Microeconomic Theory I

Profit maximization and cost minimization

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Economic Profit

- **A firm uses inputs j = 1…,m to make products i = 1,…n.**
- **Output levels are $y_1,\ldots,y_n$.**
- **Input levels are $x_1,\ldots,x_m$.**
- **Product prices are $p_1,\ldots,p_n$.**
- **Input prices are $w_1,\ldots,w_m$.**

The Competitive Firm

- **The competitive firm takes all output prices $p_1,\ldots,p_n$ and all input prices $w_1,\ldots,w_m$ as given constants.**

Economic Profit

- The economic profit generated by the production plan $(x_1,\ldots,x_m,y_1,\ldots,y_n)$ is

$$\Pi = p_1 y_1 + \cdots + p_n y_n - w_1 x_1 - \cdots w_m x_m.$$

Economic Profit

- **Output and input levels are typically flows.**
- **E.g. x_1 might be the number of labor units used per hour.**
- **And y_3 might be the number of cars produced per hour.**
- **Consequently, profit is typically a flow also; e.g. the number of dollars of profit earned per hour.**

Economic Profit

- **How do we value a firm?**
- **Suppose the firm's stream of periodic economic profits is $\Pi_0, \Pi_1, \Pi_2, \ldots$ and r is the rate of interest.**
- **Then the present-value of the firm's economic profit stream is**

$$PV = \Pi_0 + \frac{\Pi_1}{1+r} + \frac{\Pi_2}{(1+r)^2} + \cdots$$

Economic Profit

- **A competitive firm seeks to maximize its present-value.**
- **How?**

Economic Profit

- **Suppose the firm is in a short-run circumstance in which $x_2 \equiv \tilde{x}_2$.**
- **Its short-run production function is**

$$y = f(x_1, \tilde{x}_2).$$

Economic Profit

- **Suppose the firm is in a short-run circumstance in which $x_2 \equiv \tilde{x}_2$.**
- **Its short-run production function is**

$$y = f(x_1, \tilde{x}_2).$$

- **The firm's fixed cost is $FC = w_2\tilde{x}_2$ and its profit function is**

$$\Pi = py - w_1x_1 - w_2\tilde{x}_2.$$

Short-Run Iso-Profit Lines

- **A $\$\Pi$ iso-profit line contains all the production plans that provide a profit level $\$\Pi$.**
- **A $\$\Pi$ iso-profit line's equation is**

$$\Pi \equiv py - w_1x_1 - w_2\tilde{x}_2.$$

Short-Run Iso-Profit Lines

- **A $\$\Pi$ iso-profit line contains all the production plans that yield a profit level of $\$\Pi$.**
- **The equation of a $\$\Pi$ iso-profit line is**

$$\Pi \equiv py - w_1x_1 - w_2\tilde{x}_2.$$

- **I.e.**

$$y = \frac{w_1}{p}x_1 + \frac{\Pi + w_2\tilde{x}_2}{p}.$$

Short-Run Iso-Profit Lines

$$y = \frac{w_1}{p} x_1 + \frac{\Pi + w_2 \tilde{x}_2}{p}$$

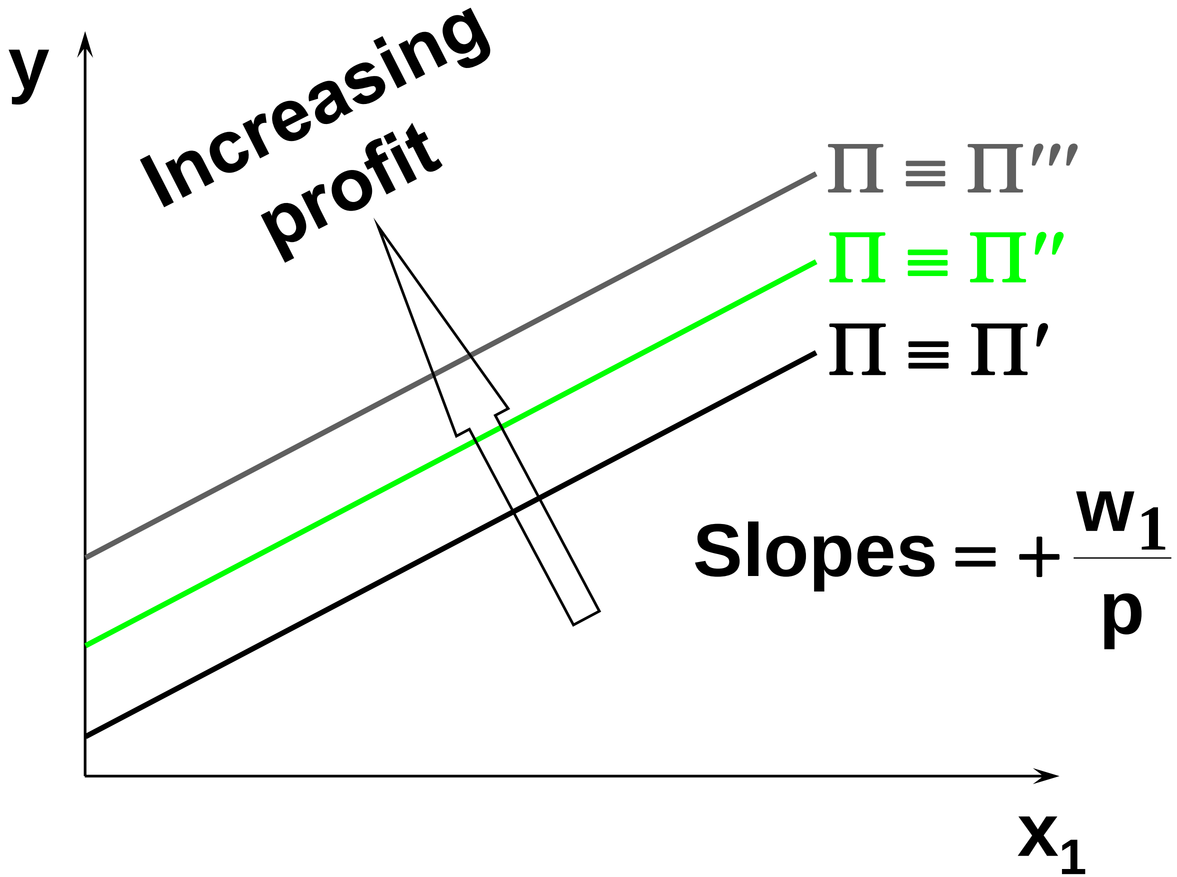
has a slope of

$$+ \frac{w_1}{p}$$

and a vertical intercept of

$$\frac{\Pi + w_2 \tilde{x}_2}{p}.$$

Short-Run Iso-Profit Lines



Short-Run Profit-Maximization

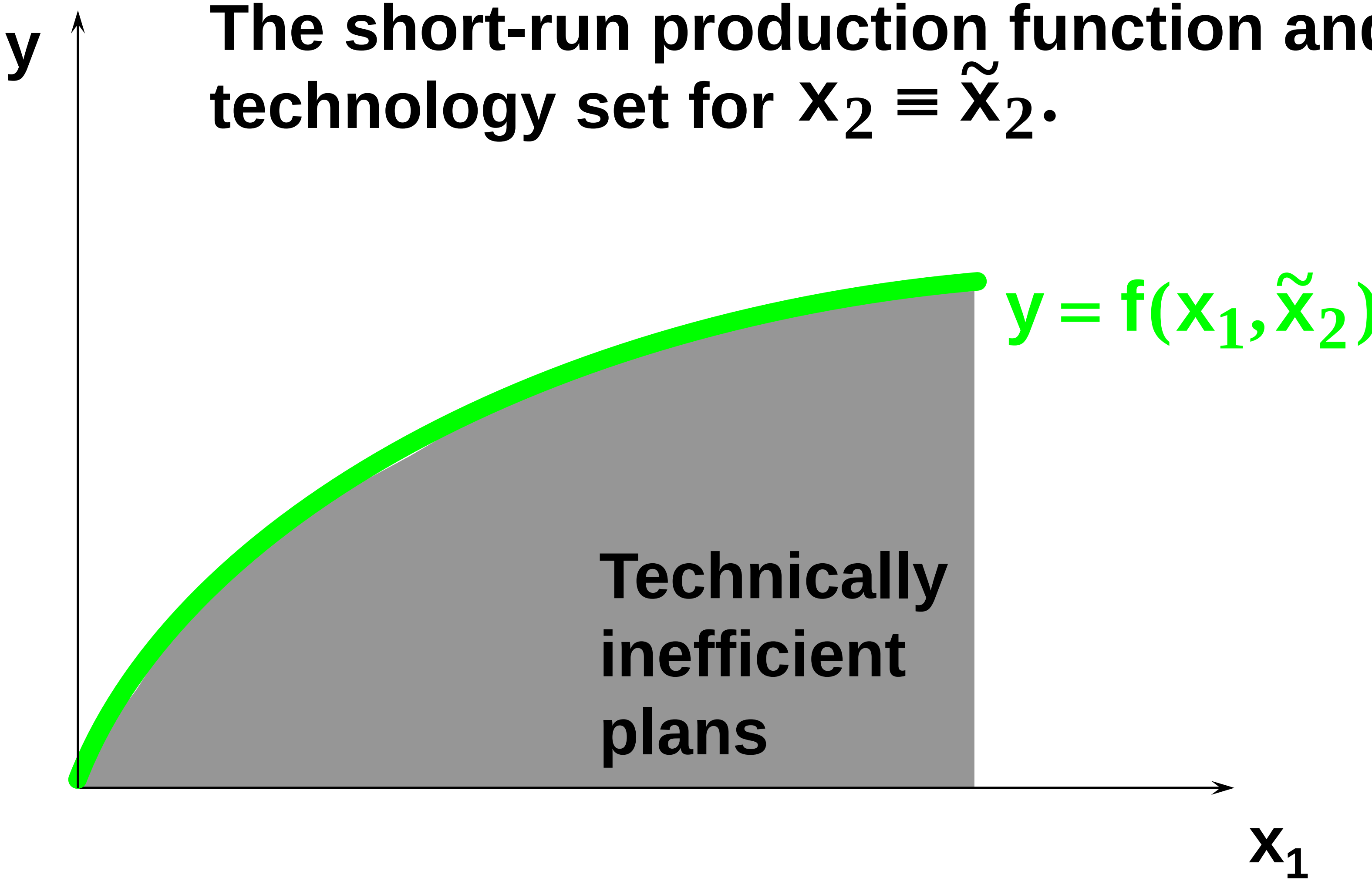
- **The firm's problem is to locate the production plan that attains the highest possible iso-profit line, given the firm's constraint on choices of production plans.**
- **Q: What is this constraint?**

Short-Run Profit-Maximization

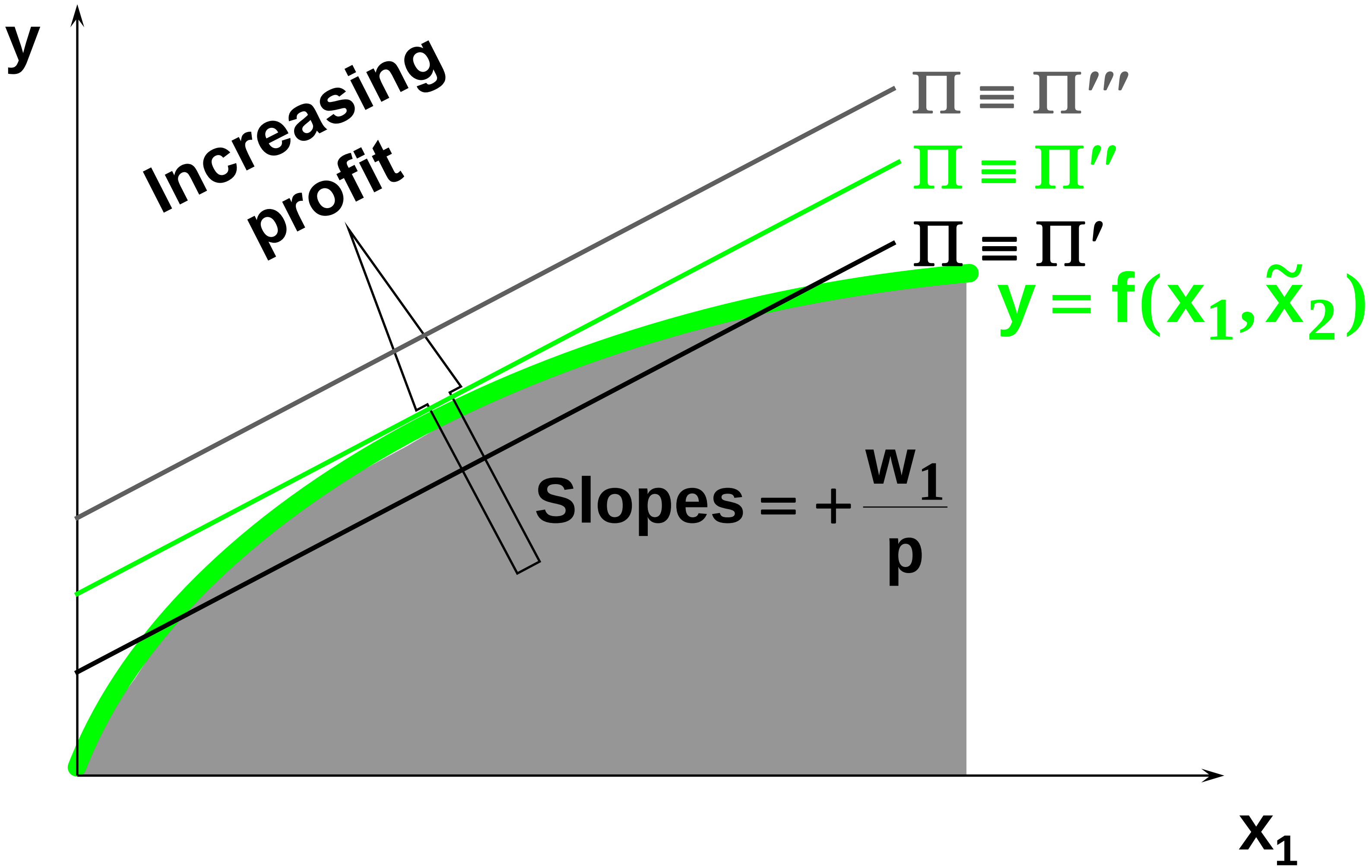
- **The firm's problem is to locate the production plan that attains the highest possible iso-profit line, given the firm's constraint on choices of production plans.**
- **Q: What is this constraint?**
- **A: The production function.**

Short-Run Profit-Maximization

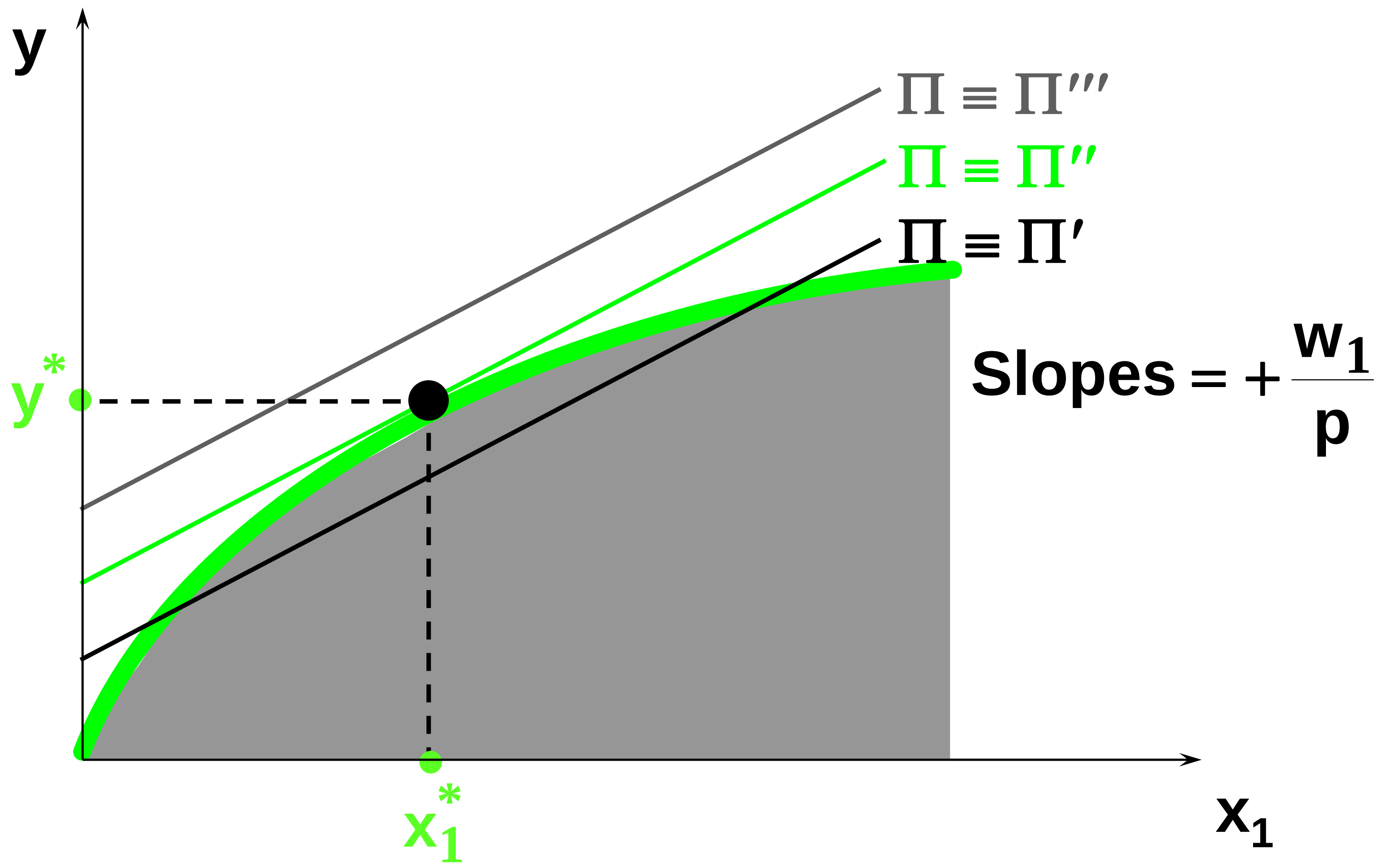
The short-run production function and technology set for $x_2 \equiv \tilde{x}_2$.



Short-Run Profit-Maximization

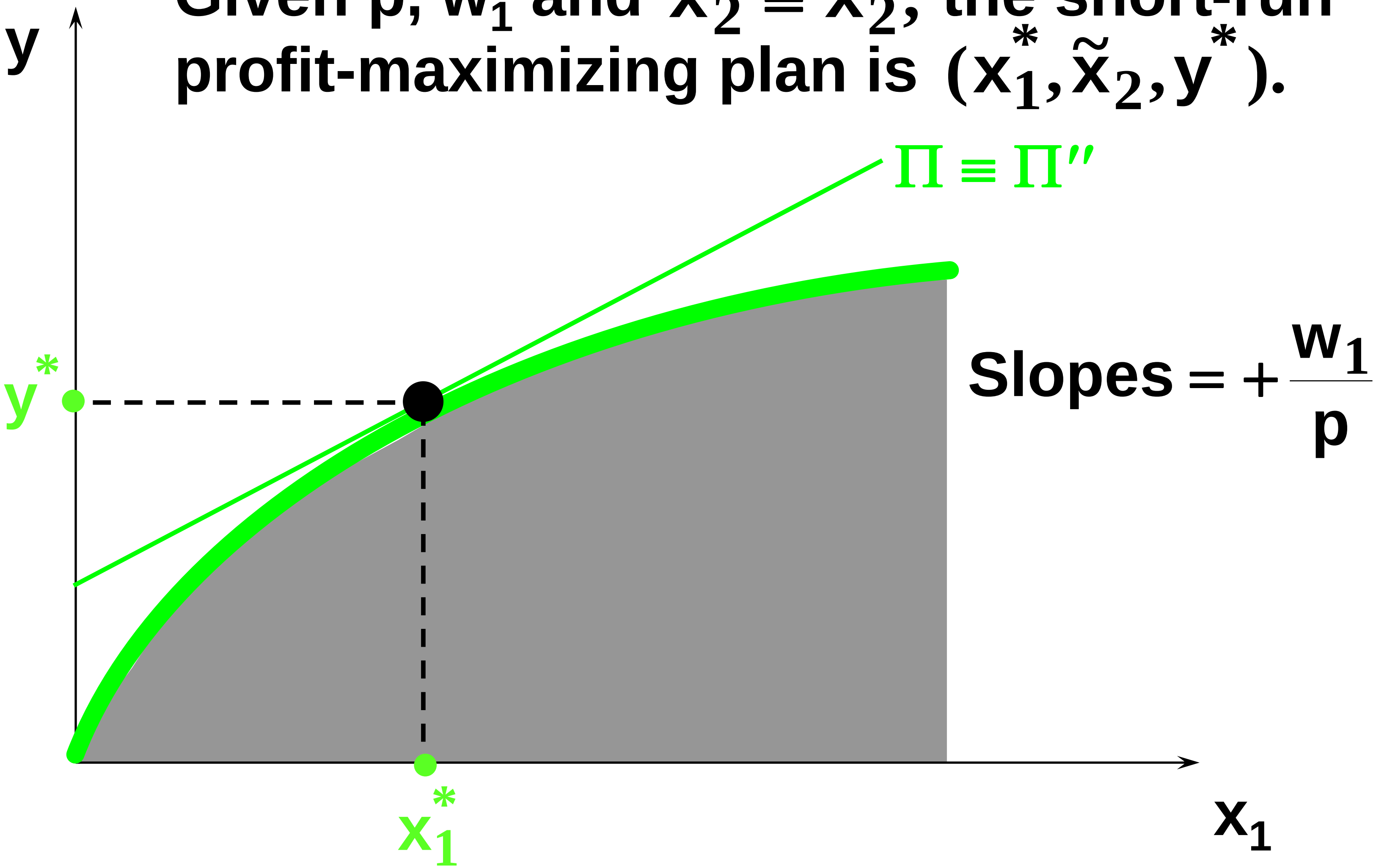


Short-Run Profit-Maximization



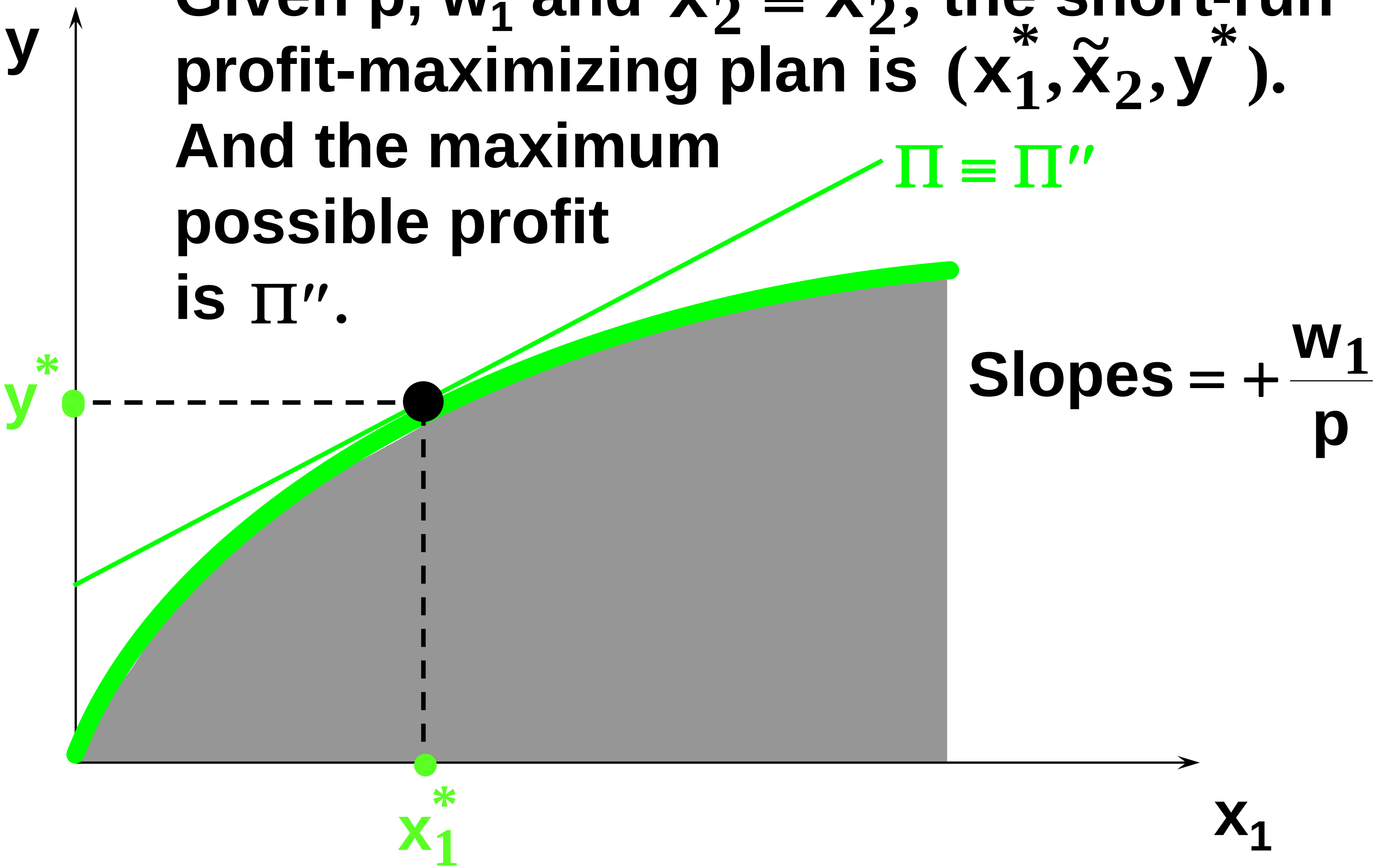
Short-Run Profit-Maximization

Given p, w_1 and $x_2 \equiv \tilde{x}_2$, the short-run profit-maximizing plan is $(x_1^*, \tilde{x}_2, y^*)$.



