been clarified or enriched.

In sum, in the above, a student's problematic perception of $|g|$, $<g>$ and the group operation were gradually revealed. Interrelated linguistic (abbreviated use of the term) and conceptual (static and operational duality) interpretations of the student's difficulty with the notion of order of an element were given. Further, the operation of generating $<g>$ seemed to be problematically perceived by the student who uses expressions like 'times' and 'powers of' vaguely and at times interchangeably.

From a teaching point of view, the dialectics between tutor and student illustrate a communicational gap which leaves the question, whether the student's perceptions have been enriched, open. In particular, it could perhaps have been more helpful if the tutor had recognised the student's need for the proof of $|g| = |<g>|$ – say, by building up $|g|$ and $|<g>|$ separately, with examples (e.g. with calculations in (Z_5, +) or with an exploration of the four subgroups of $C_3 x\ C_3$ of order 3), and then illustrating the satisfactory result that these two numbers are always the same. In the same vein, it could perhaps have been more helpful if the tutor had elaborated further the student's use of 'times': this is a word used for multiplication and for repetition and in the lectures its use might have included an expression such as "how many times do we have to multiply g by itself to reach e?" that had then been transferred inaccurately by the student.

Episode 2: Bestowing meaning on the concept of coset through ambivalent uses of geometric images

As addressed in Harel and Kaput's work on Object-Valued Operators (1991), students often construct concept images of cosets via familiar geometric figures. The students' intensive need to resort to this familiarity originates largely, not to the 'fairly simple' relationships between certain mathematical objects (Leron and Dubinsky, 1995) that characterise theorems such as the Homomorphism Theorems or Lagrange's Theorem (their consistently problematic conceptualisation by students is reported in (Hazzan and Leron, 1996)) but to the abstract nature of the mathematical objects involved. For example, Lagrange's Theorem is about 'easy' things such as "one number dividing the other, two sets having the same cardinality or being disjoint, etc.". "The students' difficulty", they conclude, "may be largely due to their confusion about the nature of cosets", to not "seeing these cosets as objects to be measured, counted and compared". These observations on the students' conceptualising processes resonate with the evidence presented in this Episode.

Background to the episode. In the beginning of the tutorial student Camille declares her confusion with the notion of equivalence classes defined by "$a \sim b$ when $f(a) = f(b)$ where G is a group and $f: G \rightarrow G$" (Note: in more accurate terms Camille ought to have said 'if and only if' instead of 'when' and also specify what f is: a homomorphism from G to G or an injection as suggested in the Episode). The tutor draws

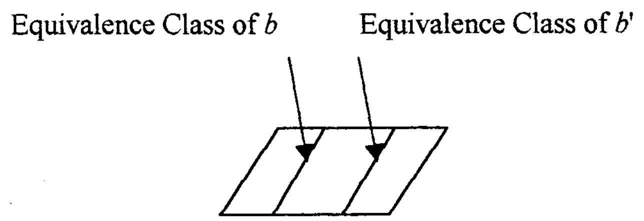

Figure 1.

to illustrate that "if b and b' are in different equivalence classes then $f(b)$ and $f(b')$ are different". In the subsequent discussion (C1-C5 and T1-T3, omitted here) Camille appears puzzled with Figure 1 ("Why are they all straight lines?", she enquires). Because these lines "do not mean anything", the tutor replaces Figure 1 with Figure 2, one with 'squigges' as the equivalence classes:

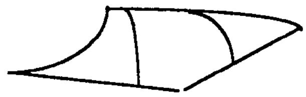

Figure 2.

Figure 1 is the tutor's image of equivalence classes where an element *a* of the domain is a dot and its equivalence class (defined as the set of elements in the domain that are mapped on the same value as *a*) is a line segment. The metaphorical elements of this image however seem to escape Camille who interprets Figure 1 literally. In the *Episode* below she offers analogous interpretations with regard to the notion of *coset*.

The Episode. Subsequently the discussion of the correspondence between the elements of a group and their equivalence classes evokes in Camille a query on another correspondence: 'the 1–1 correspondence between the conjugates of *x* and *x*'. Remarkably Camille demonstrates precise knowledge of the relevant definitions (centraliser, conjugate) as well as a relation between the two concepts (unlike most of the students who were incapable of reproducing definitions of even simpler group-theoretical constructs mentioned in the lectures). The tutor initiates a discussion of the 1–1 relation between cosets of the centraliser and conjugates of *x* but Camille is quiet and looks sceptical. Then she asks:

C6: What are cosets materially?