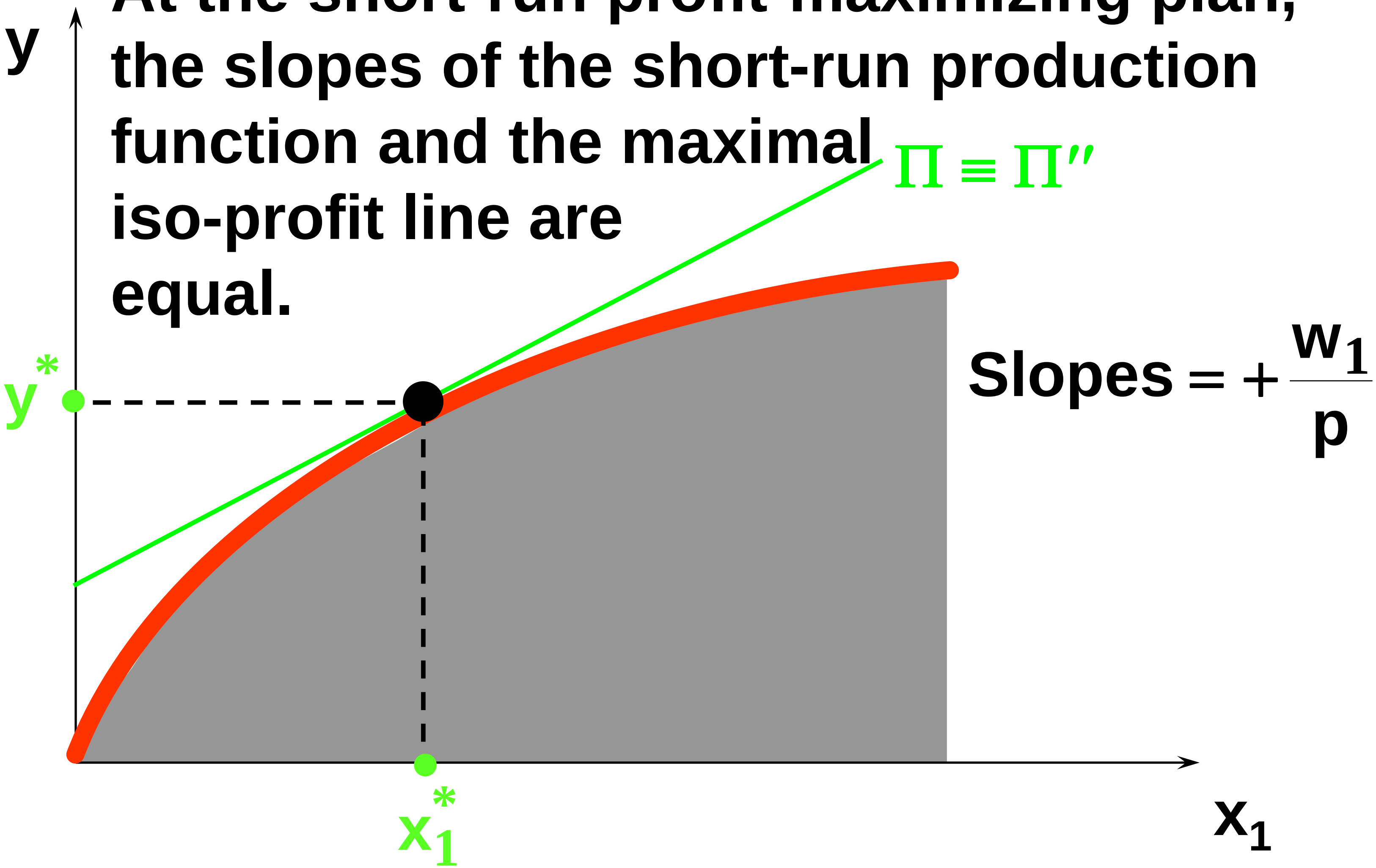
Short-Run Profit-Maximization

Given p, w_1 and $x_2 \equiv \tilde{x}_2$, the short-run profit-maximizing plan is $(x_1^*, \tilde{x}_2, y^*)$. And the maximum possible profit is Π''.



Short-Run Profit-Maximization

At the short-run profit-maximizing plan, the slopes of the short-run production function and the maximal iso-profit line are equal.



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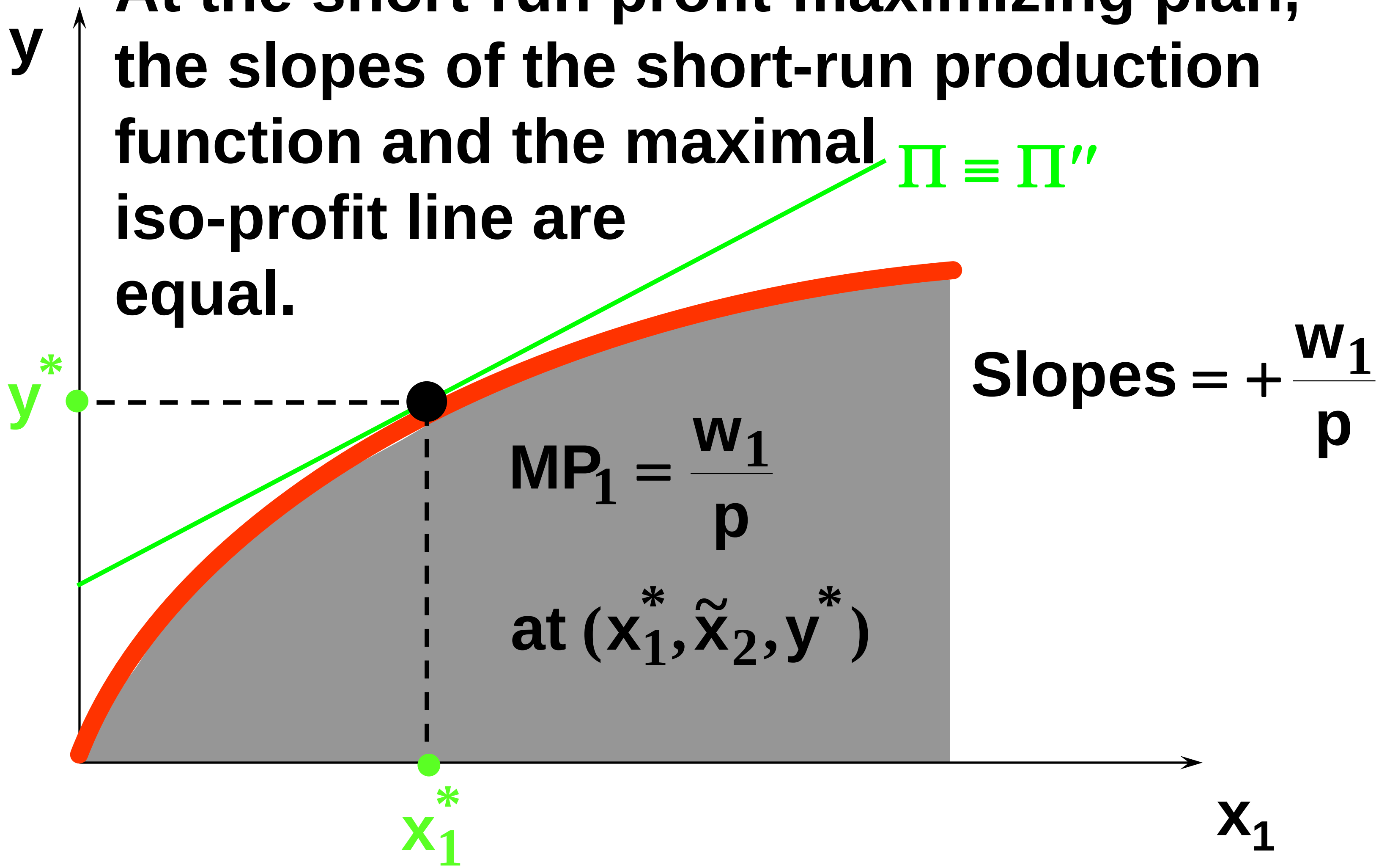

Short-Run Profit-Maximization

$$MP_1 = \frac{w_1}{p} \quad \Leftrightarrow \quad p \times MP_1 = w_1$$

$p \times MP_1$ is the marginal revenue product of input 1, the rate at which revenue increases with the amount used of input 1.

If $p \times MP_1 > w_1$ then profit increases with x_1.

If $p \times MP_1 < w_1$ then profit decreases with x_1.

Short-Run Profit-Maximization; A Cobb-Douglas Example

Suppose the short-run production function is $y = x_1^{1/3}\tilde{x}_2^{1/3}$.

The marginal product of the variable input 1 is

$$MP_1 = \frac{\partial y}{\partial x_1} = \frac{1}{3}x_1^{-2/3}\tilde{x}_2^{1/3}.$$

The profit-maximizing condition is

$$MRP_1 = p \times MP_1 = \frac{p}{3}(x_1^*)^{-2/3}\tilde{x}_2^{1/3} = w_1.$$

Short-Run Profit-Maximization; A Cobb-Douglas Example

Solving $\frac{p}{3}(x_1^*)^{-2/3}\tilde{x}_2^{1/3} = w_1$ for x_1 gives

$$(x_1^*)^{-2/3} = \frac{3w_1}{p\tilde{x}_2^{1/3}}.$$

Short-Run Profit-Maximization; A Cobb-Douglas Example

Solving $\frac{p}{3}(x_1^*)^{-2/3}\tilde{x}_2^{1/3} = w_1$ **for** x_1 **gives**

$$(x_1^*)^{-2/3} = \frac{3w_1}{p\tilde{x}_2^{1/3}}.$$

That is,

$$(x_1^*)^{2/3} = \frac{p\tilde{x}_2^{1/3}}{3w_1}$$

Short-Run Profit-Maximization; A Cobb-Douglas Example

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That is,

$$(x_1^*)^{2/3} = \frac{p\tilde{x}_2^{1/3}}{3w_1}$$

so $$x_1^* = \left(\frac{p\tilde{x}_2^{1/3}}{3w_1}\right)^{3/2} = \left(\frac{p}{3w_1}\right)^{3/2}\tilde{x}_2^{1/2}.$$

Short-Run Profit-Maximization; A Cobb-Douglas Example

$$x_1^* = \left(\frac{p}{3w_1}\right)^{3/2} \tilde{x}_2^{1/2}$$

is the firm's short-run demand for input 1 when the level of input 2 is fixed at $\tilde{x}_2$ units.

Short-Run Profit-Maximization; A Cobb-Douglas Example

$$x_1^* = \left(\frac{p}{3w_1}\right)^{3/2} \tilde{x}_2^{1/2}$$

is the firm's short-run demand for input 1 when the level of input 2 is fixed at $\tilde{x}_2$ units.

The firm's short-run output level is thus

$$y^* = (x_1^*)^{1/3} \tilde{x}_2^{1/3} = \left(\frac{p}{3w_1}\right)^{1/2} \tilde{x}_2^{1/2}.$$

Comparative Statics of Short-Run Profit-Maximization

- **What happens to the short-run profit-maximizing production plan as the output price p changes?**

Comparative Statics of Short-Run Profit-Maximization

The equation of a short-run iso-profit line is

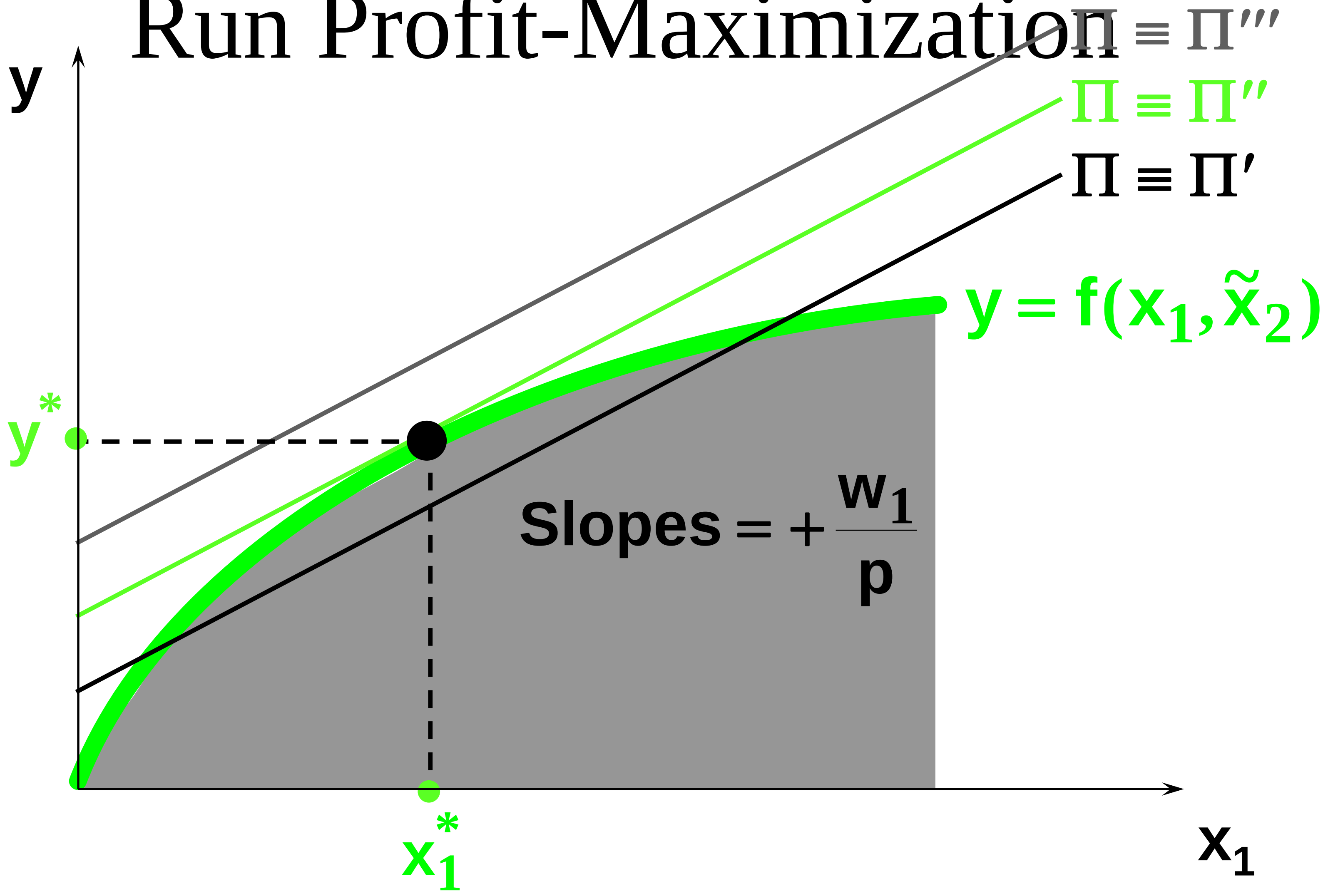
$$y = \frac{w_1}{p}x_1 + \frac{\Pi + w_2\tilde{x}_2}{p}$$

so an increase in p causes

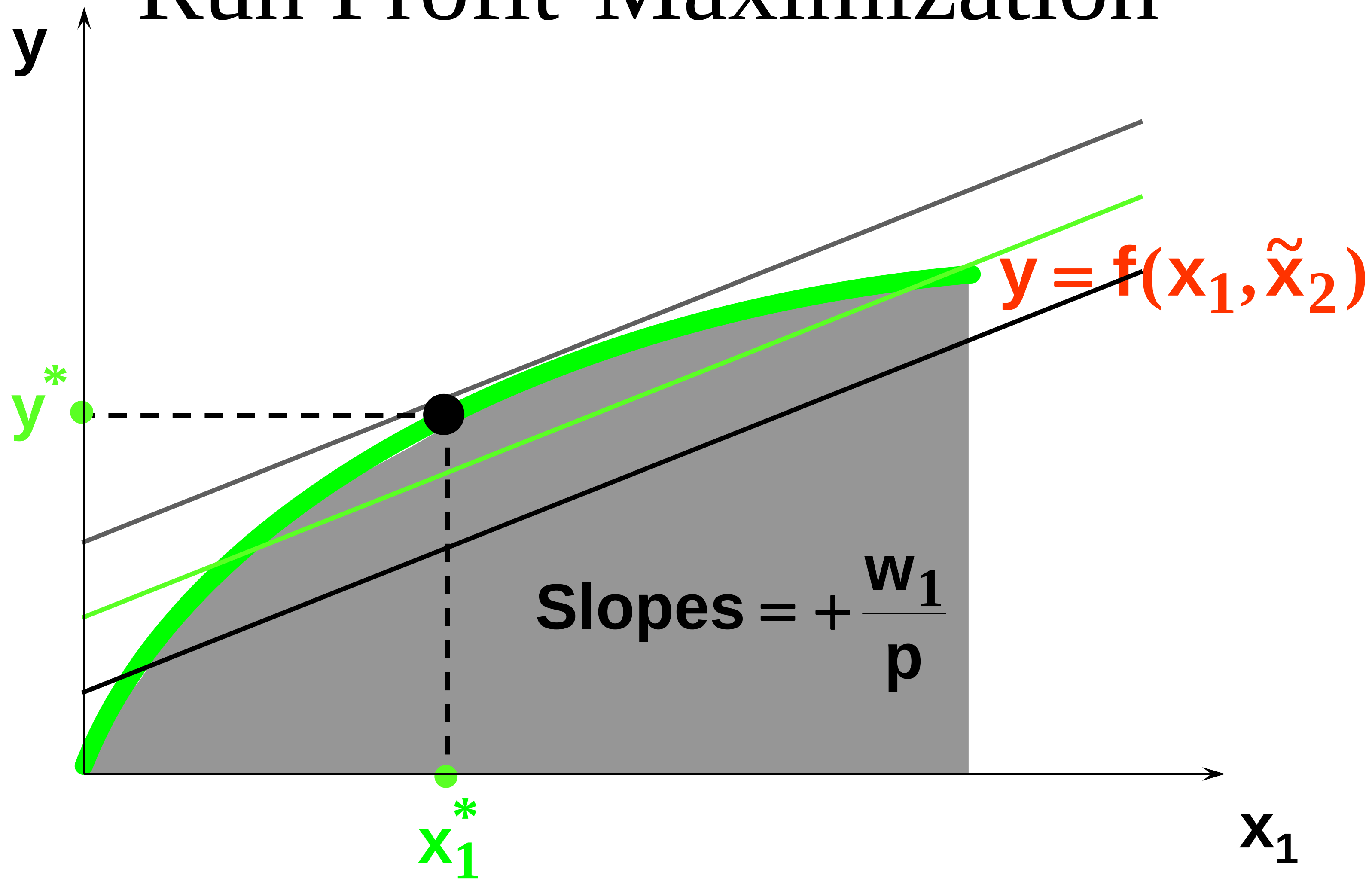
-- a reduction in the slope, and

-- a reduction in the vertical intercept.

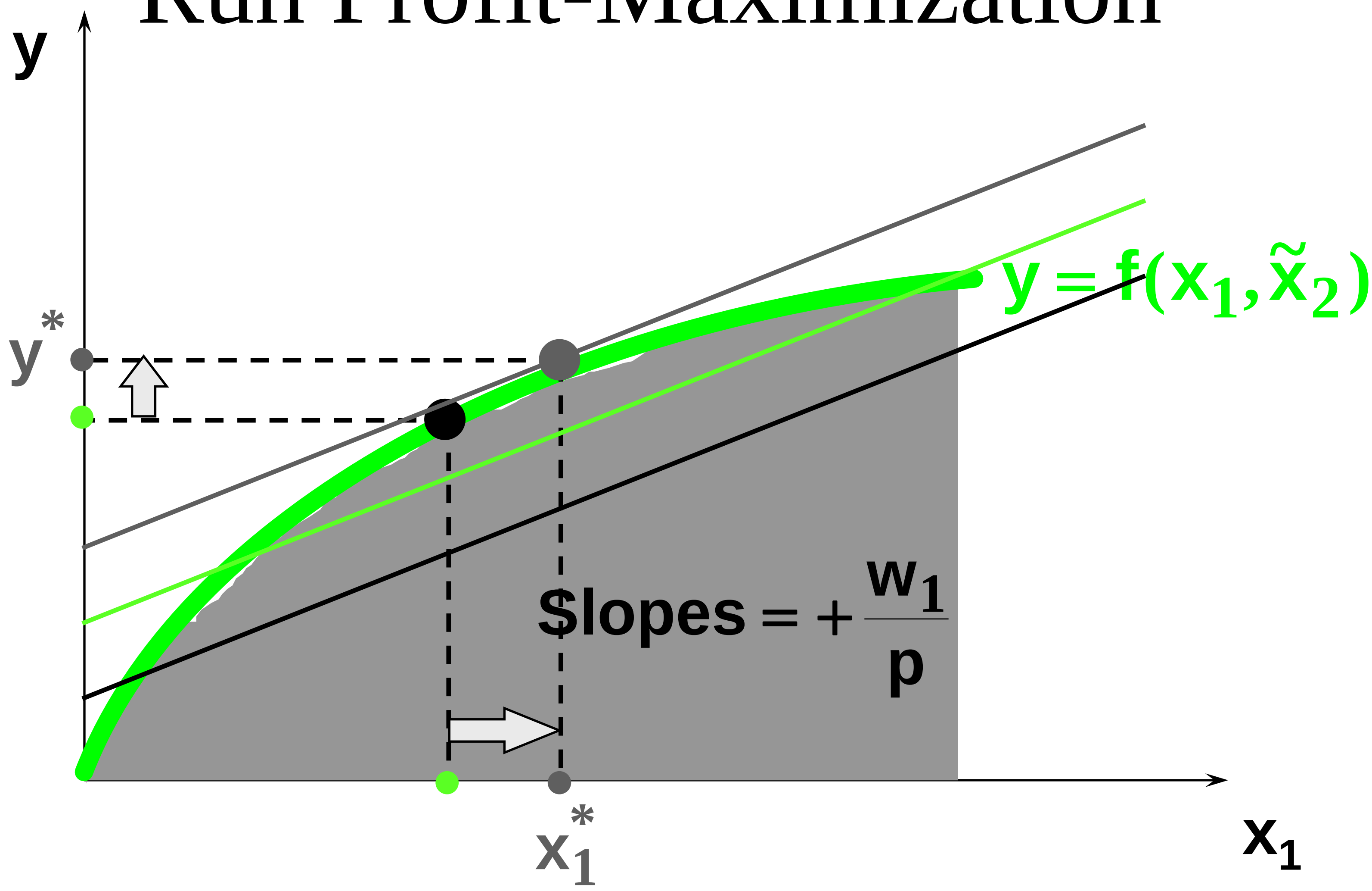
Comparative Statics of Short-Run Profit-Maximization



Comparative Statics of Short-Run Profit-Maximization



Comparative Statics of Short-Run Profit-Maximization



Comparative Statics of Short-Run Profit-Maximization

- **An increase in p, the price of the firm's output, causes**
  - **an increase in the firm's output level (the firm's supply curve slopes upward), and**
  - **an increase in the level of the firm's variable input (the firm's demand curve for its variable input shifts outward).**

Comparative Statics of Short-Run Profit-Maximization

The Cobb-Douglas example: When

$y = x_1^{1/3}\tilde{x}_2^{1/3}$ **then the firm's short-run demand for its variable input 1 is**

$$x_1^* = \left(\frac{p}{3w_1}\right)^{3/2}\tilde{x}_2^{1/2}$$

and its short-run supply is

$$y^* = \left(\frac{p}{3w_1}\right)^{1/2}\tilde{x}_2^{1/2}.$$

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x_1^* increases as p increases.

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x_1^* increases as p increases.

y^* increases as p increases.