C7: If we have a group *G* and a subgroup *H* why do we bother to find the cosets?

C8: Cosets are a group multiplied by an element in the big group.

C9: Cosets are just a moving...

T4: What do you mean by that?

T5: Because of results like this. They turn up naturally.

T6: *[hesitantly]* Yes ... it's a set...

T7: That's right. That's one way... you can look at it as translates of a subgroup... sort of multiplying *g* with everything in *H* and it shifts it...

C10: *[after a pause]* So if we have a square of size one and then the group G is like this. . .

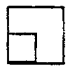

Figure 3.

T8: You have to be slightly careful. . . It is slightly. . . don't think about in. . . you're not thinking of applying it on squares, are you?

C11: So then it would have four cosets?

T9: Mmm. . . if the subgroup was a quarter of the size of the whole thing, yes. . . it would have four cosets. . . that's right.

C12: And the cosets are always the same size as the original.

T10: That's right. As we know they partition the group.

Camille, in her above mentioned demonstration of knowledge, has not used the term *coset* at all. The term occurs for the first time in the tutor's words and captures Camille's attention. Subsequently, and in the rest of the Episode, it seems that the notion of *coset* constitutes a large part of her preoccupation: C6-C12 seem to be persistent, multiple attempts to imbue it with some meaning. C6 comes through as a surprisingly philosophical and abstract question which raises a very fundamental existential issue with regard to the notion of coset: what is surprising about C6 is that it comes in the middle of the tutor's describing a quite sophisticated construction (establishing a correspondence between the cosets of the centraliser and the conjugates of an element x in a group) and shifts the conversation from the strictly and specifically mathematical (represented by the tutor) to the metamathematical. Camille has been attentively listening to the tutor's demonstration of the construction and has given the very strong impression that, throughout, she has been processing the dense information provided by the tutor. C6 however illustrates that this processing must have been motivated mostly by the desire to construct an image of coset – visual, 'material' – than consume the tutor's argument. From then on, as said earlier, C6-C12 is a series of successive attempts at interpreting the concept of coset.

C6 is a nearly platonistic enquiry on the nature of cosets as objects, as entities. Camille's entities in C6 do not necessarily act or interact. In C7 the questioning of the nature of these objects takes the form of an exploration of their *raison-d'-être*. C8 is a dissection of a coset that equates a coset with how it comes into existence. I note that so far T4-T6 do not seem to have a direct impact on Camille's generating of ideas of what a *coset* is. C9 is a geometric interpretation of C8 derived from the notion (and notation for) transformations, and in particular translations. The tutor carefully tunes in, using 'translates', a classical description (T7), but Camille accelerates her tentative condensation of her conception of coset in a geometric image in an unusual way. C10 (in parallel with her 'straight lines' in the *Background*) illustrate the thin line between a metaphorical and a literal interpretation of a picture. It is perhaps reasonable to assume here that Camille operates under the strong visual impact of the four-sided figures used by the tutor earlier in the tutorial (Figure 1 and Figure 2). The tutor is surprised and alarmed (T8) by Camille's intention to "apply [this idea] on squares". C11 is evidence that Camille is too preoccupied with her image construction to be influenced by T8 and she furthers the interpretation of her Figure 3 in a less controversial but highly ambivalent way. T9 is one more effort on the tutor's side to tune in and transform the student's images from within. Surprisingly then Camille turns in a shift to a more abstract property of cosets in which however the geometric/numerical jargon ('size' in C12) is maintained. The tutor (T10) has completely adopted Camille's metaphor and contributes another observation on cosets.

Finally Camille ceases the effort to interpret further the notion of *coset* once she acquires an image of *cosets* that is satisfying and clear to her. That Camille is content with what she has acquired can be assumed on the basis of the evidence, given during the study, that this student does not bring a conversation to an end until she acquires a satisfactory (to her) understanding. The issue that C6-C12 raise is whether the quality of the acquired perception of a coset – via these visual images – justifies Camille's eventual sense of content. Given that the tutor cautiously surrenders in adopting Camille's image but does not cross-check whether the intended (by the tutor) and the acquired (by Camille) image of a *coset* coincide, the questions raised by this issue ought to remain open.