Comparative Statics of Short-Run Profit-Maximization

- **What happens to the short-run profit-maximizing production plan as the variable input price w_1 changes?**

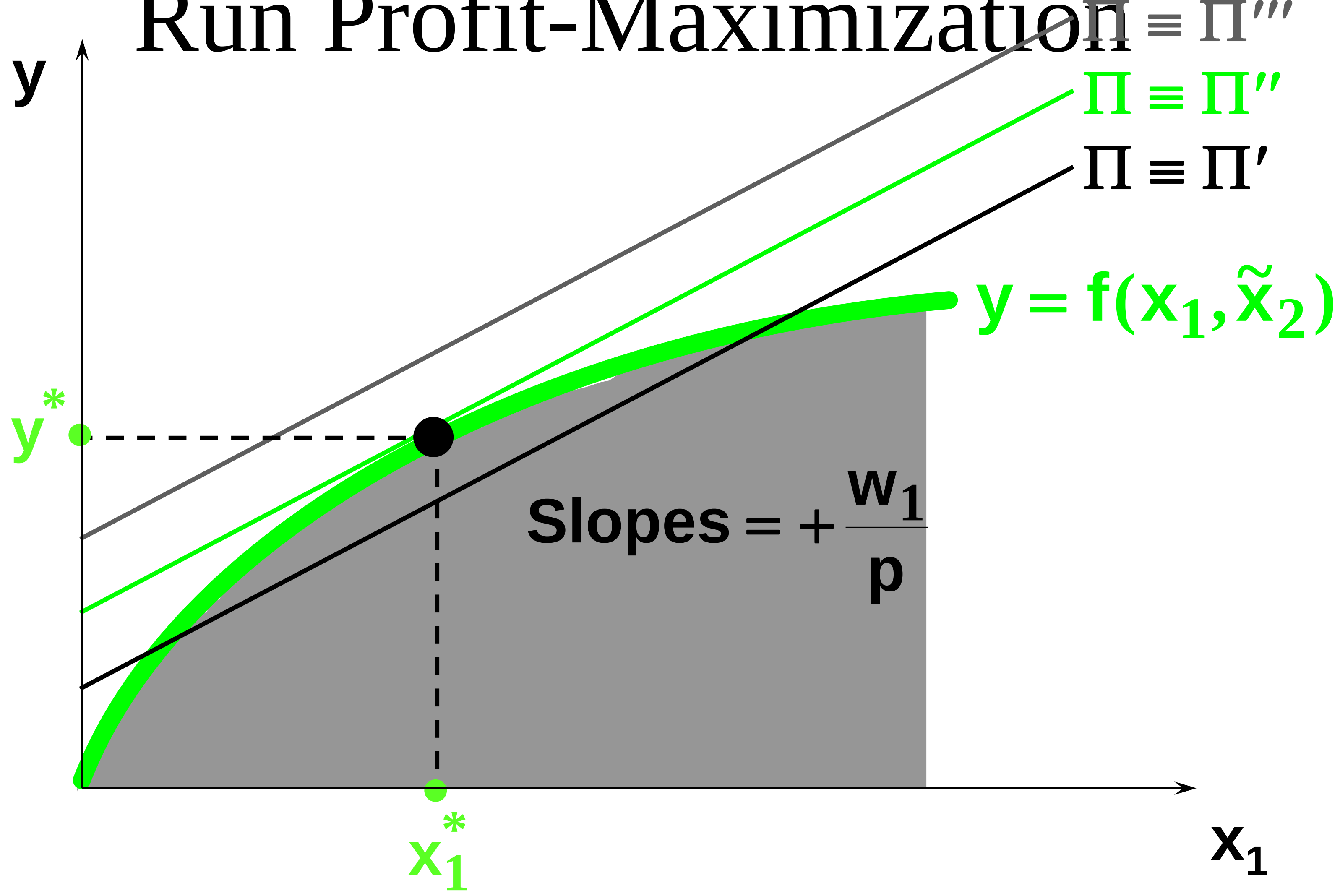
Comparative Statics of Short-Run Profit-Maximization

The equation of a short-run iso-profit line is

$$y = \frac{w_1}{p} x_1 + \frac{\Pi + w_2 \tilde{x}_2}{p}$$

so an increase in w_1 causes
- **an increase in the slope, and**
- **no change to the vertical intercept.**

Comparative Statics of Short-Run Profit-Maximization



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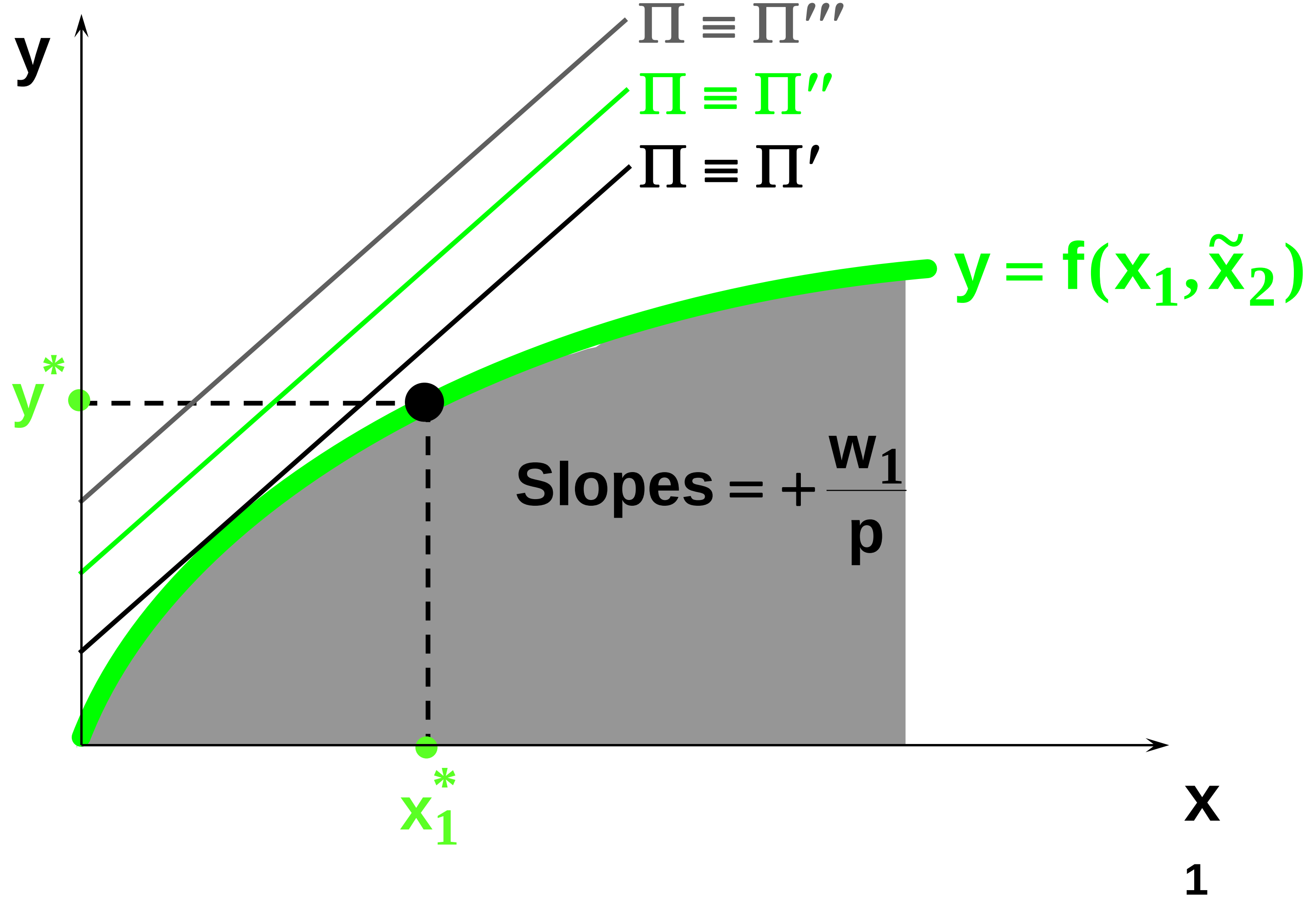
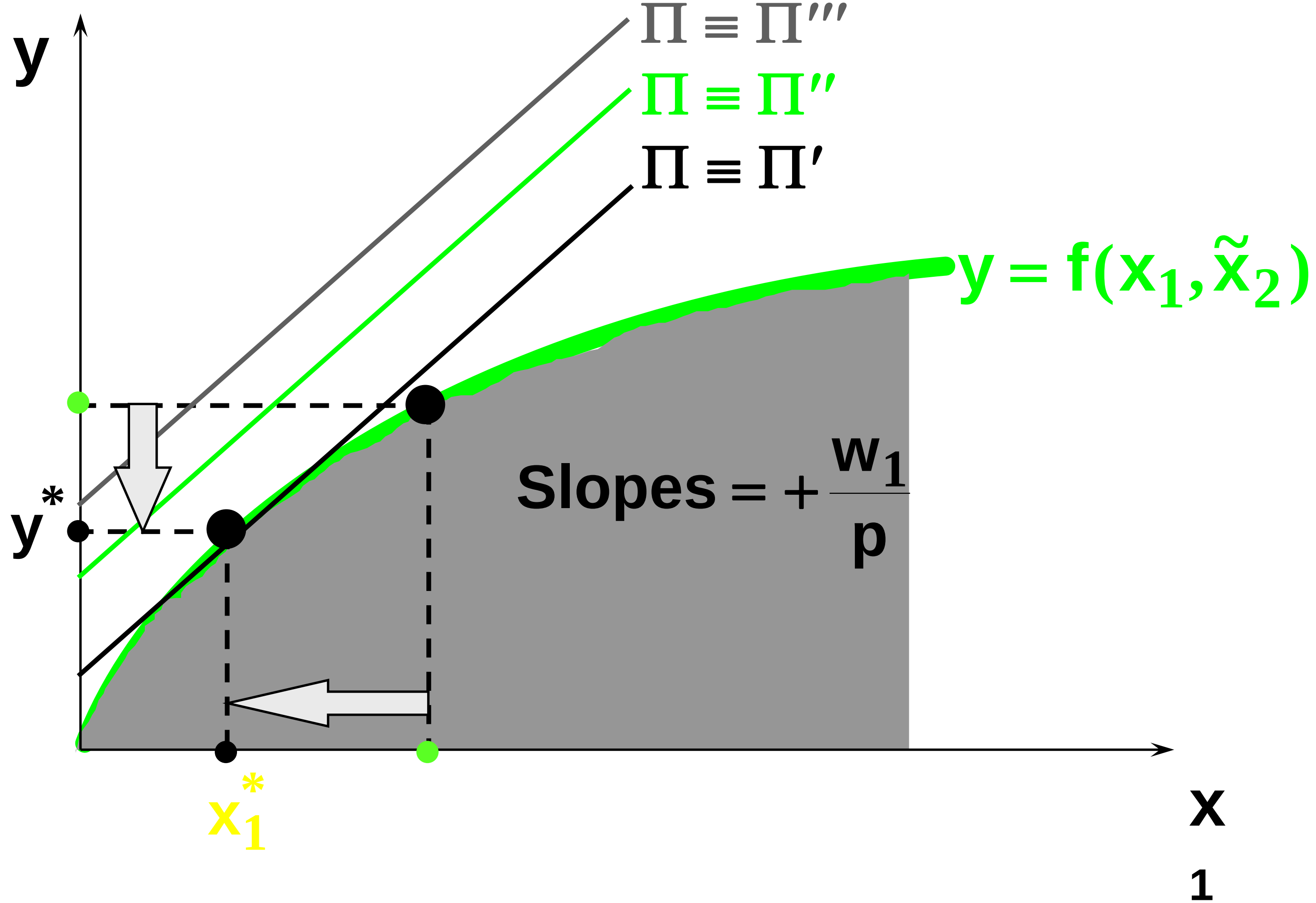


Comparative Statics of Short-Run Profit-Maximization



Comparative Statics of Short-Run Profit-Maximization

- **An increase in w_1, the price of the firm's variable input, causes**
  - **a decrease in the firm's output level (the firm's supply curve shifts inward), and**
  - **a decrease in the level of the firm's variable input (the firm's demand curve for its variable input slopes downward).**

Comparative Statics of Short-Run Profit-Maximization

The Cobb-Douglas example: When $y = x_1^{1/3}\tilde{x}_2^{1/3}$ then the firm's short-run demand for its variable input 1 is

$$x_1^* = \left(\frac{p}{3w_1}\right)^{3/2}\tilde{x}_2^{1/2}$$

and its short-run supply is

$$y^* = \left(\frac{p}{3w_1}\right)^{1/2}\tilde{x}_2^{1/2}.$$

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x_1^* decreases as w_1 increases.

Comparative Statics of Short-Run Profit-Maximization

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and its short-run supply is

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x_1^* **decreases as w_1 increases.**

y^* **decreases as w_1 increases.**

Long-Run Profit-Maximization

- Now allow the firm to vary both input levels.
- Since no input level is fixed, there are no fixed costs.

Long-Run Profit-Maximization

- **Both x_1 and x_2 are variable.**
- **Think of the firm as choosing the production plan that maximizes profits for a given value of x_2, and then varying x_2 to find the largest possible profit level.**

Long-Run Profit-Maximization

The equation of a long-run iso-profit line is

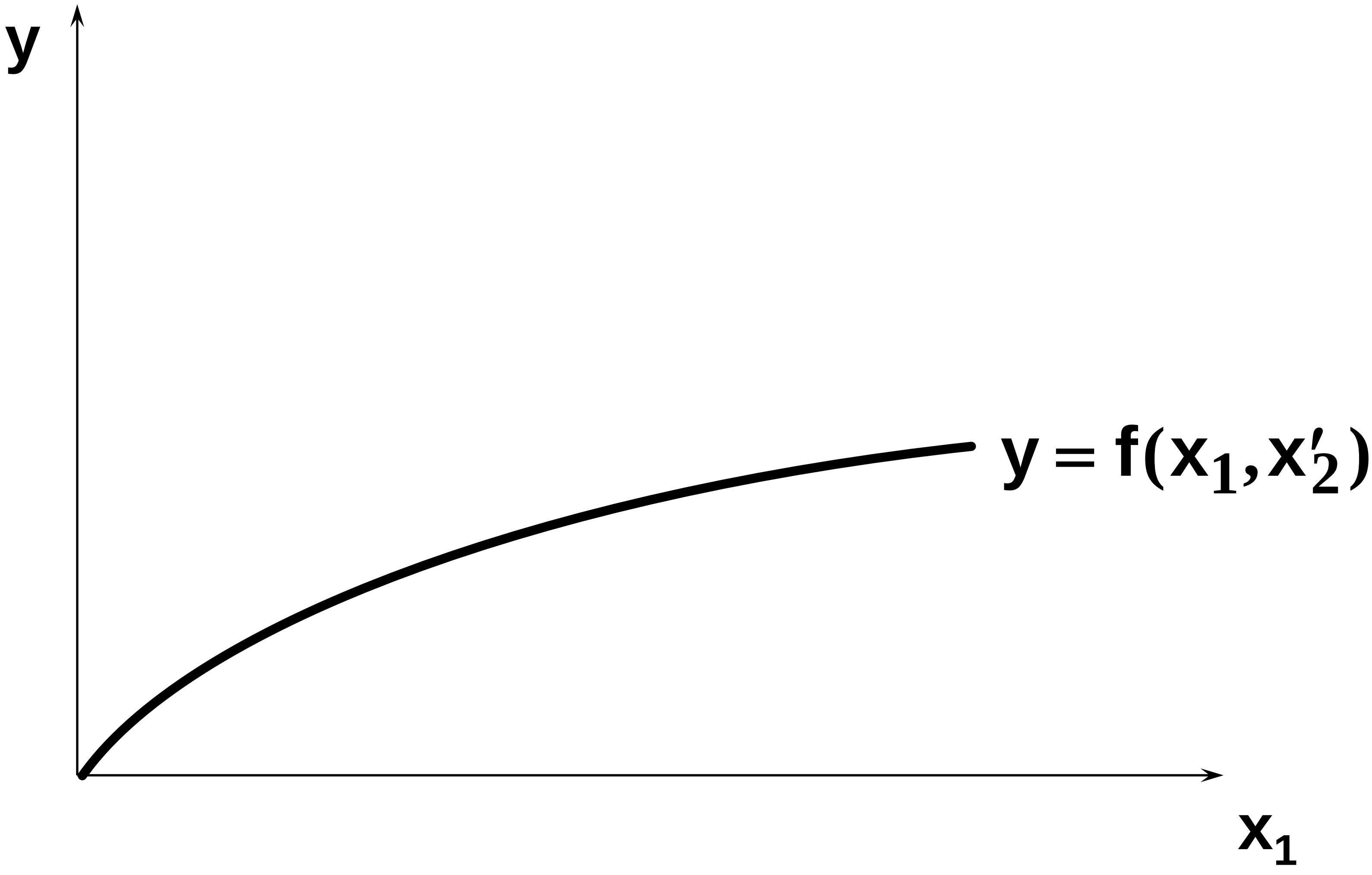
$$y = \frac{w_1}{p} x_1 + \frac{\Pi + w_2 x_2}{p}$$

so an increase in x_2 causes

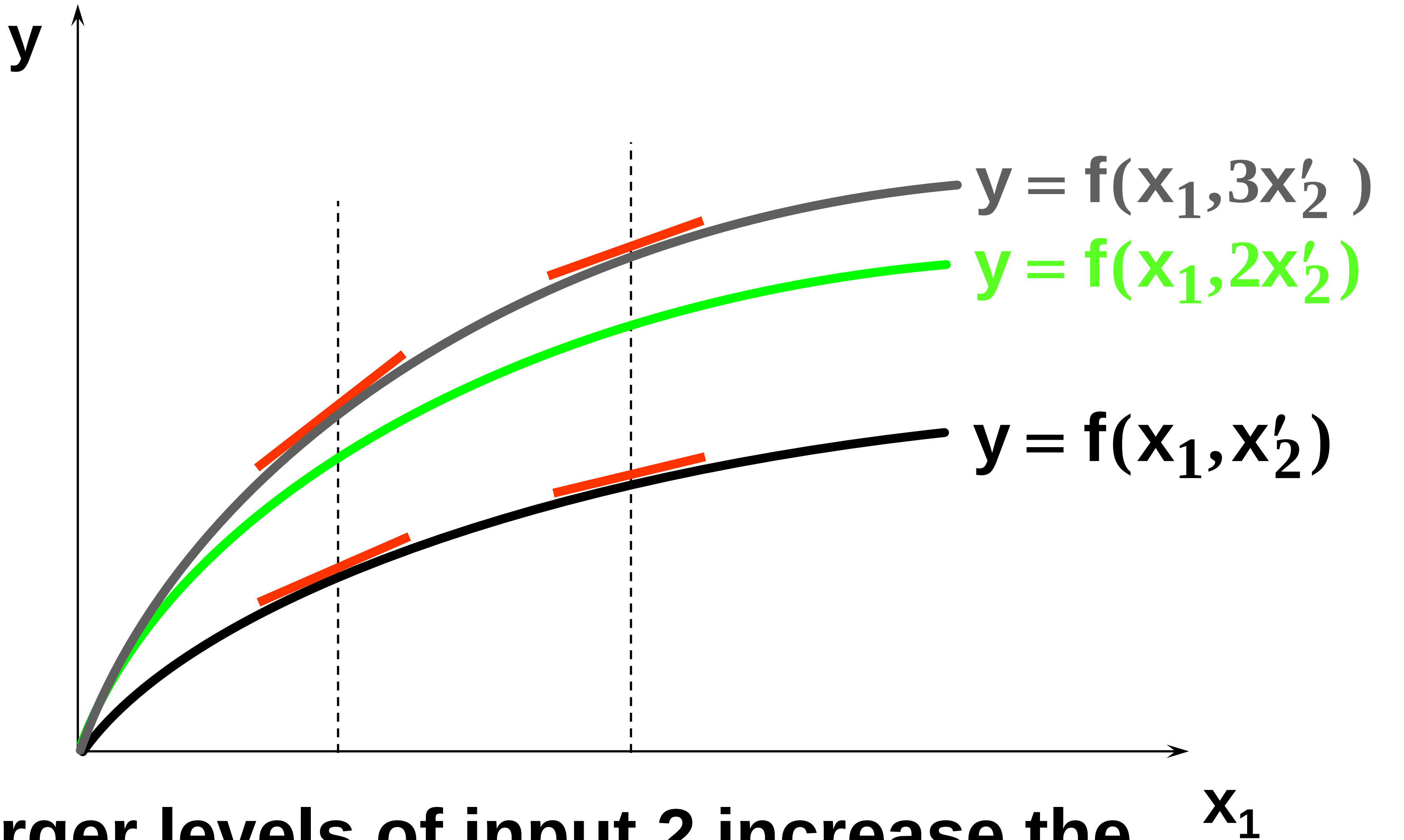
-- no change to the slope, and

-- an increase in the vertical intercept.

Long-Run Profit-Maximization

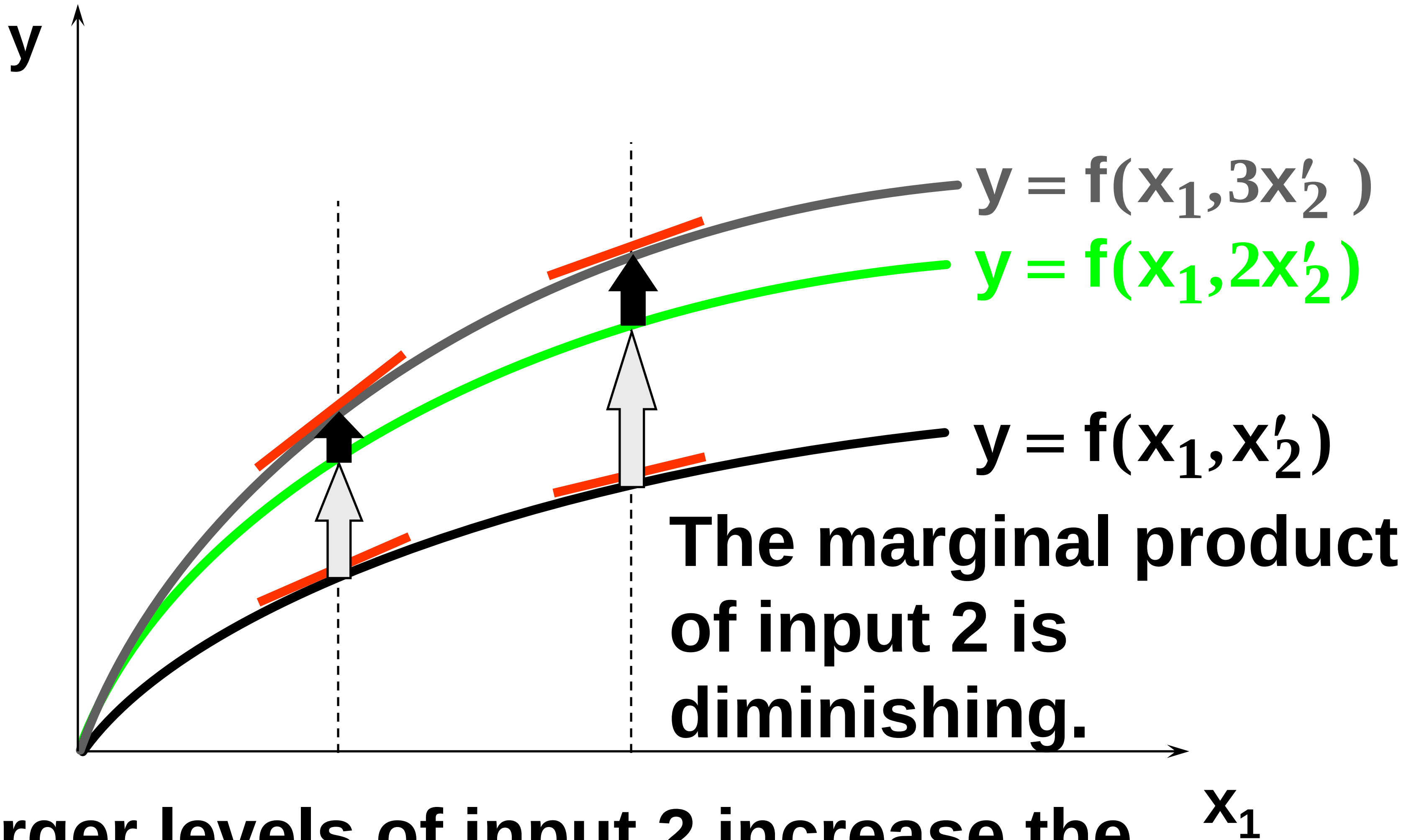


Long-Run Profit-Maximization



Larger levels of input 2 increase the productivity of input 1.

Long-Run Profit-Maximization



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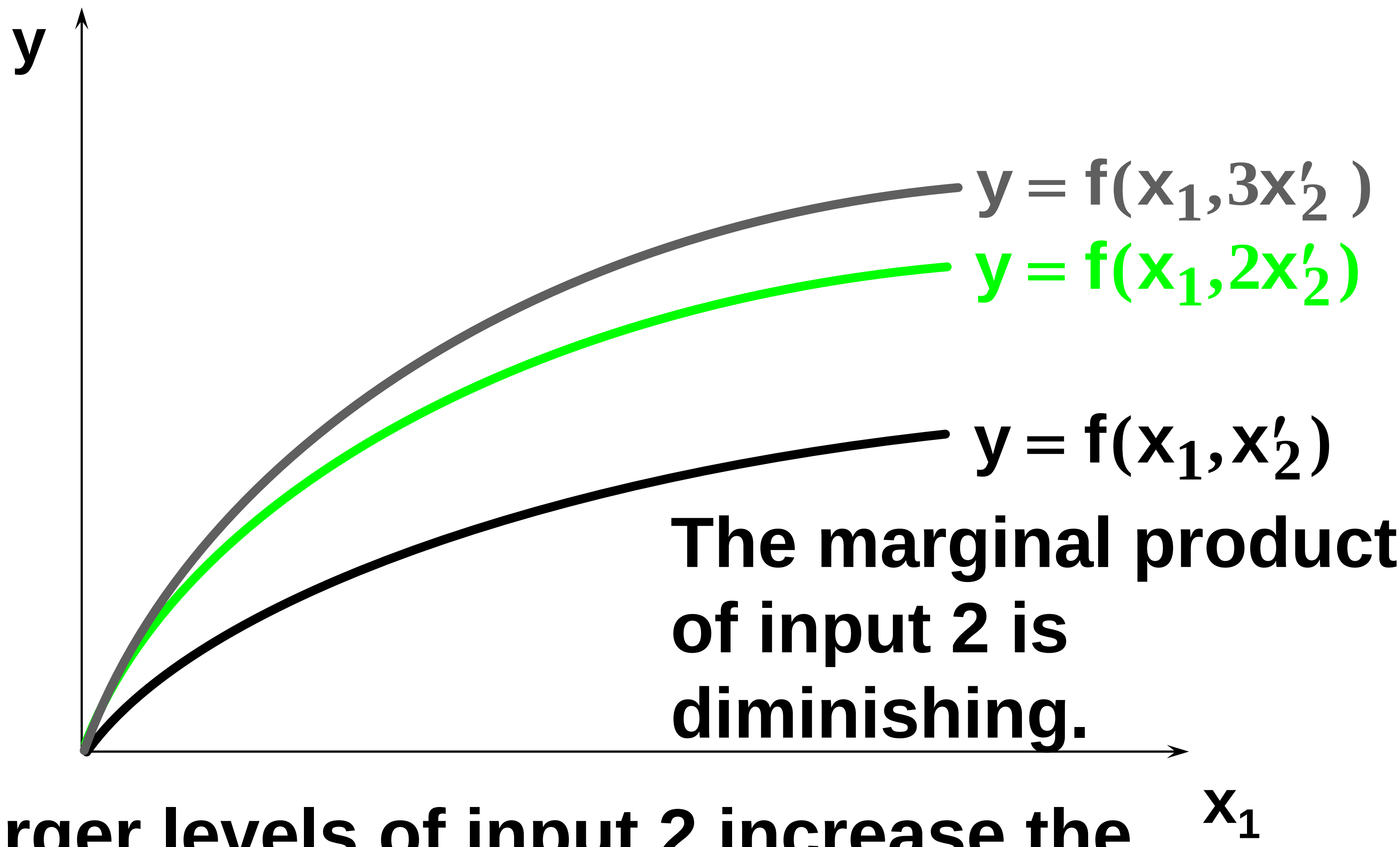
Long-Run Profit-Maximization

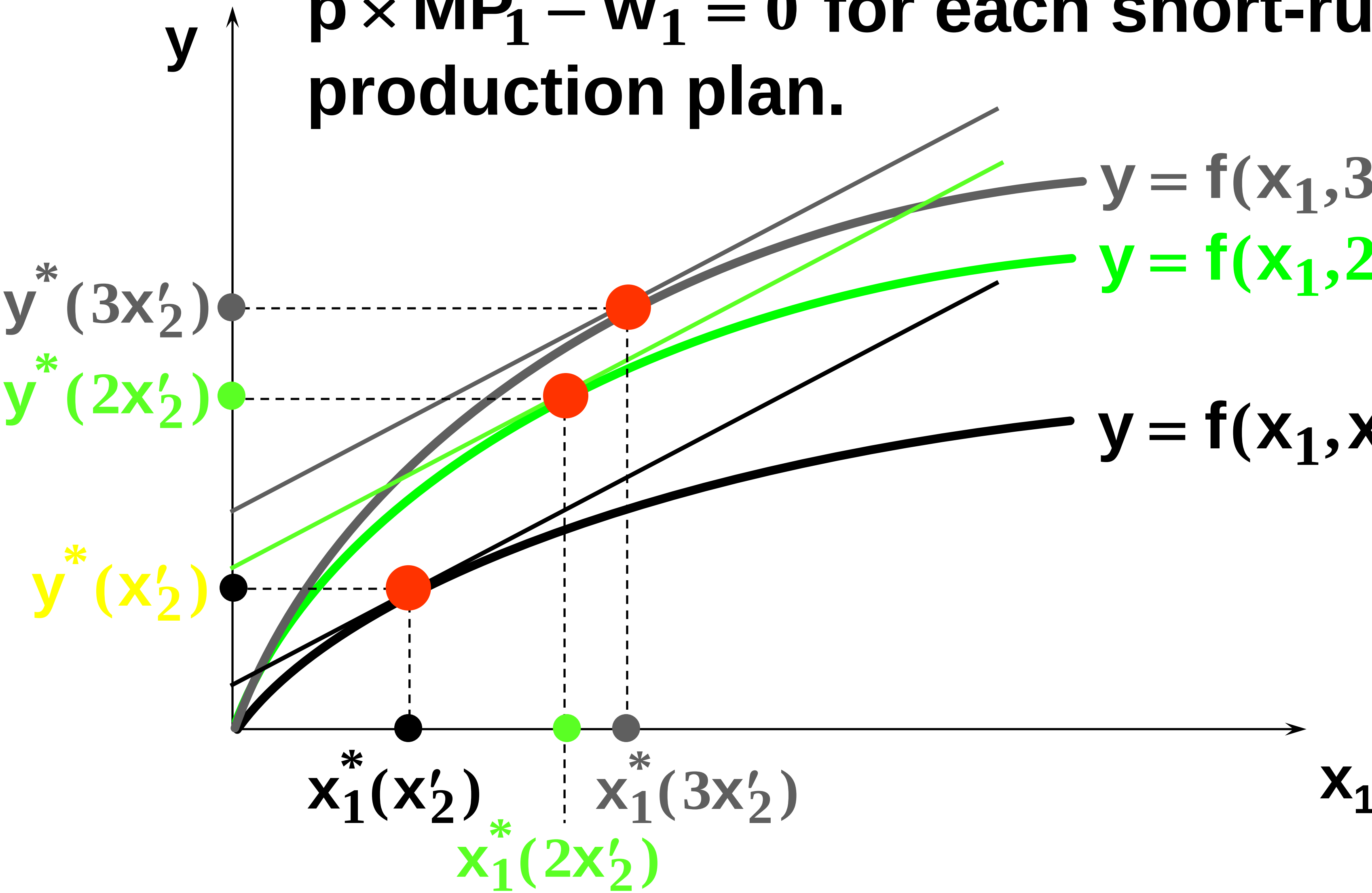


The marginal product of input 2 is diminishing.

Larger levels of input 2 increase the productivity of input 1.

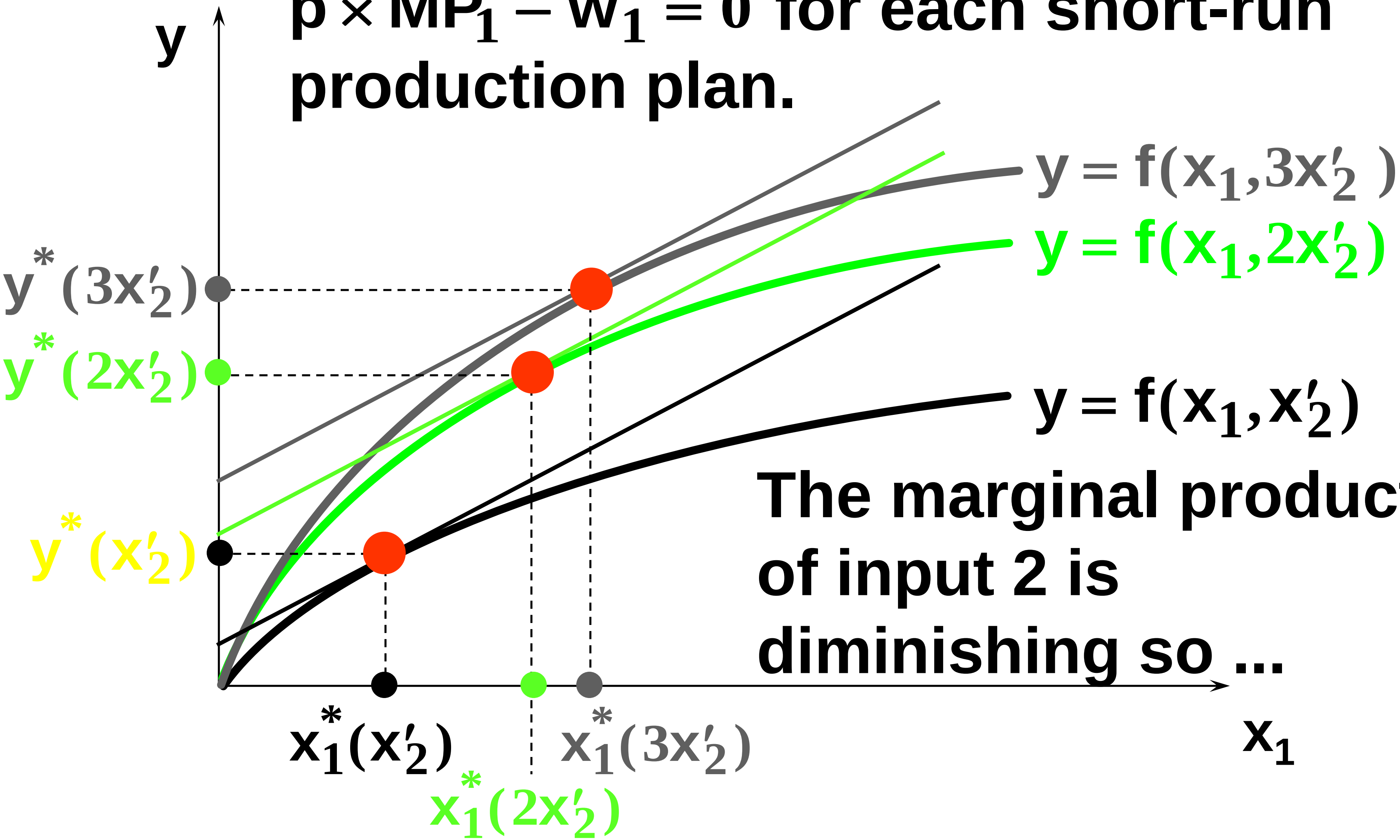
Long-Run Profit-Maximization

$p \times MP_1 - w_1 = 0$ for each short-run production plan.



Long-Run Profit-Maximization

$p \times MP_1 - w_1 = 0$ **for each short-run production plan.**



The marginal product of input 2 is diminishing so ...

Long-Run Profit-Maximization

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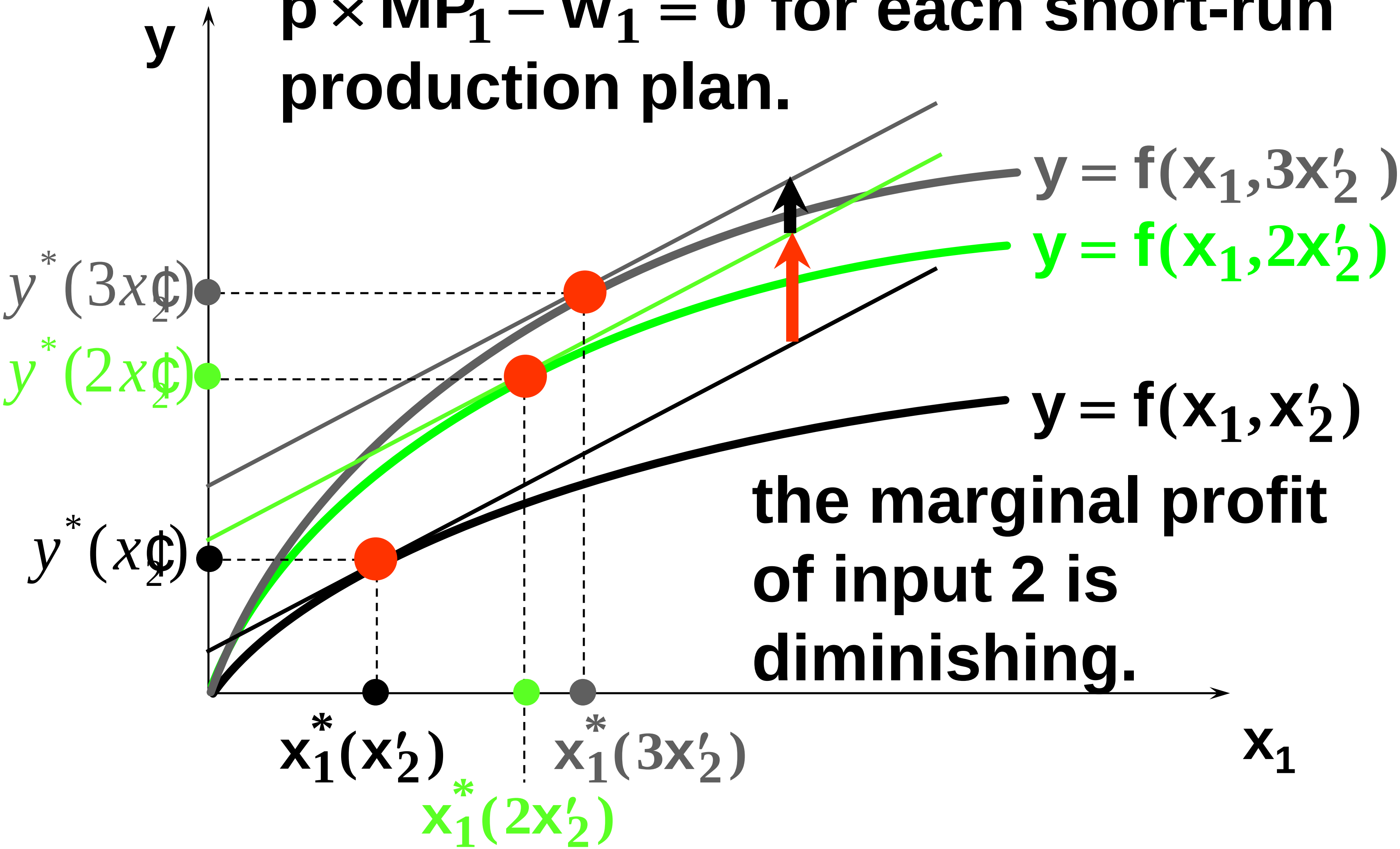

the marginal profit of input 2 is diminishing.

Long-Run Profit-Maximization

- **Profit will increase as x_2 increases so long as the marginal profit of input 2**

$$p \times MP_2 - w_2 > 0.$$

- **The profit-maximizing level of input 2 therefore satisfies**

$$p \times MP_2 - w_2 = 0.$$

Long-Run Profit-Maximization

- Profit will increase as x_2 increases so long as the marginal profit of input 2

$$p \times MP_2 - w_2 > 0.$$

- The profit-maximizing level of input 2 therefore satisfies

$$p \times MP_2 - w_2 = 0.$$

- And $p \times MP_1 - w_1 = 0$ is satisfied in any short-run, so ...

Long-Run Profit-Maximization

- The input levels of the long-run profit-maximizing plan satisfy

$$p \times MP_1 - w_1 = 0 \quad \text{and} \quad p \times MP_2 - w_2 = 0.$$

- That is, marginal revenue equals marginal cost for all inputs.

Long-Run Profit-Maximization

The Cobb-Douglas example: When $y = x_1^{1/3}\tilde{x}_2^{1/3}$ then the firm's short-run demand for its variable input 1 is

$$x_1^* = \left(\frac{p}{3w_1}\right)^{3/2} \tilde{x}_2^{1/2}$$

and its short-run supply is

$$y^* = \left(\frac{p}{3w_1}\right)^{1/2} \tilde{x}_2^{1/2}.$$

Short-run profit is therefore …

Long-Run Profit-Maximization

$$\Pi = py^* - w_1x_1^* - w_2\tilde{x}_2$$

$$= p\left(\frac{p}{3w_1}\right)^{1/2}\tilde{x}_2^{1/2} - w_1\left(\frac{p}{3w_1}\right)^{3/2}\tilde{x}_2^{1/2} - w_2\tilde{x}_2$$

Long-Run Profit-Maximization

$$\Pi = py^* - w_1x_1^* - w_2\tilde{x}_2$$

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$$= p\left(\frac{p}{3w_1}\right)^{1/2}\tilde{x}_2^{1/2} - w_1\frac{p}{3w_1}\left(\frac{p}{3w_1}\right)^{1/2} - w_2\tilde{x}_2$$

Long-Run Profit-Maximization

$$\Pi = py^* - w_1 x_1^* - w_2 \tilde{x}_2$$

$$= p\left(\frac{p}{3w_1}\right)^{1/2} \tilde{x}_2^{1/2} - w_1 \left(\frac{p}{3w_1}\right)^{3/2} \tilde{x}_2^{1/2} - w_2 \tilde{x}_2$$

$$= p\left(\frac{p}{3w_1}\right)^{1/2} \tilde{x}_2^{1/2} - w_1 \frac{p}{3w_1} \left(\frac{p}{3w_1}\right)^{1/2} - w_2 \tilde{x}_2$$

$$= \frac{2p}{3}\left(\frac{p}{3w_1}\right)^{1/2} \tilde{x}_2^{1/2} - w_2 \tilde{x}_2$$

Long-Run Profit-Maximization

$$\Pi = py^* - w_1x_1^* - w_2\tilde{x}_2$$

$$= p\left(\frac{p}{3w_1}\right)^{1/2}\tilde{x}_2^{1/2} - w_1\left(\frac{p}{3w_1}\right)^{3/2}\tilde{x}_2^{1/2} - w_2\tilde{x}_2$$

$$= p\left(\frac{p}{3w_1}\right)^{1/2}\tilde{x}_2^{1/2} - w_1\frac{p}{3w_1}\left(\frac{p}{3w_1}\right)^{1/2} - w_2\tilde{x}_2$$

$$= \frac{2p}{3}\left(\frac{p}{3w_1}\right)^{1/2}\tilde{x}_2^{1/2} - w_2\tilde{x}_2$$

$$= \left(\frac{4p^3}{27w_1}\right)^{1/2}\tilde{x}_2^{1/2} - w_2\tilde{x}_2.$$

Long-Run Profit-Maximization

$$\Pi = \left(\frac{4p^3}{27w_1}\right)^{1/2} \tilde{x}_2^{1/2} - w_2\tilde{x}_2.$$

What is the long-run profit-maximizing level of input 2? Solve

$$0 = \frac{\partial \Pi}{\partial \tilde{x}_2} = \frac{1}{2}\left(\frac{4p^3}{27w_1}\right)^{1/2} \tilde{x}_2^{-1/2} - w_2$$

to get $$\tilde{x}_2 = x_2^* = \frac{p^3}{27w_1w_2^2}.$$

Long-Run Profit-Maximization

What is the long-run profit-maximizing input 1 level? Substitute

$$x_2^* = \frac{p^3}{27w_1w_2^2} \quad \text{into} \quad x_1^* = \left(\frac{p}{3w_1}\right)^{3/2} \tilde{x}_2^{1/2}$$

to get

Long-Run Profit-Maximization

What is the long-run profit-maximizing input 1 level? Substitute

$$x_2^* = \frac{p^3}{27w_1w_2^2} \quad \text{into} \quad x_1^* = \left(\frac{p}{3w_1}\right)^{3/2} \tilde{x}_2^{1/2}$$

to get

$$x_1^* = \left(\frac{p}{3w_1}\right)^{3/2} \left(\frac{p^3}{27w_1w_2^2}\right)^{1/2} = \frac{p^3}{27w_1^2w_2}.$$

Long-Run Profit-Maximization

What is the long-run profit-maximizing output level? Substitute

$$x_2^* = \frac{p^3}{27w_1w_2^2} \quad \text{into} \quad y^* = \left(\frac{p}{3w_1}\right)^{1/2} \tilde{x}_2^{1/2}$$

to get

Long-Run Profit-Maximization

What is the long-run profit-maximizing output level? Substitute

$$x_2^* = \frac{p^3}{27w_1w_2^2} \quad \text{into} \quad y^* = \left(\frac{p}{3w_1}\right)^{1/2} \tilde{x}_2^{1/2}$$

to get

$$y^* = \left(\frac{p}{3w_1}\right)^{1/2} \left(\frac{p^3}{27w_1w_2^2}\right)^{1/2} = \frac{p^2}{9w_1w_2}.$$

Long-Run Profit-Maximization

So given the prices p, w_1 and w_2, and the production function $y = x_1^{1/3} x_2^{1/3}$

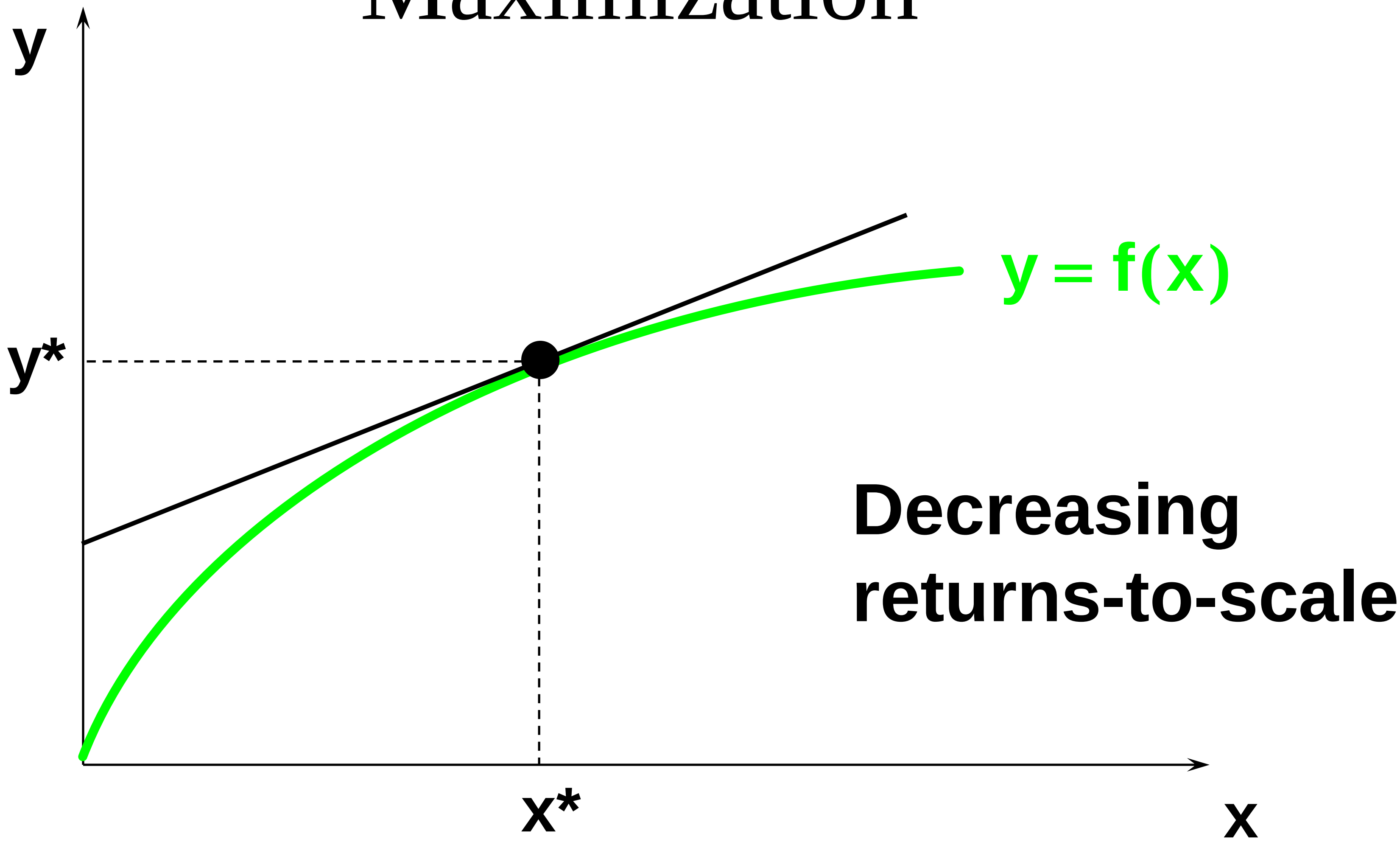
the long-run profit-maximizing production plan is

$$(x_1^*, x_2^*, y^*) = \left(\frac{p^3}{27 w_1^2 w_2}, \frac{p^3}{27 w_1 w_2^2}, \frac{p^2}{9 w_1 w_2} \right).$$

Returns-to-Scale and Profit-Maximization

- **If a competitive firm's technology exhibits decreasing returns-to-scale then the firm has a single long-run profit-maximizing production plan.**

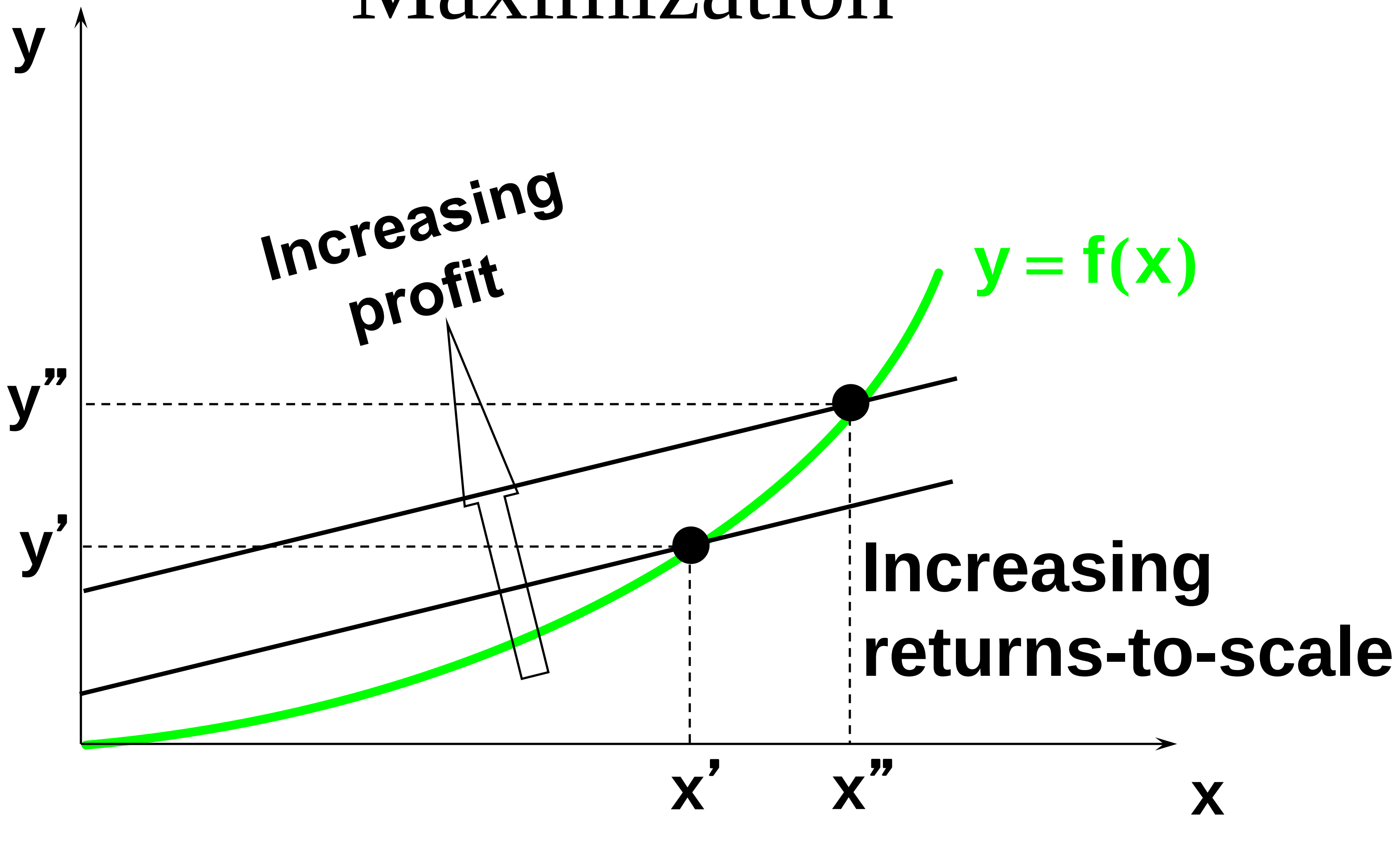
Returns-to Scale and Profit-Maximization



Returns-to-Scale and Profit-Maximization

- **If a competitive firm's technology exhibits exhibits increasing returns-to-scale then the firm does not have a profit-maximizing plan.**