In sum, in the above, a student, who exhibits a remarkable memory of the definitions of the concepts involved in the discussion, is engaging in a meaning bestowing process with regard to the notion of coset. The student asks the tutor about the raison-d'-être of the concept and her efforts are characterised by a tendency to use images of familiar regular geometric shapes in order to construct a mental image of new concepts (equivalence classes as parallel straight lines, cosets as squares). Evidence was given that these geometric images are interpreted literally by the student. This raises the issue of a potential cognitive danger built in their use – despite their undeniable, and widely acknowledged in the literature, pedagogical value.

From a teaching point of view, the tutor has demonstrated a certain degree of flexibility in thinking in the terms of the student's images (actually it is the tutor who sparks off the use of geometric images in this tutorial) but, in the end, there doesn't seem to exist any guarantee that the particular use of these images has resulted in the tutor's intended concept image of the notion of coset. The student's plea may have been for concrete illustrations ('materially'?) of the utility of the new concept of coset but this plea here seems to have remained unresponded to.

Episode 3: The first isomorphism theorem for groups as a container of compressed conceptual difficulties

The concept of isomorphism is "but a formal expression of many general ideas about similarity and difference, most notably the idea that two things which are different, may be viewed as similar under an appropriate act of abstraction" (Leron, Hazzan and Zazkis, 1994, 1995). According to these authors, the difficulty with this act of abstraction lies at the heart of the students' difficulty in understanding the relationship a group isomorphism defines. Their work highlights the influence of two especially complex notions: function and an existential quantifier. As far as the latter is concerned, the more objectified the conception of an existential quantifier is, the more appropriated the students' activity in defining isomorphisms turns out to be. Resonant to the study reported here are their observations on the former: a discourse on functions as actions, processes and objects and, emphatically, in terms of their domain and range is absolutely essential. This issue is elaborated in this Episode and dealt with beyond the context of Group Theory in (Nardi, 2000).

Background to the Episode. The evidence I use below originates in four quite similar tutorials given consecutively to 4 pairs of students by the same tutor (Tutorial 1: Eleanor (her pair, Camille, was absent)), Tutorial 2: Cary (silent throughout) and Beth, Tutorial 3: Cleo and Patricia, Tutorial 4: Abidul and Frances) in which they prove the First Isomorphism Theorem for Groups:

Let G, G' be groups and ϕ: $G \to G'$ a homomorphism. If $K = ker\phi$ then: $G|K \sim Im\phi$.
Outline of proof: since $K \lhd G$, $G|K$ can be defined.
Then Ψ: $G|K \to Im\phi$, where $\Psi(Kg) = \phi(g)$
is an isomorphism, namely Ψ is a well-defined, 1–1 and onto homomorphism.

The Episode. In the following, I use extracts from the discussion of the proof in the four tutorials in order to highlight issues relating to retrieval of the theorem as a necessary but not sufficient condition for meaningful understanding (section (i)). These issues have emerged in almost identical fashion in the context of Lagrange's Theorem but, for conciseness, only the data from the above Theorem are used here. I also highlight issues relating to problematic perceptions of the various types of mapping in the context of Group Theory (ϕ and Ψ in sections (ii) and (iii)).

(i) Memory retrieval as a necessary but not sufficient condition for meaningful understanding

Successful retrieval of the theorem varies in the four tutorials. Except Beth, the other students either remain silent or contribute very weak associations: Eleanor hesitantly says that she has associated the theorem with a normal subgroup. The tutor insists that it is a particular normal subgroup and Eleanor remembers kerϕ. In Tutorial 3 Patricia says that she can only recall that the theorem is related to cosets and a mapping which she says she thinks is an isomorphism. The tutor corrects: it is a homomorphism. She then asks for the conclusion of the theorem. Patricia replies:

P1: G . . . divided by . . . kernel . . . er, . . . f . . . does this [$\sim$] actually mean equals?
T1: Isomorphic. It means isomorphic.

Patricia's association with 'division' (possibly invoked by the use of '$|$' in the statement of the theorem) is accompanied by her query on the meaning of $\sim$. I note that similar evidence was given by these students with regard to *Lagrange's Theorem*. Patricia's interpretation ($\sim$ means =) can be perceived as meaningful (if = is taken as meaning 'is the same as') but it is inaccurate: $G|K$ contains sets of sets and Imϕ contains elements of the group. It nevertheless reflects the student's muddled perception of $\sim$ stemming from the diverse meanings that this symbol has in various mathematical contexts (e.g. in Fourier Series). As elaborated below, Patricia's interpretation of $\sim$ as = conveys how problematic the perception of the newly introduced notion of isomorphism is.

Generally the students find it difficult to retrieve the theorem. The tutor, most notably in Tutorial 4, the last in the series, sounds alarmed and firmly reminds the students (as in the context of *Lagrange's Theorem* in evidence omitted here) of the importance of "a theorem with a name attached to it". The tutor's firm and strict recommendation is a pragmatic and not an epistemological argument. In fact the students have given contradictory evidence as to the power of persuasion of the different arguments: so – whilst often not satisfied at all with pragmatic explanations and asking for the *raison-d'-être* of most newly-introduced concepts, for instance Camille in Episode 2 – in most other occasions they seem to feel motivated to explore and understand some new concepts simply on the basis of their highly probable appearance on an exam paper. I do not see these two motivational forces as mutually exclusive: however it seems to me that epistemological rather than pragmatic motivation is more likely to induce a more reflective approach of the concepts.

Incidentally I note that the students reproduce with relative ease information that has been given to them in catchy phrases in previous tutorials, for example:

$$Kg_1 = Kg_2 \text{ if and only if } g_1 g_2^{-1} \in K$$

the 'obvious map' from $G|K$ to G is to map Kg to g

Unfortunately immediate retrieval does not precede immediate, deep or in fact *any* understanding of what the phrases mean. It remains at best an automatic and effective act, at worst an act void of meaning that fosters a false impression of achievement.

Similarly the students apply, for instance, the rule of coset product ($Kgh = KgKh$) correctly but cannot make any decision about how to use it in order to support the completion of the proof. They comfortably provide answers to questions that constitute progressive steps of the proof as long as these steps have been pre-designed by the tutor. Only once the tutor attempts a more global comment regarding the internal structure and resonance of the *First Isomorphism Theorem* (Tutorial 3) when she says that the 'point' of the proof lies in the application of the homomorphic mechanism for Ψ.

Another feeble but noteworthy sign of instrumental understanding of the homomorphic property is given by Abidul in Tutorial 4 when she helps Frances (who is 'stuck' with $\phi(g^{-1})$) by pointing out that 'it [the $^{-1}$] doesn't matter', therefore $\phi(g^{-1})$ can be written as $\phi(g)^{-1}$. Abidul's metaphorical expression ('doesn't matter') reflects an understanding of the mechanics of ϕ and therefore a grasp of a process that is characteristic to homomorphisms.

Finally, the tutor's comment on the mechanism of the proof as 'very standard' of proofs involving quotient groups is an attempt to generalise the approach used in the proof and hint at its potential as a methodological tool. In fact the 'standardisation' comes from the fact that the mapping $Kg \to g$, which she has been calling the 'obvious map', is used quite often in group-theoretical proofs, when cosets need to be mapped to the elements of the group. She rarely makes similar comments.

(ii) An incident with kerϕ reveals problematic conceptions of mapping

In Tutorial 3, the students were in difficulty with the meaning of $g_1 g_2^{-1} \in K$. As a result, the tutor initiates a discussion of the definition of K = kerϕ and asks what a kernel is:

C1: It maps the elements at zero.
T2: Not zero!
C2: Identity.
T3: [*writes down the definition 'properly': kerϕ =$\{g \in G$ /. She leaves it unfinished and asks*]
Such that?
C3: Maps them to the identity.
T4: Which means something is equal to the identity.
C4: [after a pause] g does.
T5: When you say something goes to the identity, is equal to the identity, what do you mean by that?
C5: ϕ does.

The tutor initially objects to Cleo's use of the term 'zero' for the identity element of G (but does not pick up on her use of 'at' which, nevertheless, Cleo does not repeat in C3). Cleo quickly corrects (C2) – this is a common terminological mistake that the students habitually make and equally habitually correct when prompted by the tutor. Then, in the explication of the definition (T3), it becomes evident that C1 was phrased in a rather problematic way not only because of the student's use of the term 'zero': in Cleo's sentences the subject of the verb 'maps' in C1 and C3 is not clear at all. T4 is an interpretation that equates Cleo's 'maps the elements to the identity' with 'something is equal to identity'.