Returns-to Scale and Profit-Maximization



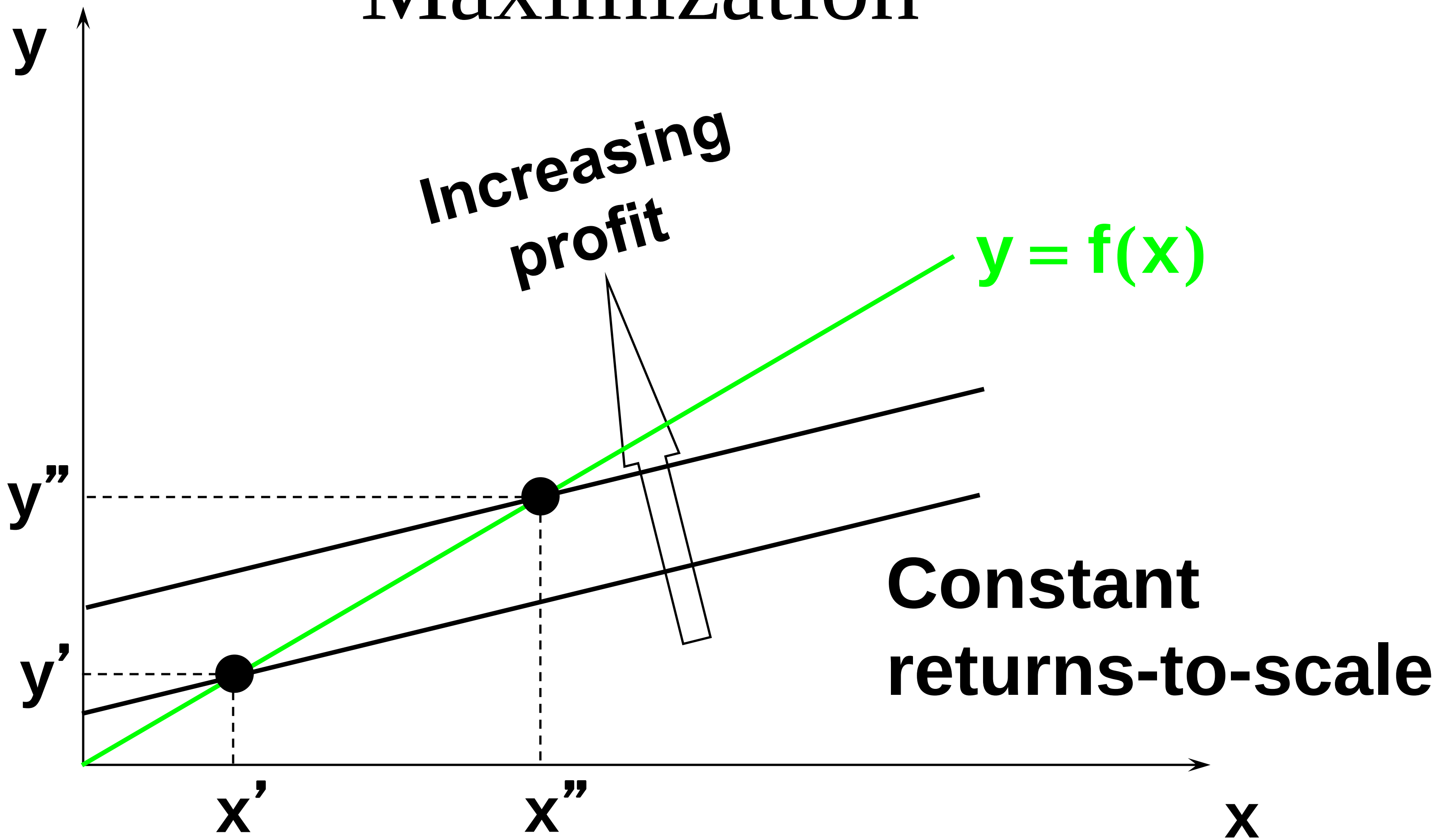
Returns-to-Scale and Profit-Maximization

- **So an increasing returns-to-scale technology is inconsistent with firms being perfectly competitive.**

Returns-to-Scale and Profit-Maximization

- **What if the competitive firm's technology exhibits constant returns-to-scale?**

Returns-to Scale and Profit-Maximization



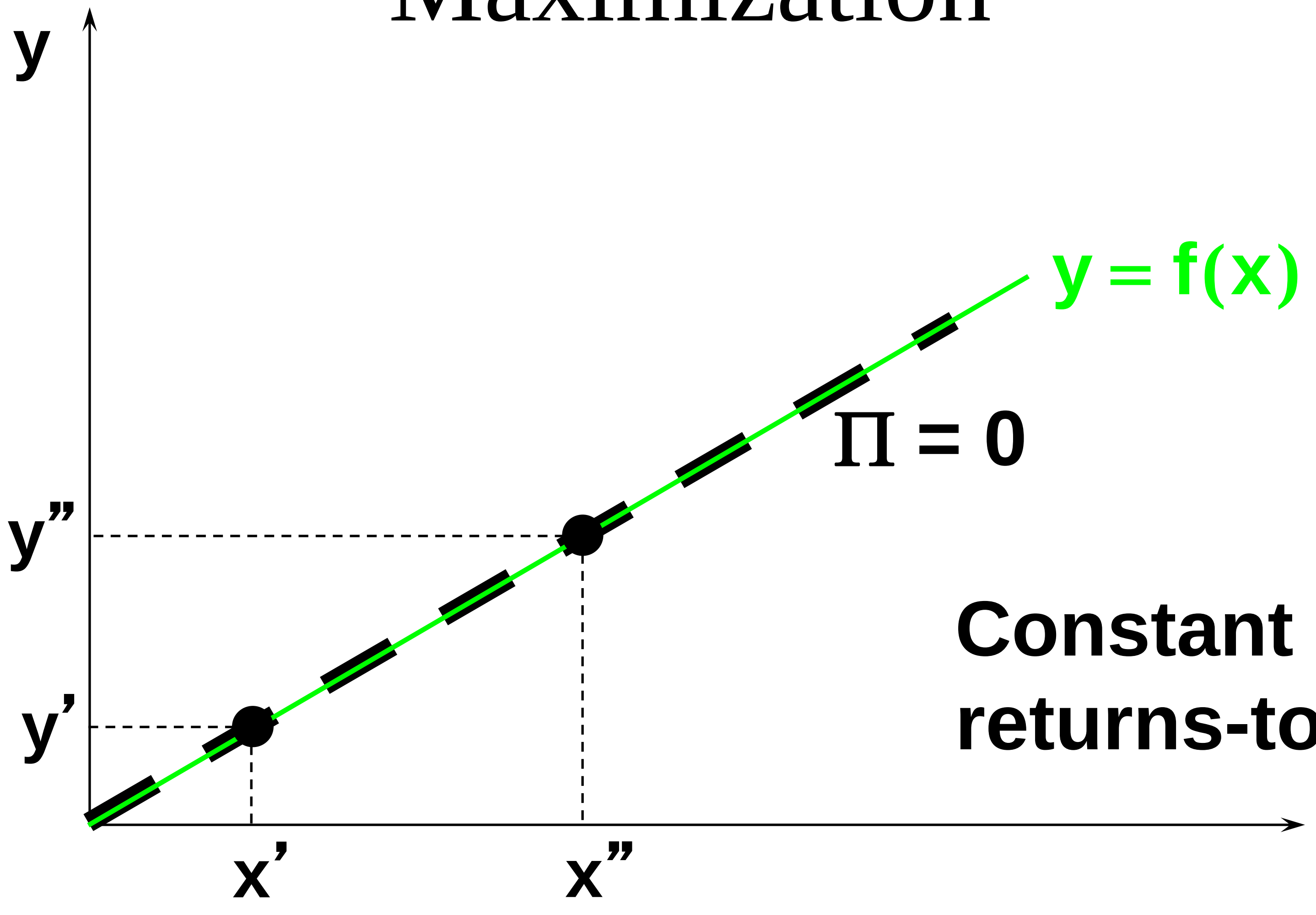
Returns-to Scale and Profit-Maximization

- **So if any production plan earns a positive profit, the firm can double up all inputs to produce twice the original output and earn twice the original profit.**

Returns-to Scale and Profit-Maximization

- Therefore, when a firm’s technology exhibits constant returns-to-scale, earning a positive economic profit is inconsistent with firms being perfectly competitive.
- Hence constant returns-to-scale requires that competitive firms earn economic profits of zero.

Returns-to Scale and Profit-Maximization



Constant returns-to-scale

Revealed Profitability

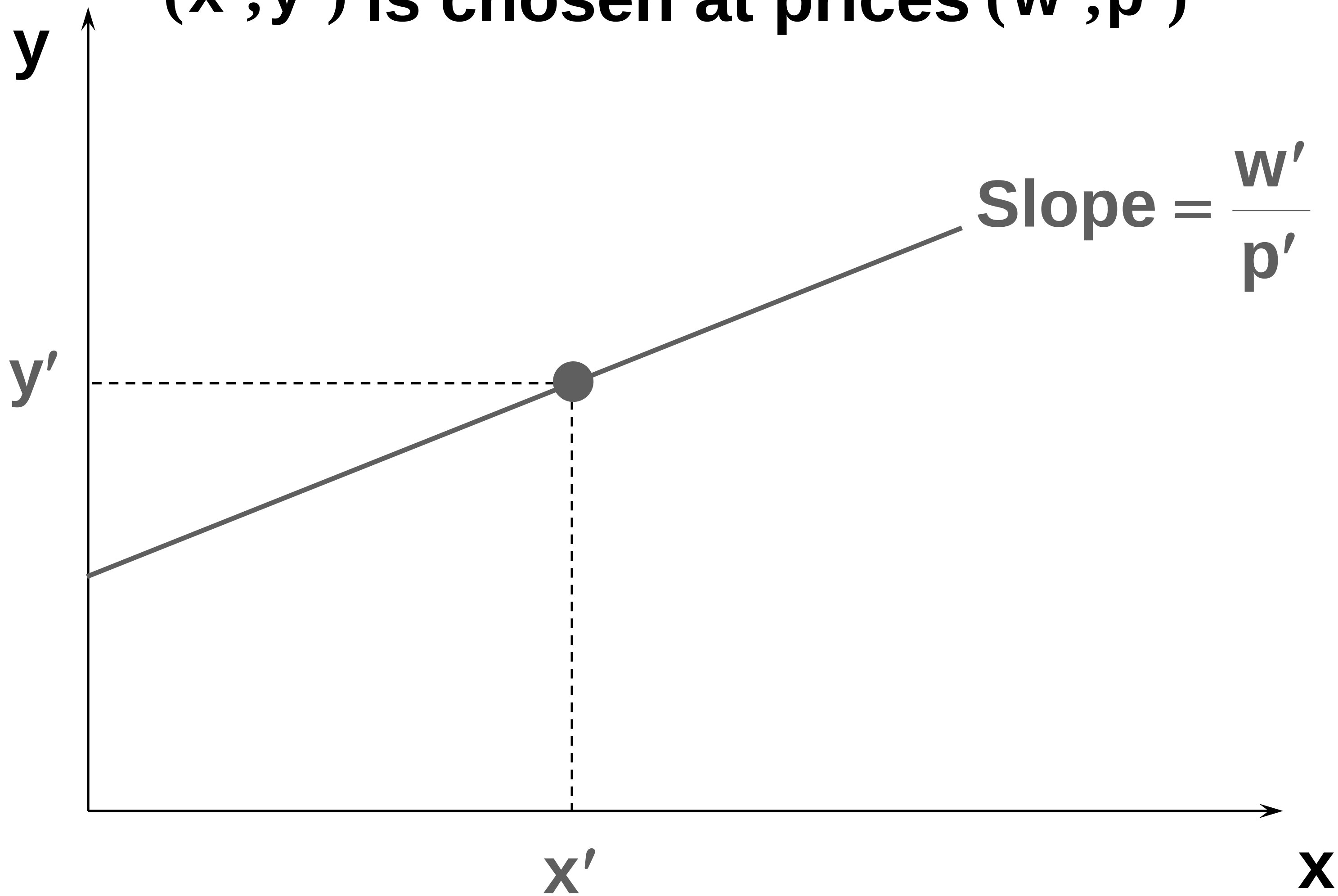
- **Consider a competitive firm with a technology that exhibits decreasing returns-to-scale.**
- **For a variety of output and input prices we observe the firm's choices of production plans.**
- **What can we learn from our observations?**

Revealed Profitability

- **If a production plan (x',y') is chosen at prices (w',p') we deduce that the plan (x',y') is revealed to be profit-maximizing for the prices (w',p').**

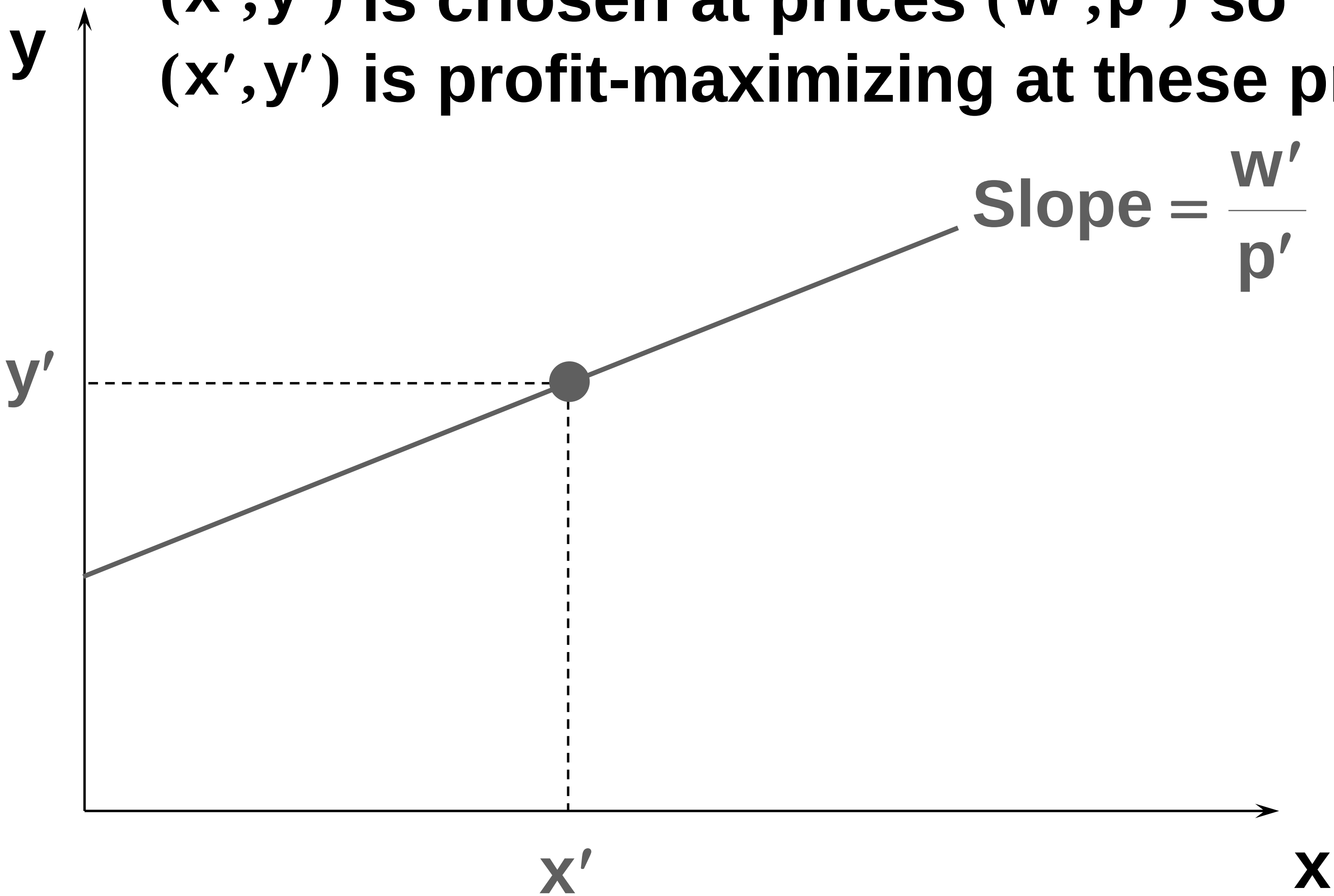
Revealed Profitability

(x',y') is chosen at prices (w',p')



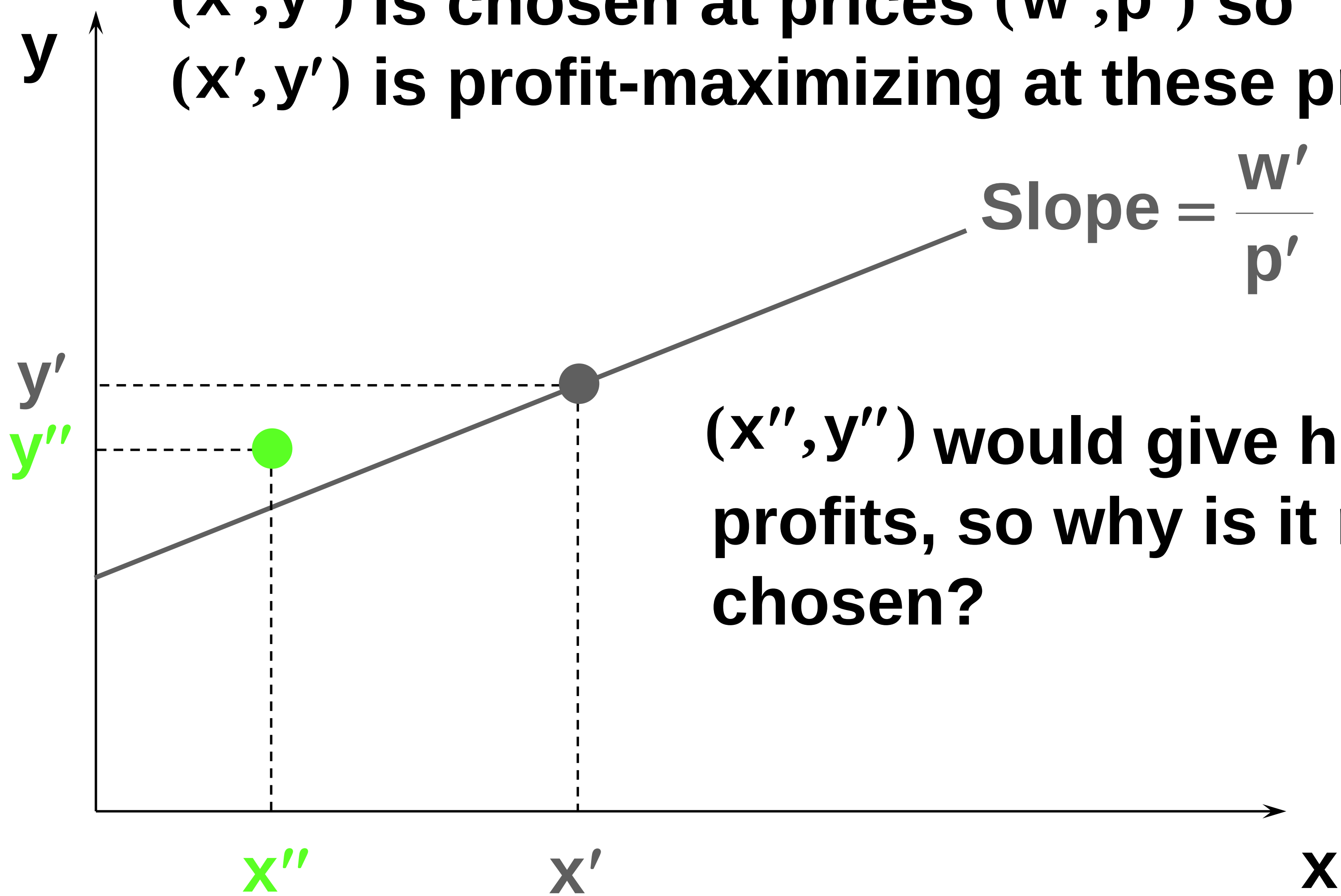
Revealed Profitability

(x', y') **is chosen at prices** (w', p') **so** (x', y') **is profit-maximizing at these prices.**



Revealed Profitability

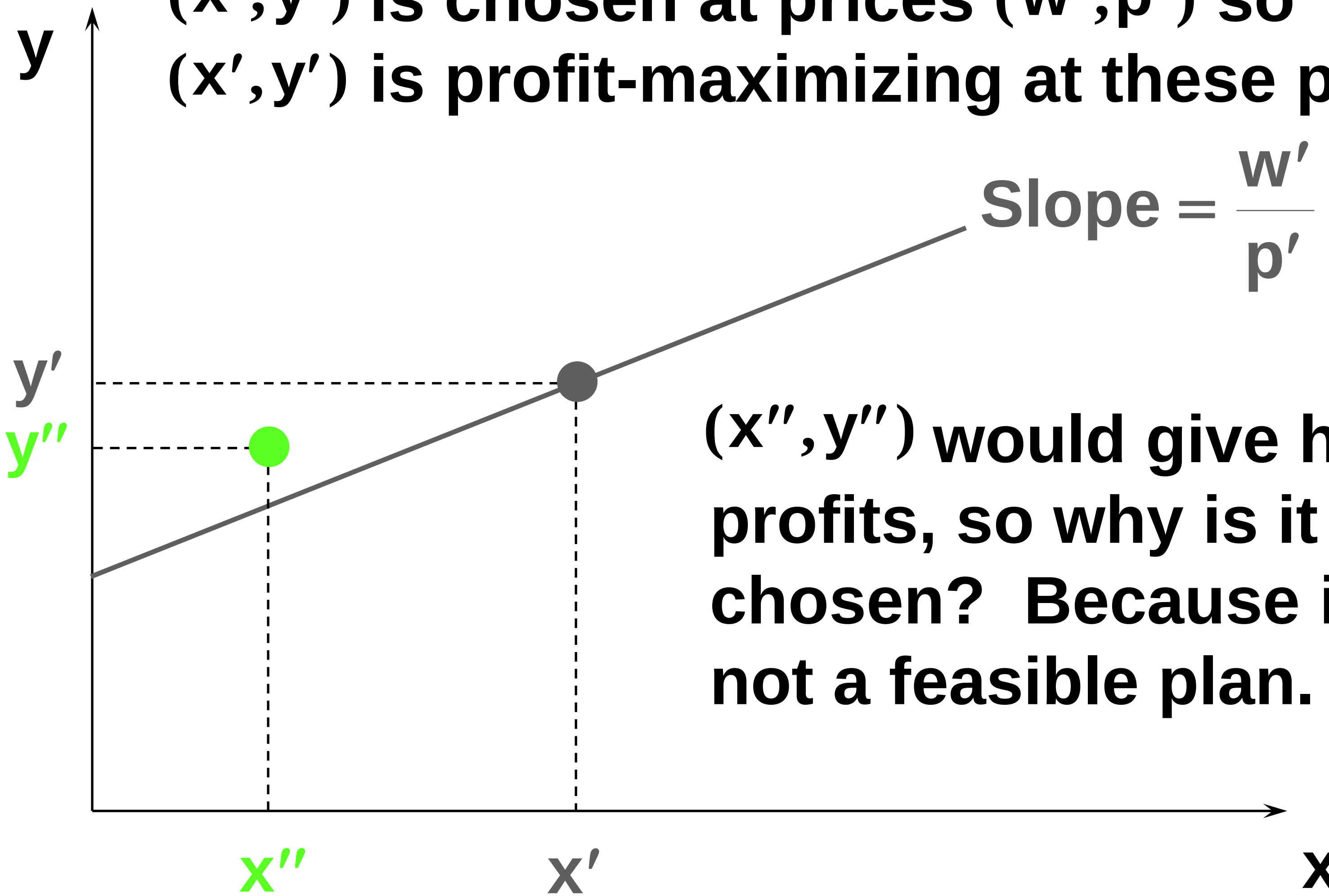
(x', y') **is chosen at prices** (w', p') **so** (x', y') **is profit-maximizing at these prices.**



(x'', y'') **would give higher profits, so why is it not chosen?**

Revealed Profitability

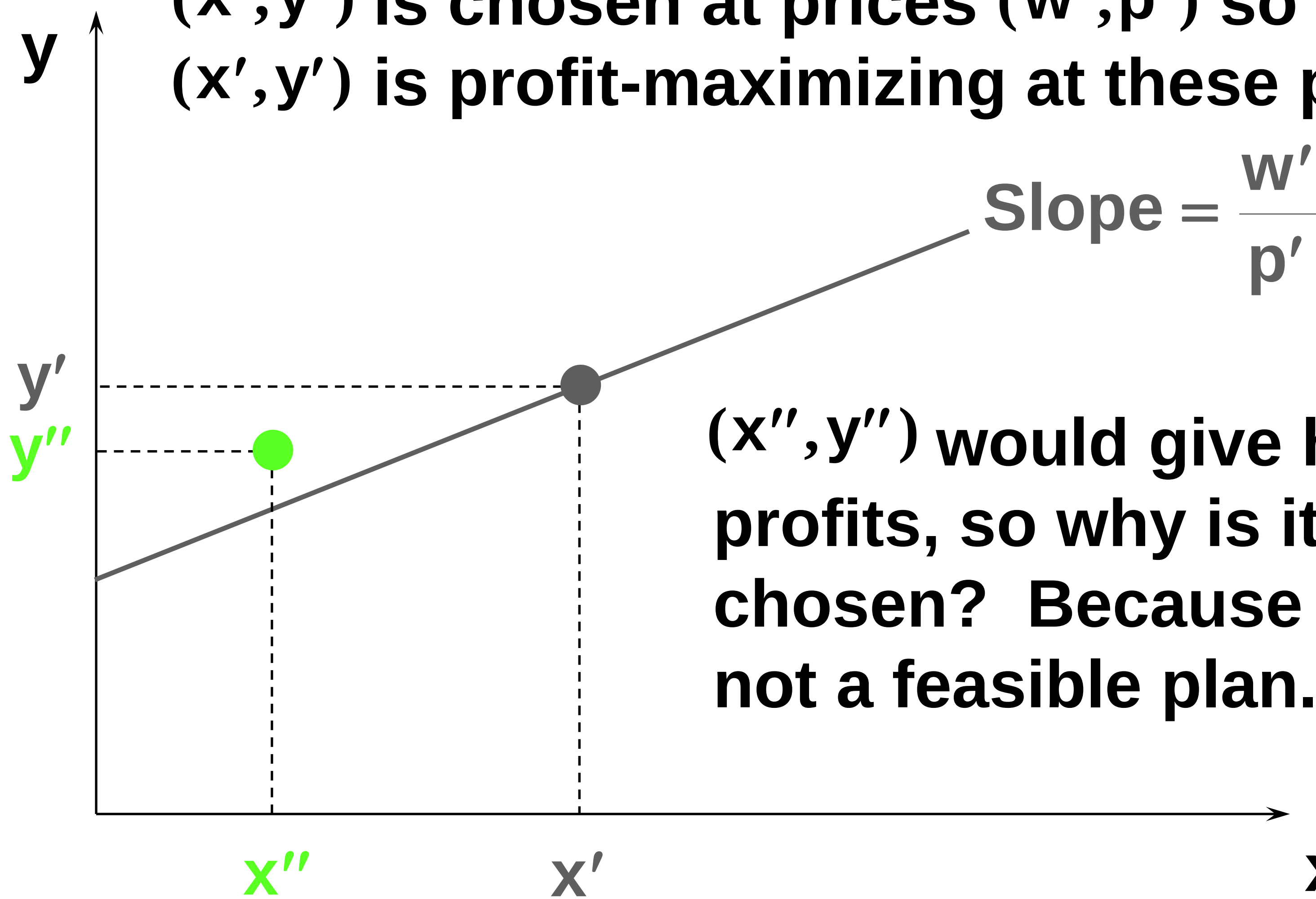
(x',y') is chosen at prices (w',p') so (x',y') is profit-maximizing at these prices.