C4 possibly means "element g goes to the identity element" and C5 that "it goes via ϕ". The tutor does not explore any further Cleo's grammatical state of mind with regard to the definition of kerϕ and completes the writing on her own initiative. Therefore it remains an open question whether Cleo's perception of ϕ as a mapping has been illuminated by the exchange of short verses in C1-C5. As in section (iii) below, the student's words reflect an undecided perception of mapping as a machine. In this perception however it is not clear what is mapped where. Cleo's pronoun ('it') as the subject of the verb 'maps' (C1 and C3), as well as her substituting the verbs (C4 and C5) with 'does', are lexical substitutions that possibly reflect and determine the ambiguity of her thinking.

(iii) The problematic $\Leftarrow$ direction of $Kg_1 = Kg_2 \Leftrightarrow \phi(g_1) = \phi(g_2)$ (*) and the properties of an isomorphism

Except Frances and Beth, the rest of the students have problems in interpreting the $\Leftarrow$ direction of (*) as the definition of 1–1 correspondence. Eleanor confuses it with onto but soon changes her mind and in Tutorial 3 a discussion is triggered that reveals a more general confusion.

The tutor points out that $\Rightarrow$ of (*) means Ψ is well-defined. What about $\Leftarrow$? she asks. The students are silent. Then they both mumble about $\phi(g_1) = \phi(g_2)$ saying something about ϕ. The tutor stresses that their answer must be related to Ψ, not ϕ. The students look lost. The tutor then repeats that they have to prove that Ψ is an isomorphism. The students are still silent and the tutor asks what an isomorphism is.

P2: It is a homomorphism.
T6: Yes? And. . .?
P3: It has to be 1–1. . .
T7: And. . .?
P4: A bijection. . .
T8: Yes. . .
P5: Onto.
T9: Yes. And, in other words, which bit have we proved here? You've got a choice: you can say it's a homomorphism, it's 1–1 and it's onto [*laughter*] Which bit have we proved? [*pause*] Is it the bit that says it is a homomorphism?
P6 and C6: [*after a pause*] No.
C7: Prove. . . er,. . . oh, it's this direction that proves it's a well-defined operation. . . so it's the other one that proves it's 1–1.

First the students are interpreting (*) as providing information about homomorphism ϕ, when in fact (*) provides information about the well-definedness and the 1–1 property of Ψ. The students are deceived possibly because their interpretation is an interpretation by appearances. Moreover through P2-P6 Patricia and Cleo clarify, via the tutor's very leading questions (for example: the multiple choice question in T9), their definition of an isomorphism. I note that, in this case of very directed questioning, the students' difficulties with the properties of an isomorphism (for example: onto and 1–1) are not really explored because the teaching is solely oriented towards the elicitation of the answers that will further the progression of the proof.

Similarly in Tutorial 2 Beth appears to be severely concerned and confused as to the information contained in (*) as well as the homomorphic property of Ψ. Like Patricia and Cleo, she does not carry out the switch from Ψ to ϕ flexibly and cannot understand how manipulating ϕ can lead to an understanding of the properties of Ψ.

Finally in Tutorial 4, Abidul's statement is another piece of evidence of the mechanical, detached from conceptual understanding, conceptualisation of isomorphism Ψ. Significantly the student starts dictating the necessary calculations for proving that Ψ is a homomorphism: instead of $\Psi\ (Kg_1Kg_2) = \Psi(Kg_1)\ \Psi(Kg_2)$, she starts dictating '$\Psi(Kg_1g_2) = \Psi(Kg_1)$' but is interrupted by the tutor. Her expression so far, and in particular the Kg_1g_2 part, seems to imply that, for her, the homomorphic property has to be proved for g_1 and g_2 and not for Kg_1 and Kg_2; in other words, as if Ψ is defined on G, not $G|K$.

Actually, the students' difficulty with the *First Isomorphism Theorem*, and by implication with a large part of the newly introduced Group Theory, lies largely in the co-ordination and understanding of the link between Ψ and ϕ, as well as the clarification about the definition of Ψ. The degree of complexity in a problem which requires a well co-ordinated manipulation of mappings between different sets is extremely high. ϕ is defined between the elements of a group (or two groups). Ψ is defined between the cosets of the kernel of ϕ and the image of ϕ. This link between Ψ and ϕ and the implications and importance of shifting back and forth from Ψ to ϕ need to be explicitly made to the learner. The shift from one level of abstraction to another is not self-evident. In the absence of a didactically illuminating decomposition of the theorem to its constituent elements, it is not surprising that the students are not capable of making these shifts to more abstract levels.

In sum, in the above, evidence was given of a number of difficulties in the conceptualisation of properties associated to the notion of mapping (homomorphic property, onto, 1–1, well-definedness of a mapping); also of the varying degrees of abstraction involved in the definition of a mapping between elements of a group or the cosets of a subgroup and the elements of the group. The high degree of abstraction and the conceptual difficulties have then been linked to the students' cognitive puzzlement in Group Theory which culminates at the introduction and proof of the First Isomorphism Theorem for Groups.

From a teaching point of view, leading questions that are intended as small steps towards eliciting from the students a completion of the proof did not seem to illuminate and dissolve their problematic perceptions. For this purpose a didactical decomposition of the constituent elements of the theorem was pointed out as a potentially helpful tool for the understanding of its content and proof.

3. SYNTHESIS

In this paper the students' first experiences of fundamental group-theoretical concepts were explored in a series of Episodes from the second term of observation.

Particularly the concept of *coset* emerged as *paradigmatically problematic*. While constructing cosets the students appeared to be in difficulty with the abstract nature of the operation between elements of a group: references to the properties of numerical operations were observed to generate a concern on the part of the tutors who discourage the students from using metaphorical expressions such as *divided by*, for example in the context of quotient groups. Somehow paradoxically, however, they do not discourage them from saying *multiply with the inverse* in the context of group operations. Similarly problematic turned out to be the use of expressions such as *times* and *powers of* – also used sometimes vaguely interchangeably by the students – with regard to cyclic groups. Of course, numerical operations have legitimate analogues in groups and these analogues, if employed cautiously (for example, so that commutativity is not illegitimately assumed), can provide a helpful grounding for an understanding of the group operation. On the other hand, the tutors' concern is understandable: the power of Group Theory lies partly in the fact that an element of a group can be a mathematical object that is not a number; and, if appropriating this fact is a crucial learning step for their students, then this step perhaps needs to be taken through a process of distancing (but, I stress, not dissociating) from a numerical context.

Linguistic condensation of meaning causes difficulties, for instance in the context of the concept of *order of an element* of a group. As an implication of the theorem $|<g>| = |g|$, the term *order of an element* of a group can be seen as an abbreviation for the term *the order of the group generated by an element*. Semantically, the abbreviation is similar: $|g|$ is the commonly used notation instead of $|<g>|$.

Another aspect of the problematic encounter with the notion of *order of an element* seems to be its static and operational duality: $|g|$ is the number of elements in $<g>$ and, at the same time, the number of times the power of g has to be taken in order to cover all the elements of $<g>$. After $|g|$ steps, the powers of g in $<g>$ start repeating themselves. So, in a sense, *order of an element* is a notion containing both information about a static characteristic of $<g>$ (its cardinality) and information about a way to construct $<g>$ (take the power of g, $|g|$ times). This type of duality is commonly seen as a source of cognitive strain for students (as suggested by most of the theories on learning advanced mathematical concepts briefly mentioned in the introduction as the theoretical tools of this study) and it is likely that *order of an element* is not an exception. Unpacking this duality then, for example making explicit the tacit theorem, $|<g>| = |g|$, mentioned above, becomes an indispensable part of the tutor's role.

The students are primarily engaged in a meaning bestowing process with regard to the newly introduced concepts: they inquire about the *raison-d'-être* of the concepts. Significantly, a number of these enquiries do not seem to be casual outcomes of the students' fatigue with abstraction. For example, coset: the enquiries were characterised by a persistent use of geometric images aiming at the construction of a meaning of the new concept (first: equivalence classes as *straight lines*, then: cosets as *squares*). A literal interpretation of these geometric images by the student may imply that these are a poor, and therefore dubious, means of image construction for some particularly abstract concepts.