(x'',y'') would give higher profits, so why is it not chosen? Because it is not a feasible plan.

Revealed Profitability

(x',y') is chosen at prices (w',p') so (x',y') is profit-maximizing at these prices.

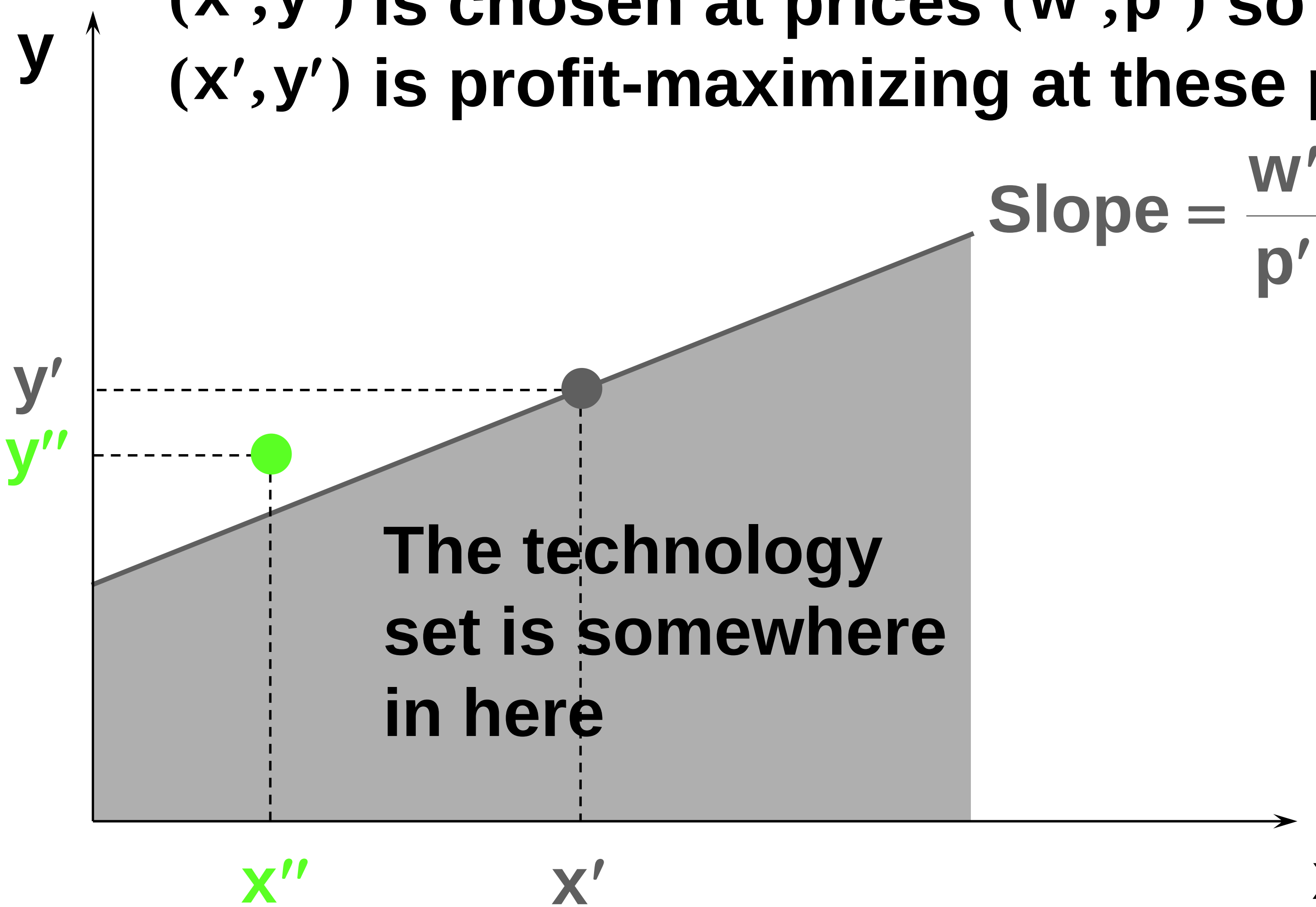


(x'',y'') would give higher profits, so why is it not chosen? Because it is not a feasible plan.

So the firm's technology set must lie under the iso-profit line.

Revealed Profitability

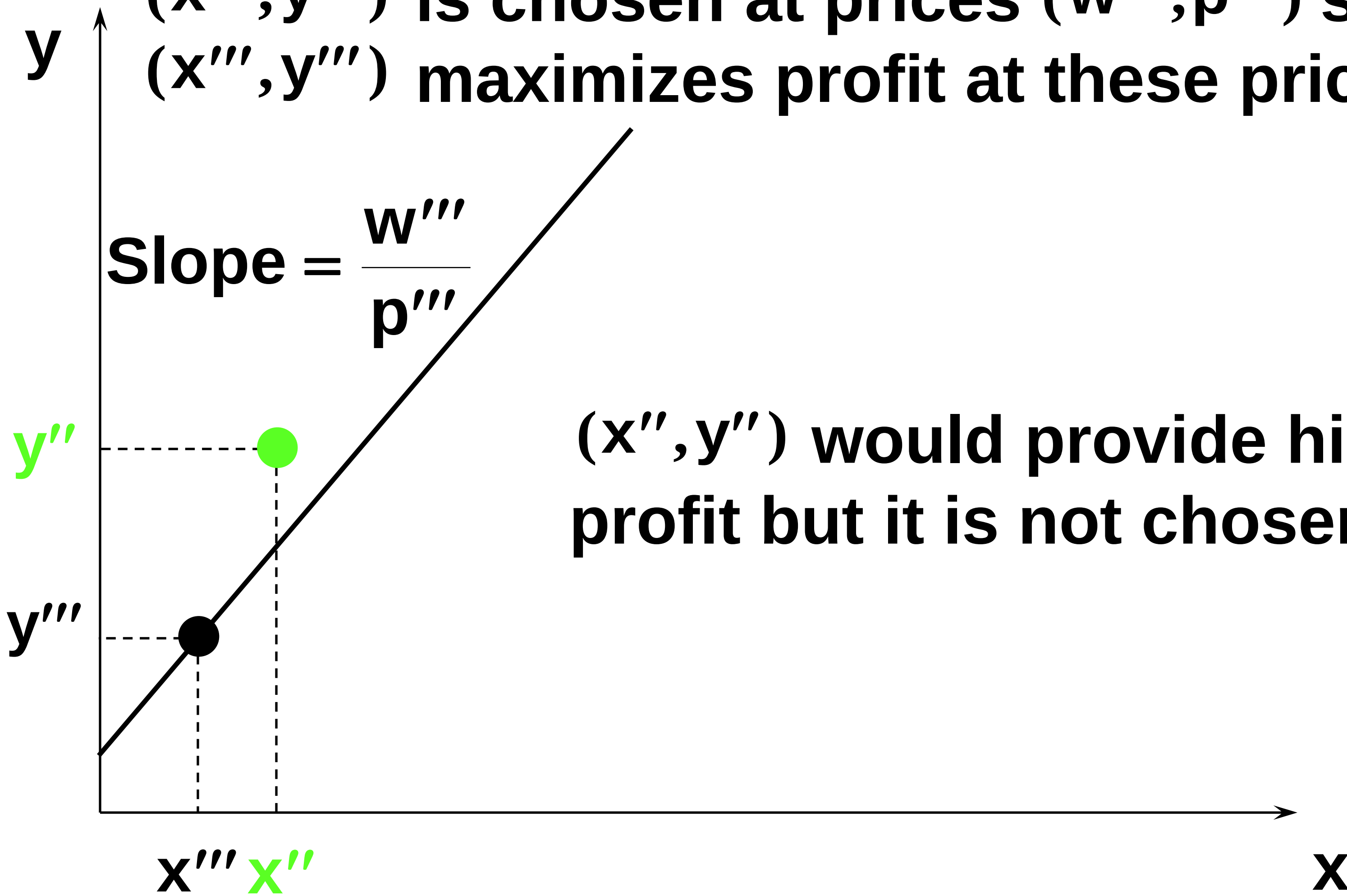
(x′,y′) is chosen at prices (w′,p′) so (x′,y′) is profit-maximizing at these prices.



So the firm's technology set must lie under the iso-profit line.

Revealed Profitability

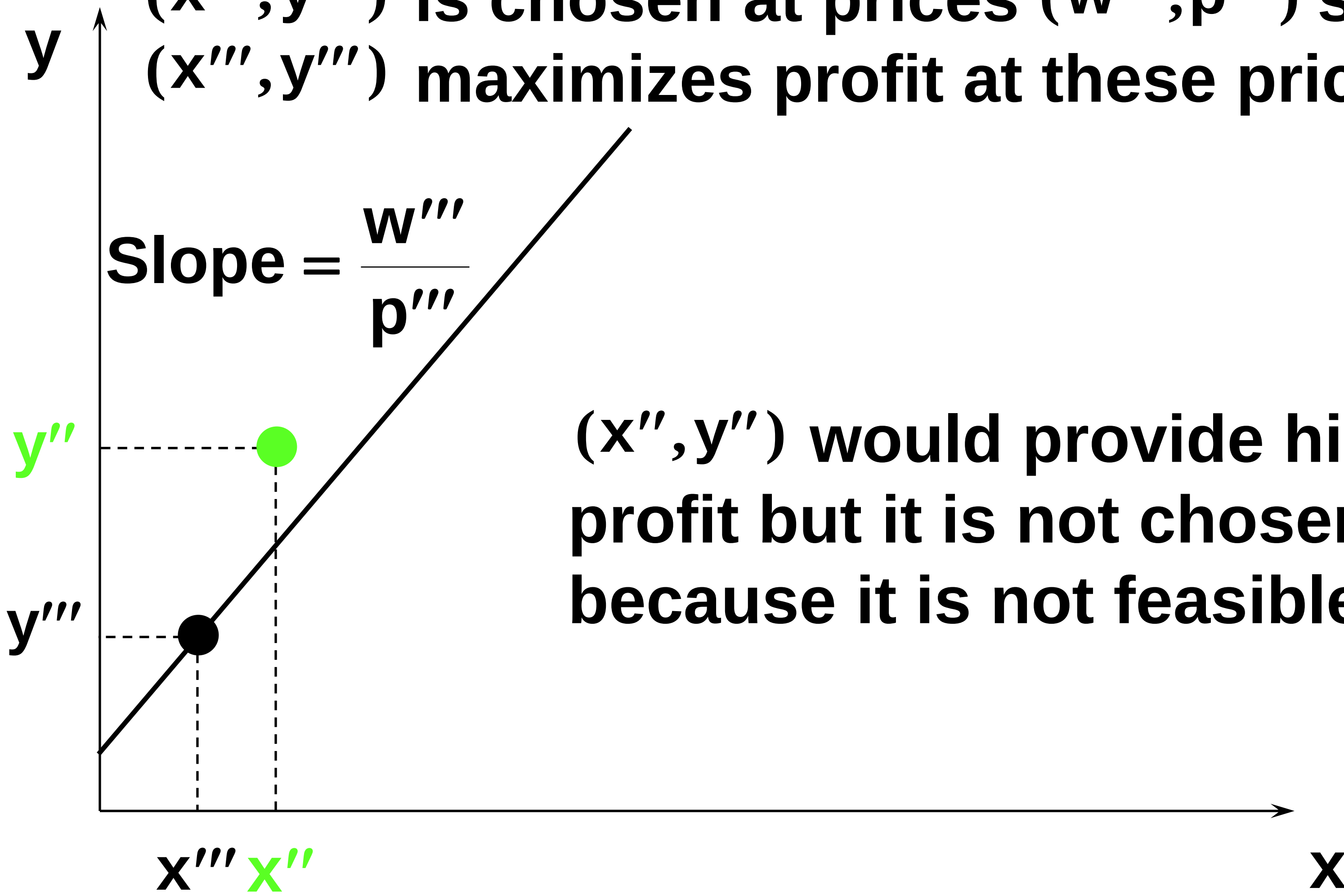
(x''', y''') **is chosen at prices** (w''', p''') **so** (x''', y''') **maximizes profit at these prices.**



(x'', y'') **would provide higher profit but it is not chosen**

Revealed Profitability

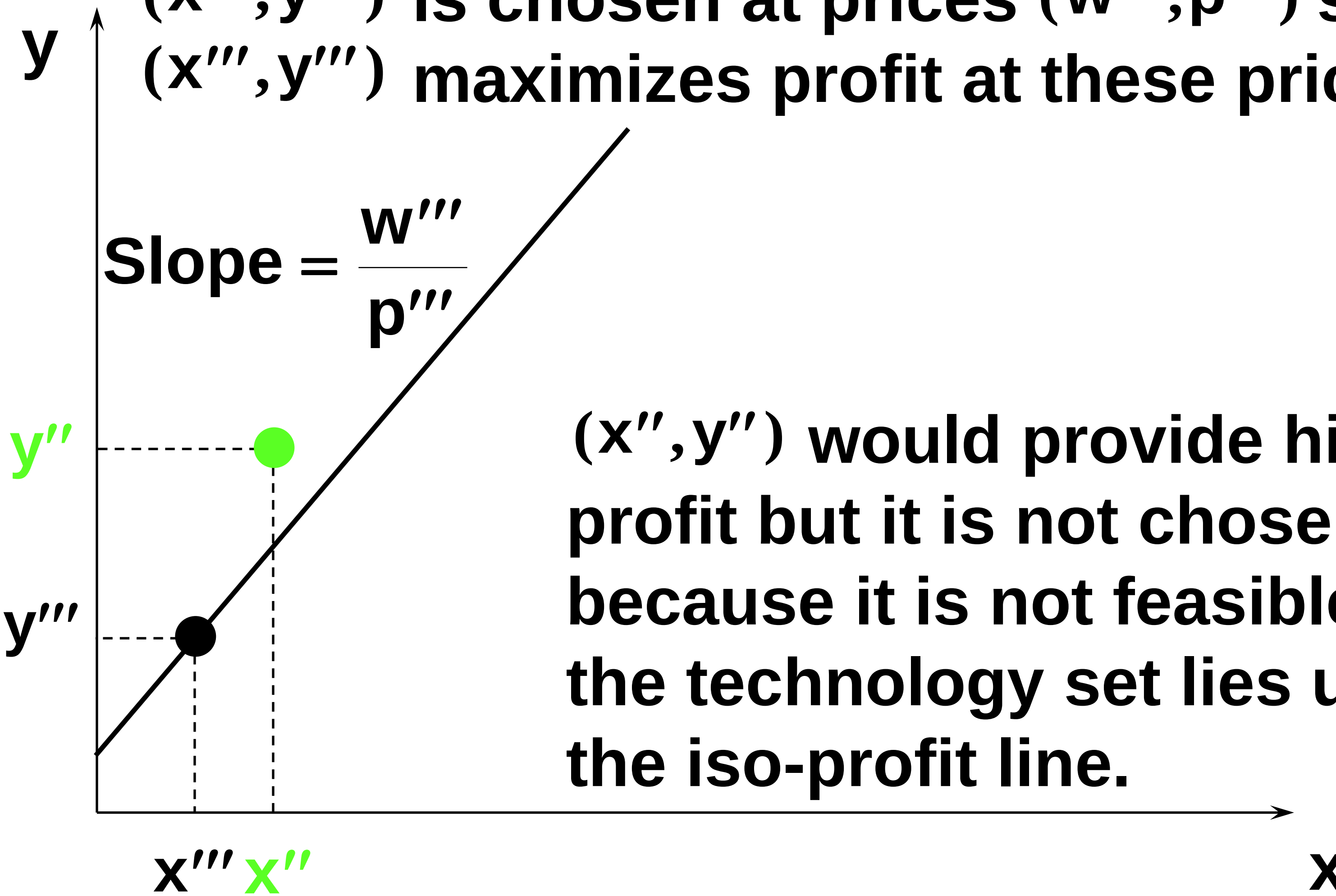
(x''', y''') is chosen at prices (w''', p''') so (x''', y''') maximizes profit at these prices.



(x'', y'') would provide higher profit but it is not chosen because it is not feasible

Revealed Profitability

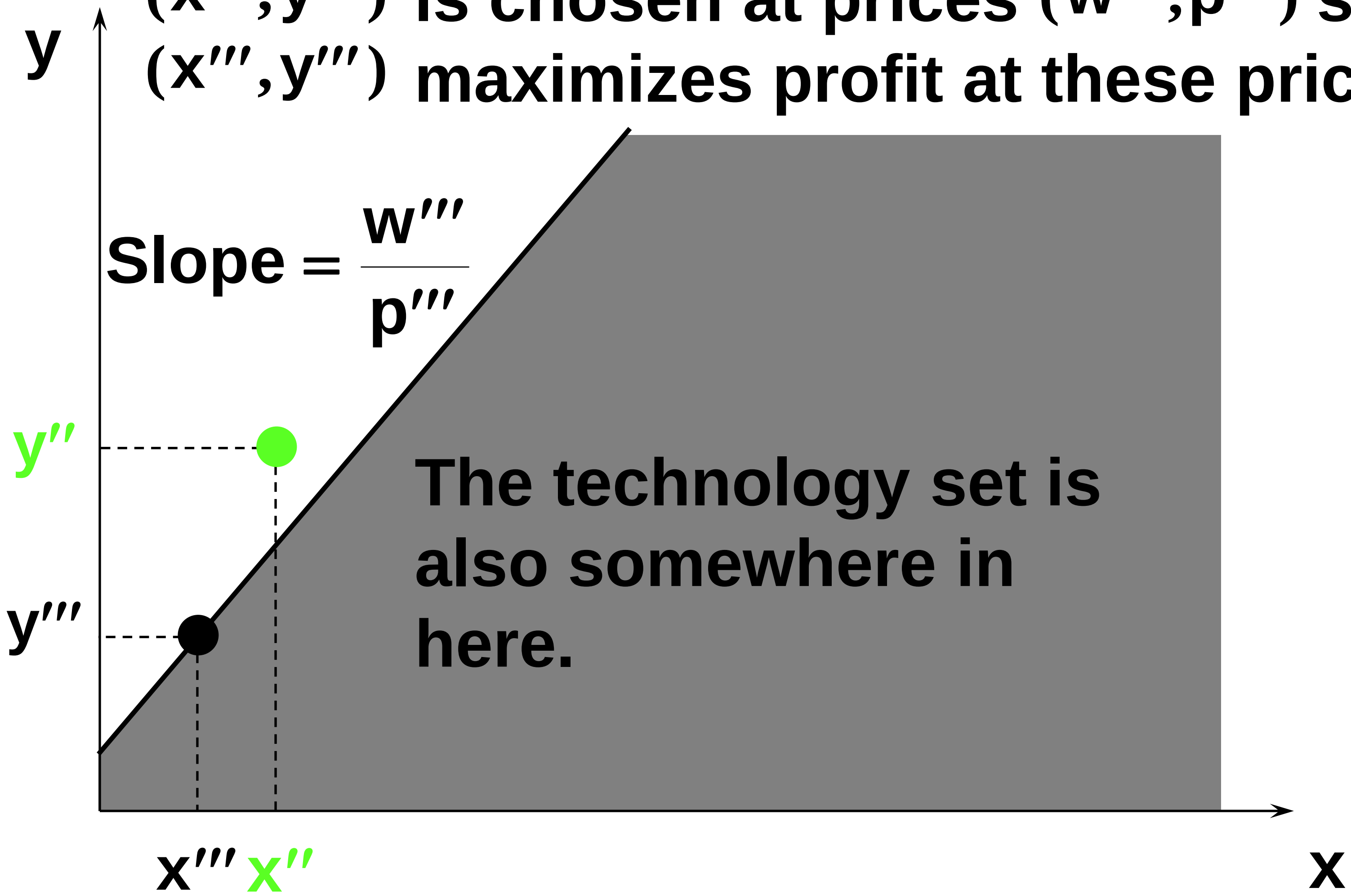
(x''',y''') is chosen at prices (w''',p''') so (x''',y''') maximizes profit at these prices.



(x'',y'') would provide higher profit but it is not chosen because it is not feasible so the technology set lies under the iso-profit line.

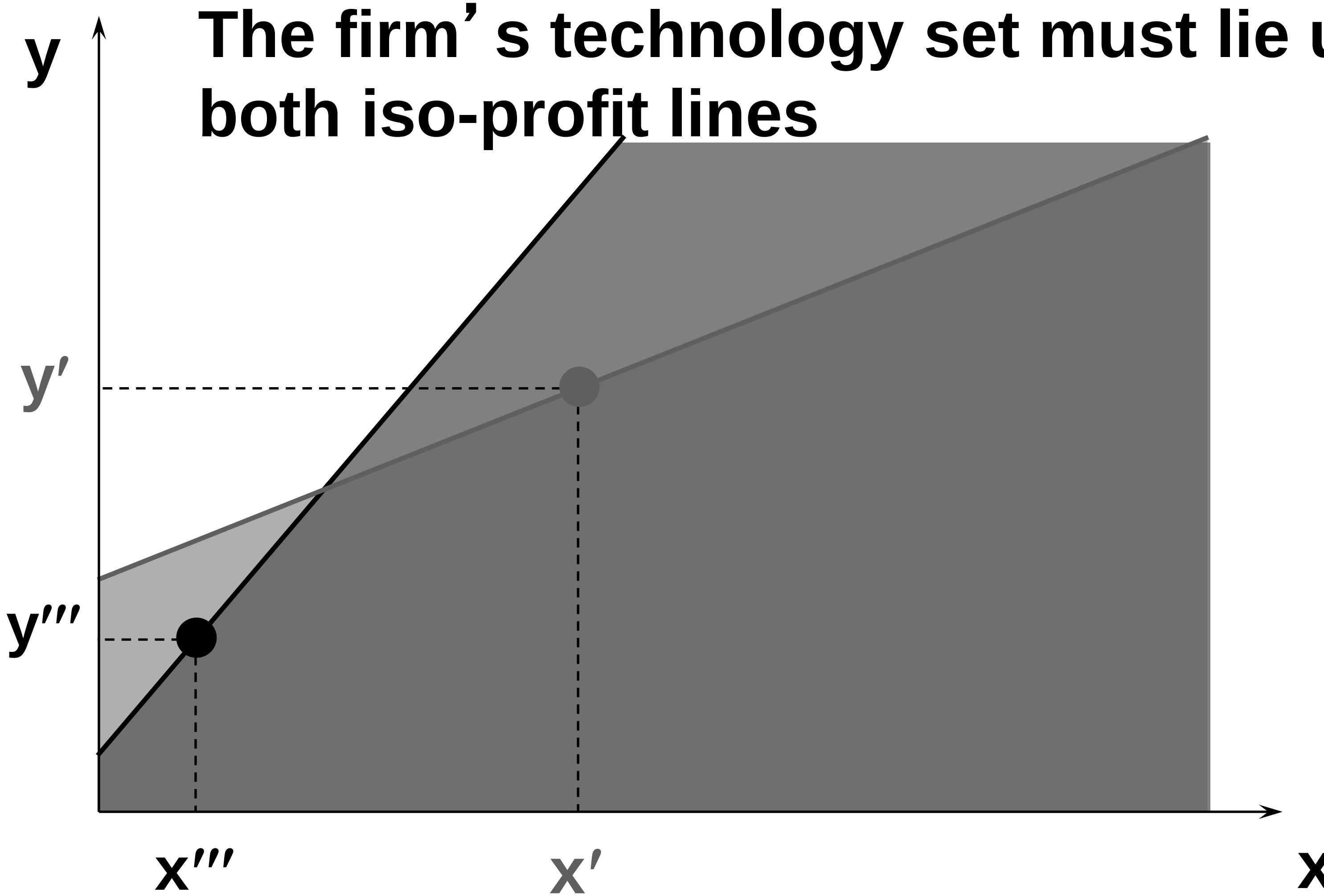
Revealed Profitability

(x''', y''') is chosen at prices (w''', p''') so (x''', y''') maximizes profit at these prices.

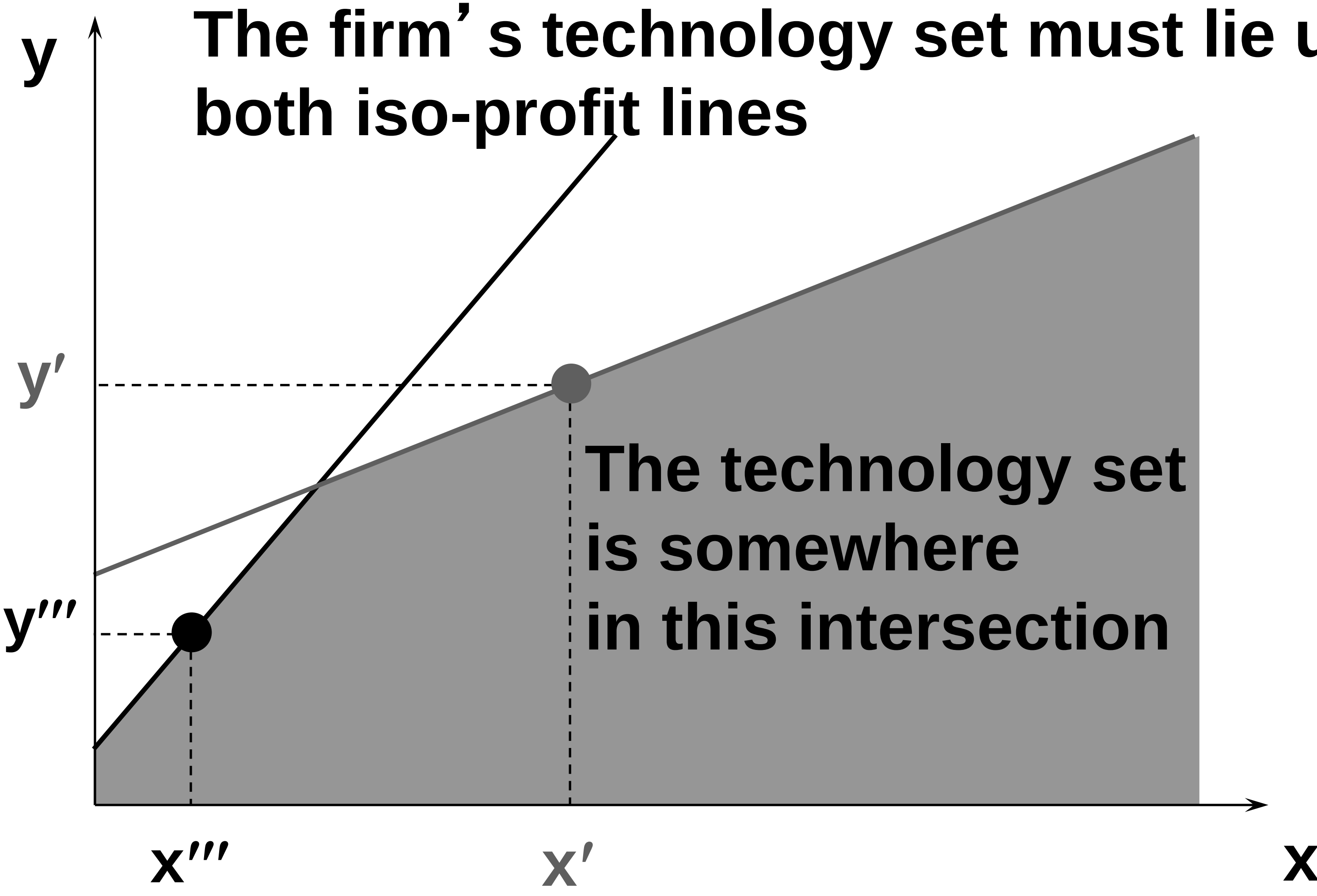


Revealed Profitability

The firm's technology set must lie under both iso-profit lines



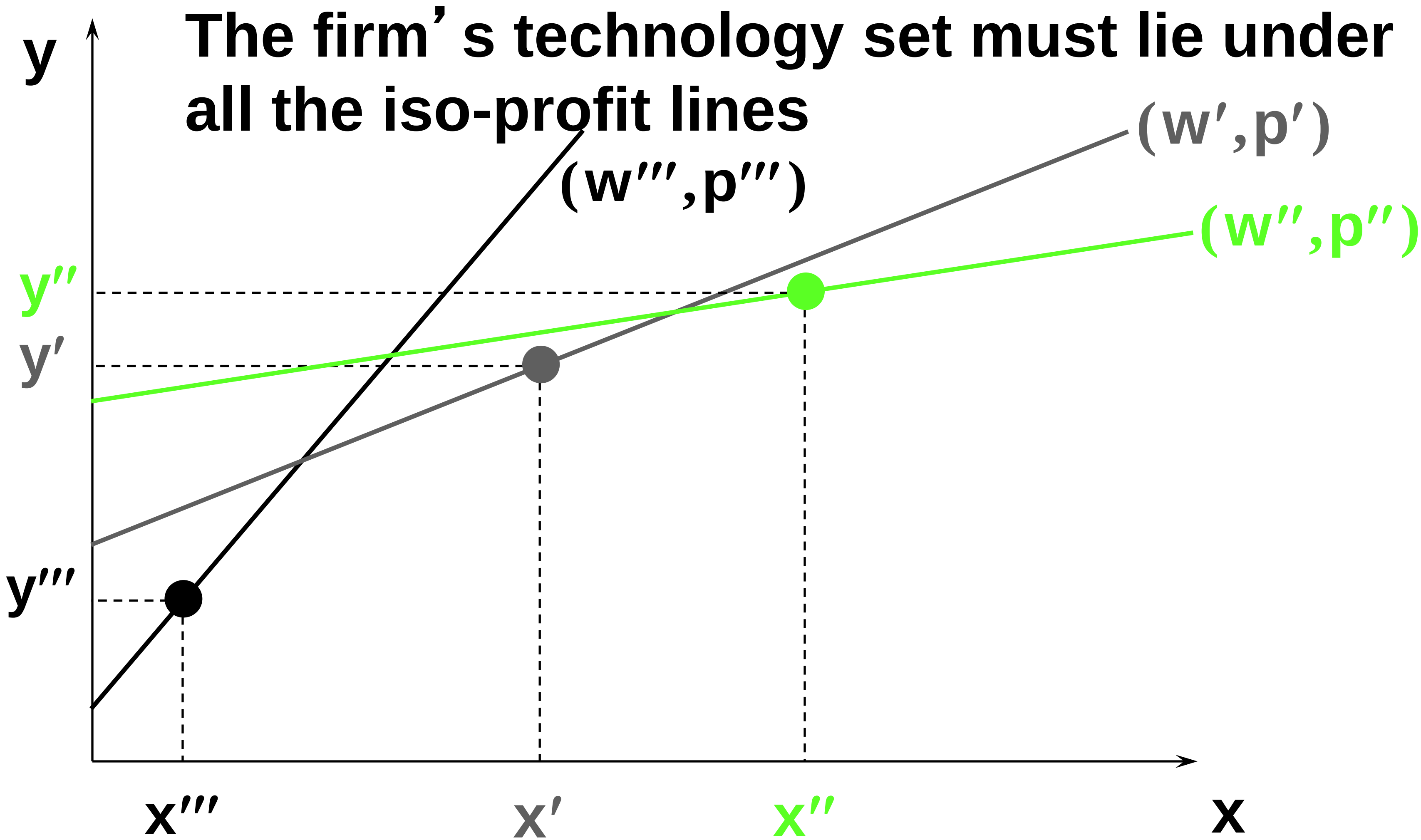
Revealed Profitability



Revealed Profitability

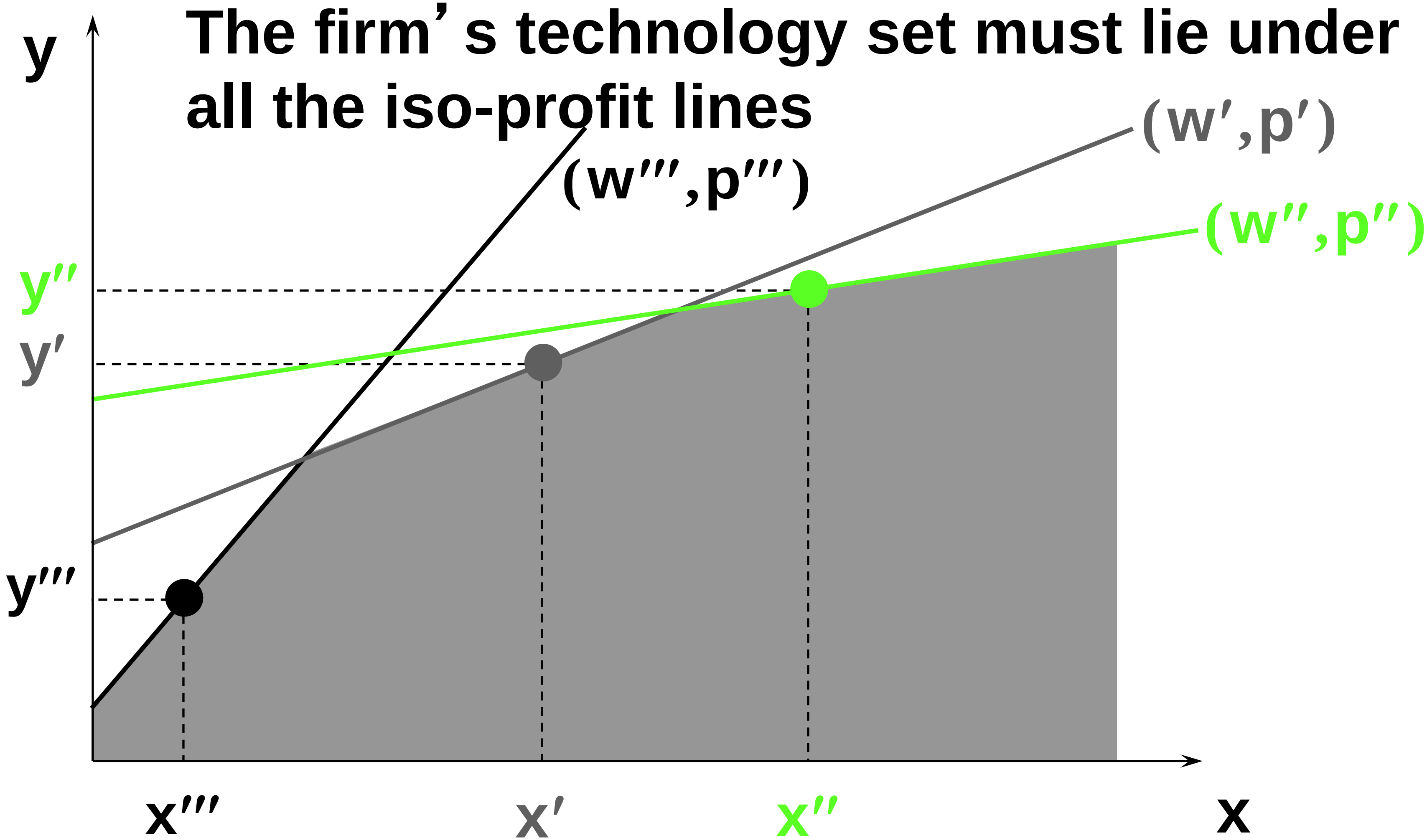
- **Observing more choices of production plans by the firm in response to different prices for its input and its output gives more information on the location of its technology set.**

Revealed Profitability



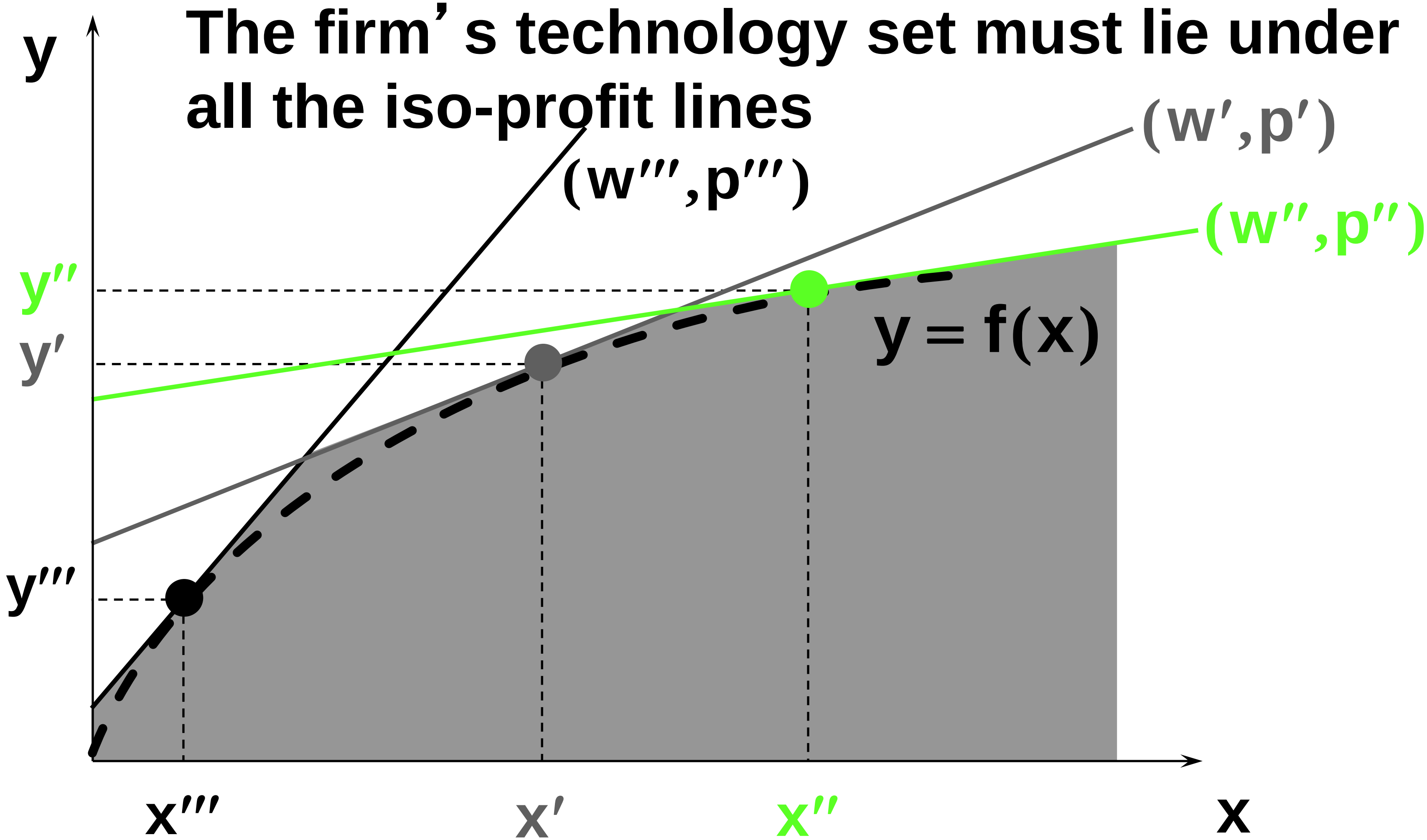
Revealed Profitability

The firm's technology set must lie under all the iso-profit lines



Revealed Profitability

The firm's technology set must lie under all the iso-profit lines

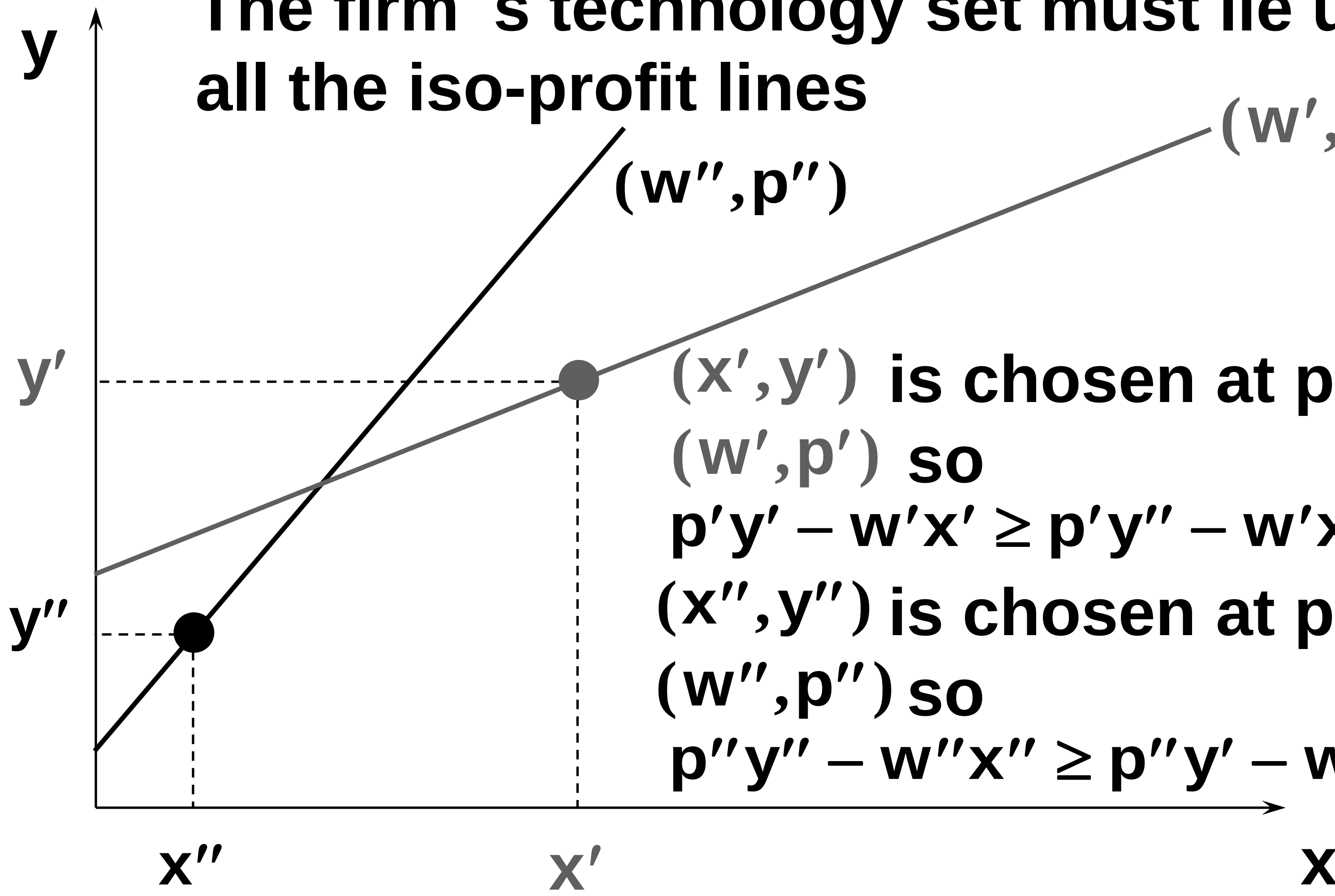


Revealed Profitability

- **What else can be learned from the firm’s choices of profit-maximizing production plans?**

Revealed Profitability

The firm's technology set must lie under all the iso-profit lines



(x', y') is chosen at prices (w', p') so

$$p'y' - w'x' \geq p'y'' - w'x''.$$

(x'', y'') is chosen at prices (w'', p'') so

$$p''y'' - w''x'' \geq p''y' - w''x'.$$

Revealed Profitability

$$p'y' - w'x' \geq p'y'' - w'x''$$ and

$$p''y'' - w''x'' \geq p''y' - w''x'$$ so

$$p'y' - w'x' \geq p'y'' - w'x''$$ and

$$-p''y' + w''x' \geq -p''y'' + w''x''$$

Adding gives

$$(p' - p'')y' - (w' - w'')x' \geq$$

$$(p' - p'')y'' - (w' - w'')x''$$

Revealed Profitability

$$(p' - p'')y' - (w' - w'')x' \geq (p' - p'')y'' - (w' - w'')x''$$

so

$$(p' - p'')(y' - y'') \geq (w' - w'')(x' - x'')$$

That is,

$$\Delta p \Delta y \geq \Delta w \Delta x$$

is a necessary implication of profit-maximization.

Revealed Profitability

$$\Delta p \Delta y \geq \Delta w \Delta x$$

is a necessary implication of profit-maximization.

Suppose the input price does not change. Then $\Delta w = 0$ and profit-maximization implies $\Delta p \Delta y \geq 0$; *i.e.*, a competitive firm's output supply curve cannot slope downward.

Revealed Profitability

$$\Delta p \Delta y \geq \Delta w \Delta x$$

is a necessary implication of profit-maximization.

Suppose the output price does not change. Then $\Delta p = 0$ and profit-maximization implies $0 \geq \Delta w \Delta x$; *i.e.*, a competitive firm's input demand curve cannot slope upward.

Cost Minimization

- A firm is a cost-minimizer if it produces any given output level $y \geq 0$ at smallest possible total cost.
- c(y) denotes the firm's smallest possible total cost for producing y units of output.
- c(y) is the firm's total cost function.

Cost Minimization

- **When the firm faces given input prices w = (w_1,w_2,…,w_n) the total cost function will be written as**

$$c(w_1,\ldots,w_n,y).$$

The Cost-Minimization Problem

- **Consider a firm using two inputs to make one output.**
- **The production function is**

  $$y = f(x_1,x_2).$$

- **Take the output level $y \geq 0$ as given.**
- **Given the input prices w_1 and w_2, the cost of an input bundle (x_1,x_2) is $w_1x_1 + w_2x_2$.**

The Cost-Minimization Problem

- **For given w_1, w_2 and y, the firm's cost-minimization problem is to solve** $\min_{x_1, x_2 \geq 0} w_1 x_1 + w_2 x_2$

  subject to $f(x_1, x_2) = y.$

The Cost-Minimization Problem

- The levels x_1*(w_1,w_2,y) and x_1*(w_1,w_2,y) in the least-costly input bundle are the firm's conditional demands for inputs 1 and 2.
- The (smallest possible) total cost for producing y output units is therefore

$$c(w_1, w_2, y) = w_1 x_1^*(w_1, w_2, y) + w_2 x_2^*(w_1, w_2, y).$$

Conditional Input Demands

- **Given w_1, w_2 and y, how is the least costly input bundle located?**
- **And how is the total cost function computed?**

Iso-cost Lines

- **A curve that contains all of the input bundles that cost the same amount is an iso-cost curve.**
- **E.g., given w_1 and w_2, the $100 iso-cost line has the equation**

$$w_1x_1 + w_2x_2 = 100.$$

Iso-cost Lines

- **Generally, given w_1 and w_2, the equation of the \$c iso-cost line is**

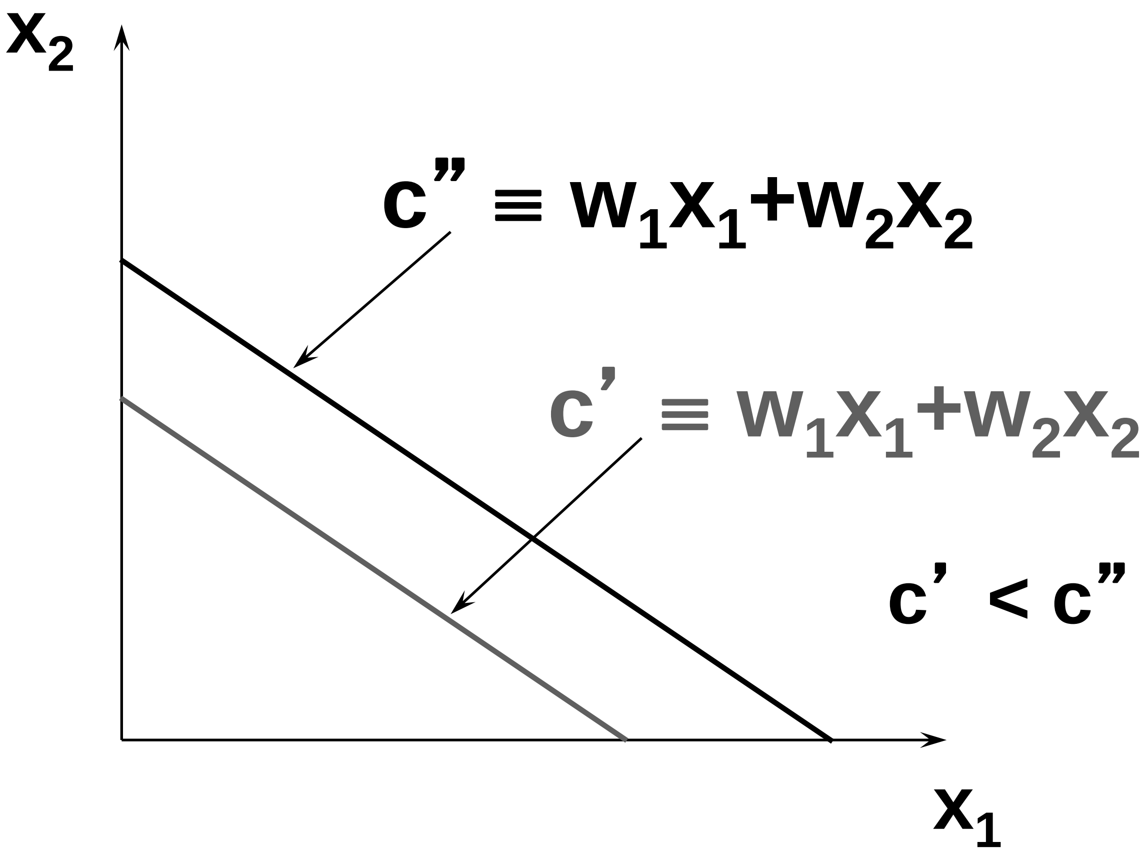
$$w_1x_1 + w_2x_2 = c$$

i.e.