The students also appeared to be in difficulty in conceptualising a mapping between elements of a group and sets of elements of the group (examples in the context of the First Isomorphism Theorem for Groups were presented). In various occasions in these tutorials, where the notion of sets of sets and of mappings between sets whose elements are gradually departing further from the simple arithmetical correspondences the students are familiar with from school mathematics, the increasing degree of abstraction as well as the students' problematic perceptions of the notions of domain and range of these mappings rendered their understanding an extremely complex process.

4. IMPLICATIONS FOR TEACHING AND CONCLUSION

The Episodes exemplified here remain rather inconclusive from a teaching point of view. This is due to the fact that their conclusion was determined conceptually in the methodology of the study by whether a learning cycle for the student in the particular tutorial seemed to have been completed. In other words, unlike, for instance,

- the work of Leron and Dubinsky (1995) which relates their work on learning with concrete teaching suggestions, or,
- Burn's (1998) direct proposals for the teaching of an introductory Abstract Algebra course,

the research reported here was intended to be primarily on the students' learning. However a few observations regarding the teaching of Group Theory did emerge – a cross-topical discussion of these observations can be found in (Nardi, 1998).

Yielding control to the learner seems to be of considerable pedagogical potential: for instance, when the tutor manages to modify a student's perspective from within, that is by adapting their point of view and challenging it with key prompting questions (Episode 2). Closed questioning, on the other hand, and highly directive instruction (Episodes 1 and 3) seems to be less efficient and a perpetrator of decontextualised algorithmic behaviour. Directive instruction is mostly based on the expert's prophecies, about, for instance, the simplicity or difficulty of a task and these prophecies can prove misleading.

A crucial didactical point, regarding the potential cognitive danger built in the use of geometric images as a visual aid for the introduction of new and abstract concepts, was made here in the context of cosets. However, flexibility on the tutor's part in thinking in the terms of the student's suggested images (Episode 2) seemed to boost a more confident demonstration of thinking on the part of the student. Negotiating appropriate images is a crucial function of teaching and the lesson in the examples used here seems to be one of clarity: that of the relationship between the intended-by-the-tutor abstract concept image and the image employed for this purpose.

The students were found to be largely engaged in an exploration of the *raisons-d'-être* of newly introduced concepts. To support this exploration, the tutors often decompose the various problematic concepts or theorems (for example the First Isomorphism Theorem for Groups in Episode 3) into their basic elements. This, even though sometimes based on the tutor's preconceptions of what constitutes the problematic elements of the concept or the theorem, seems to be quite efficient. The students seem to appreciate a new concept when it is launched as a useful apparatus, not as an ideal that exists only because of its definition.

With regard to some newly introduced theorems and in justifying their introduction (for example: First Isomorphism Theorem for Groups) the tutors often use pragmatic – as opposed to epistemological – arguments. A pragmatic argument was described as an attempt of the tutor to convince the students of the significance of a theorem by repeating that it definitely appears on exam papers because, for instance, it has a 'name attached to it' (analogous memory triggers are discussed in (Leron and Hazzan, 1997)); an epistemological argument was described as giving an existential rationale to some newly introduced concepts such as justifying the introduction of cosets as a substantial element of studying normality in Group Theory. The latter were suggested as cognitively more powerful, whereas the former were acknowledged as strong motivators.

In this paper evidence was offered of first-year mathematics undergraduates' conceptual difficulties with some fundamental concepts of Group Theory. This evidence resonates with the findings in this relatively new area of investigation on the psychology of learning advanced mathematics. The author and her colleagues are currently involved in further research in the area[2].

NOTES

1. A tutorial is a weekly 30–60 minute session given to one or two students in which the tutor and the students discuss problem sheets, concepts or theorems relevant to the week's lectures. Despite its contextual idiosyncrasies, a tutorial is a uniquely intimate learning environment which offers a naturalistic field for observing novice and expert mathematicians at work. Given that the discussions in a tutorial mostly address the students' difficulties with the various topics, the richness of this source with respect to a psychological investigation on the learning of advanced mathematics is self-evident (but was also substantiated in a Pilot Study to the doctorate carried out in the first year of the study with one Oxford College).
2. For further information: http://uea.ac.uk/∼m011

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School of Education,
University of East Anglia,
Norwich NR4 7TJ UK