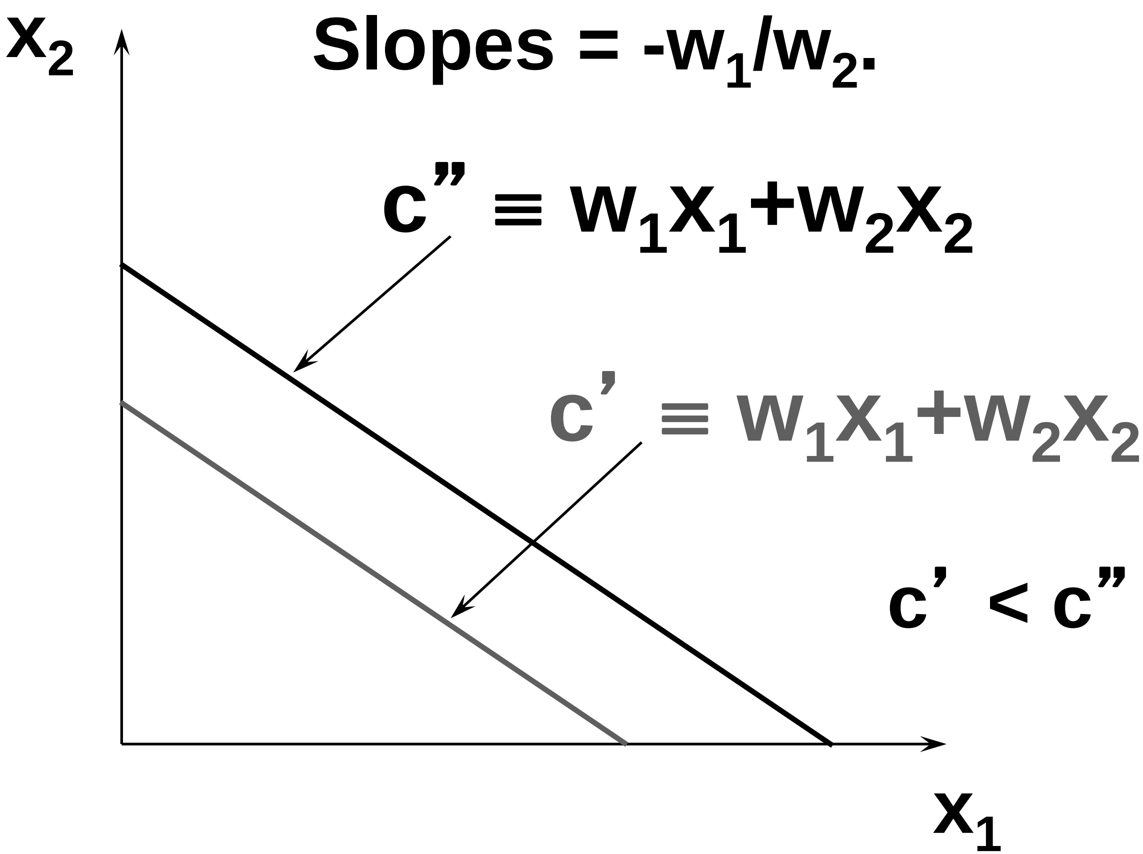
$$x_2 = -\frac{w_1}{w_2}x_1 + \frac{c}{w_2}.$$

- **Slope is - w_1/w_2.**

Iso-cost Lines

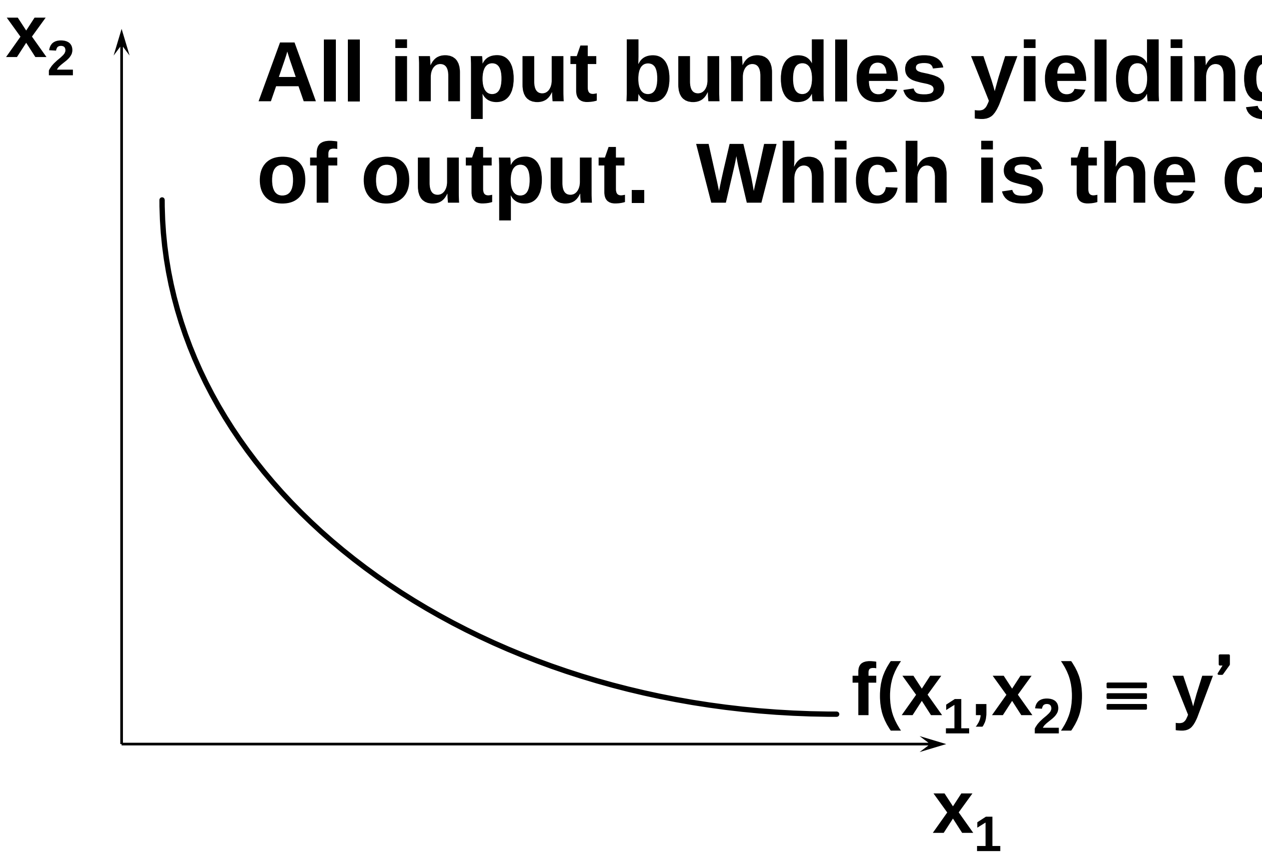


Iso-cost Lines



The y' -Output Unit Isoquant

All input bundles yielding y' units of output. Which is the cheapest?



The Cost-Minimization Problem

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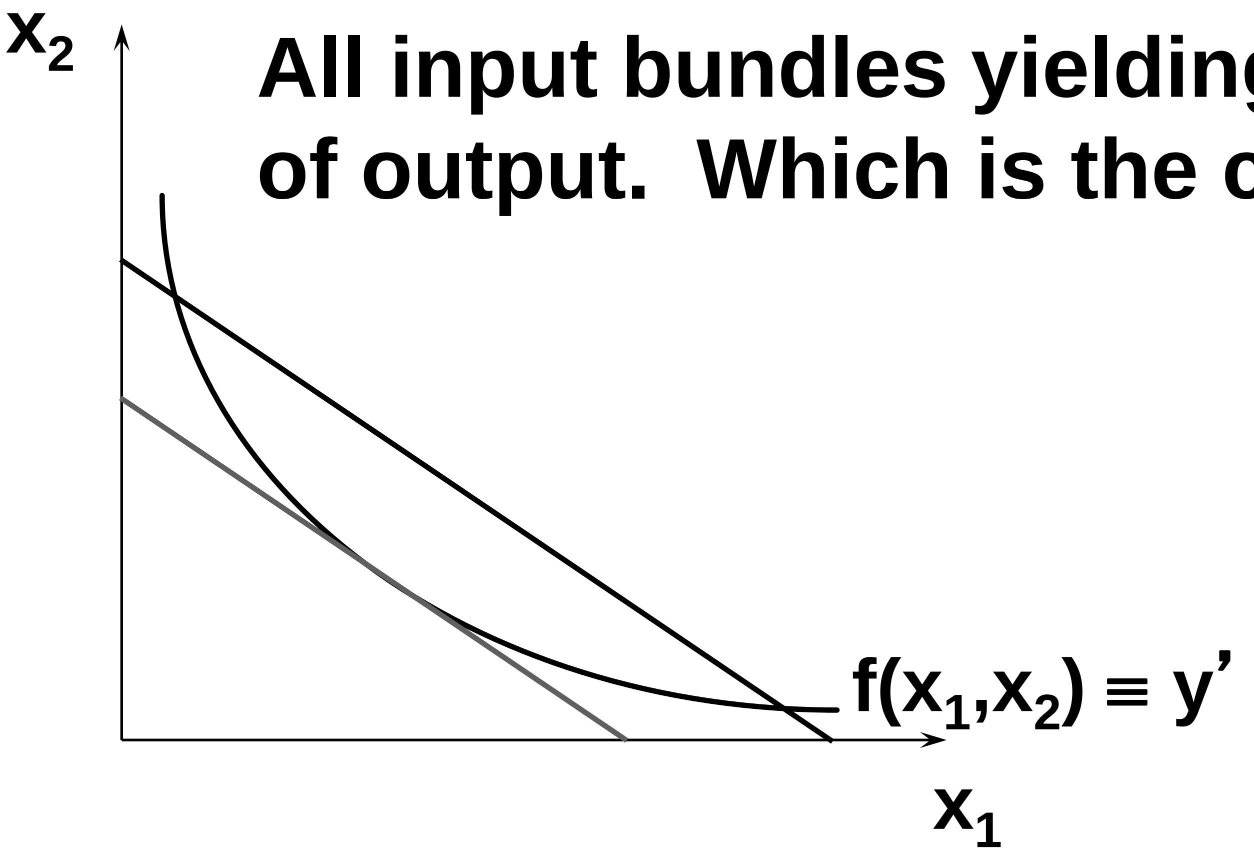

The Cost-Minimization Problem

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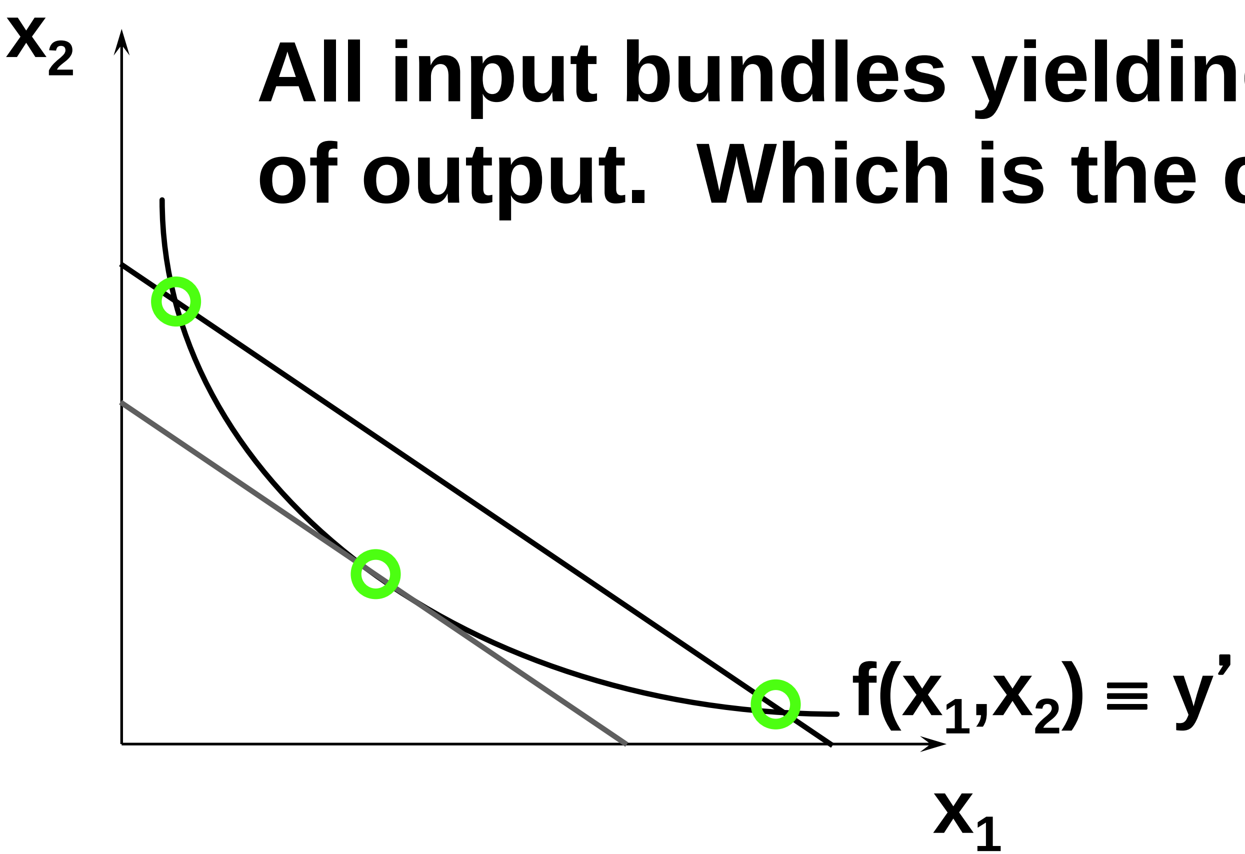
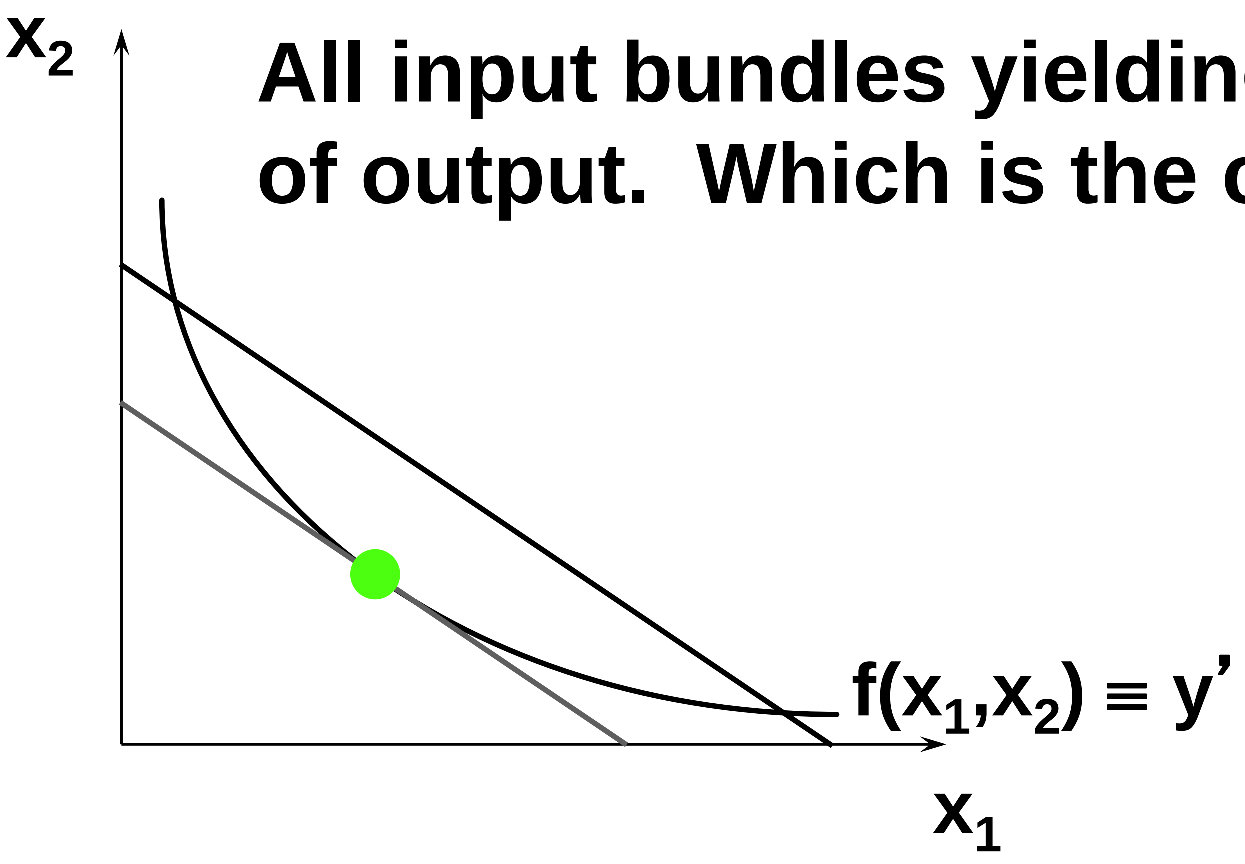


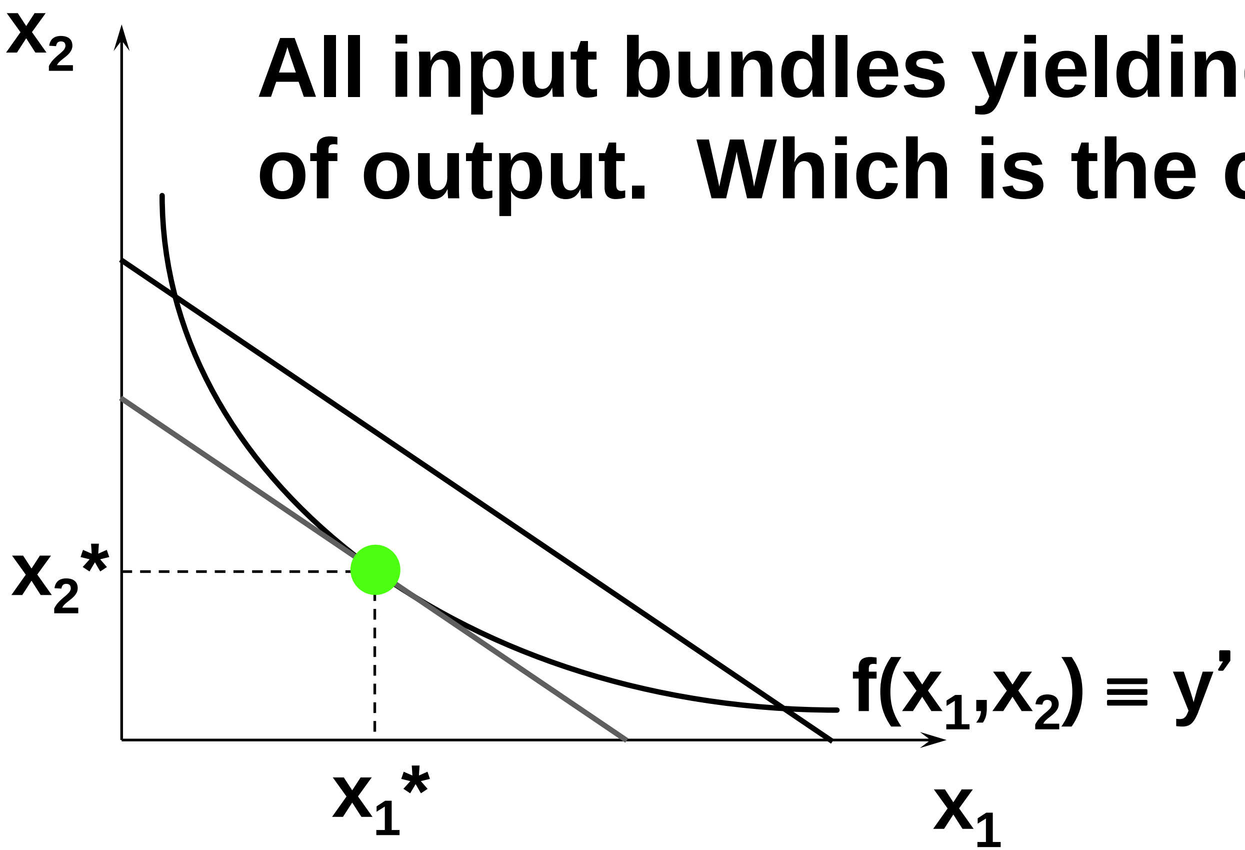
The Cost-Minimization Problem

All input bundles yielding y' units of output. Which is the cheapest?



The Cost-Minimization Problem

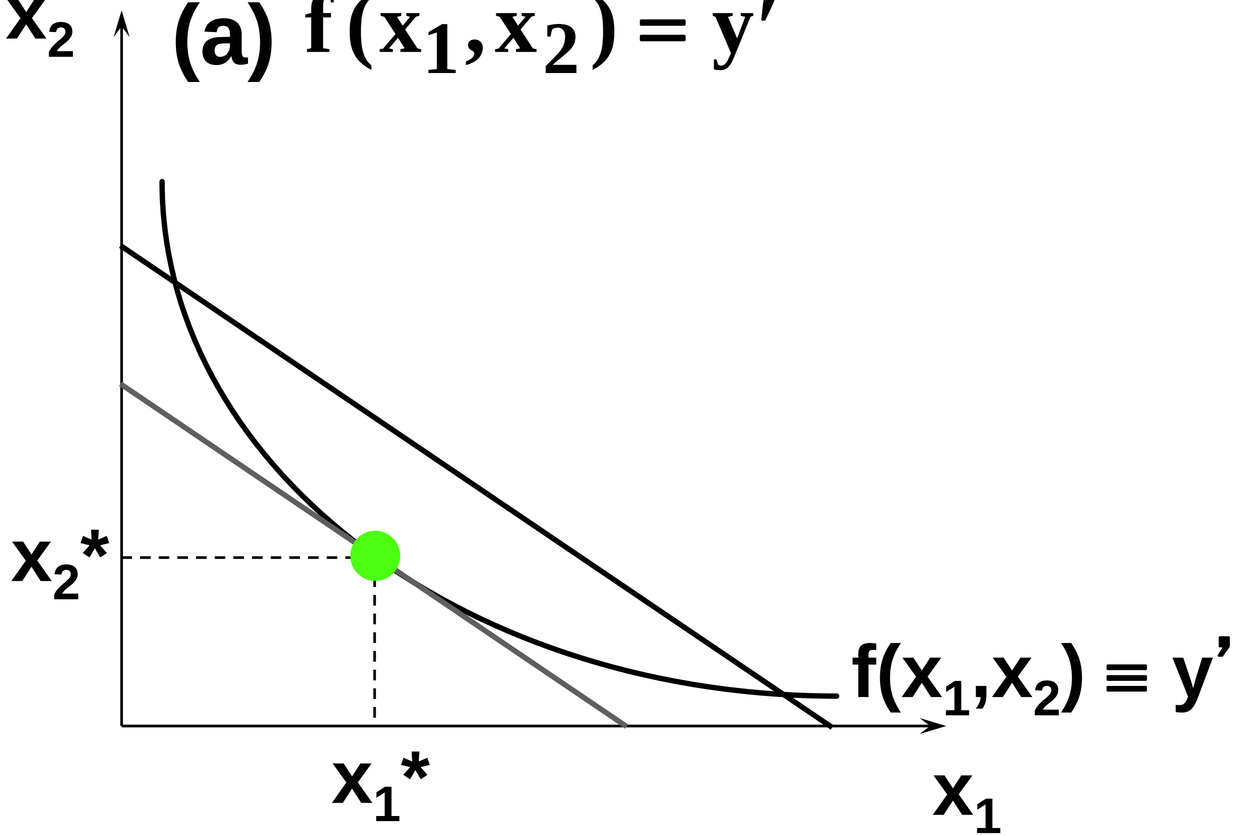
All input bundles yielding y' units of output. Which is the cheapest?



The Cost-Minimization Problem

At an interior cost-min input bundle:

(a) $f(x_1^*, x_2^*) = y'$

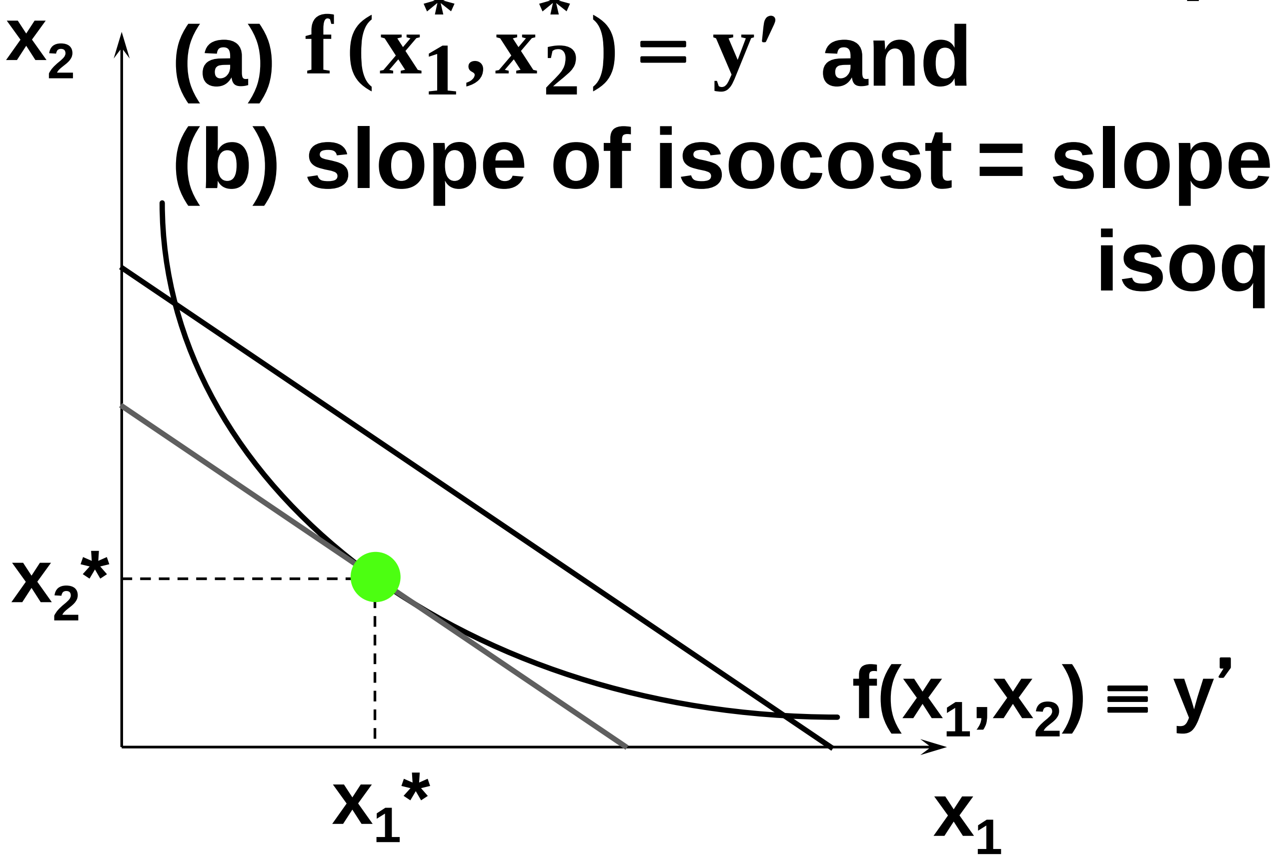


The Cost-Minimization Problem

At an interior cost-min input bundle:

(a) $f(x_1^*, x_2^*) = y'$ **and**

(b) slope of isocost = slope of isoquant



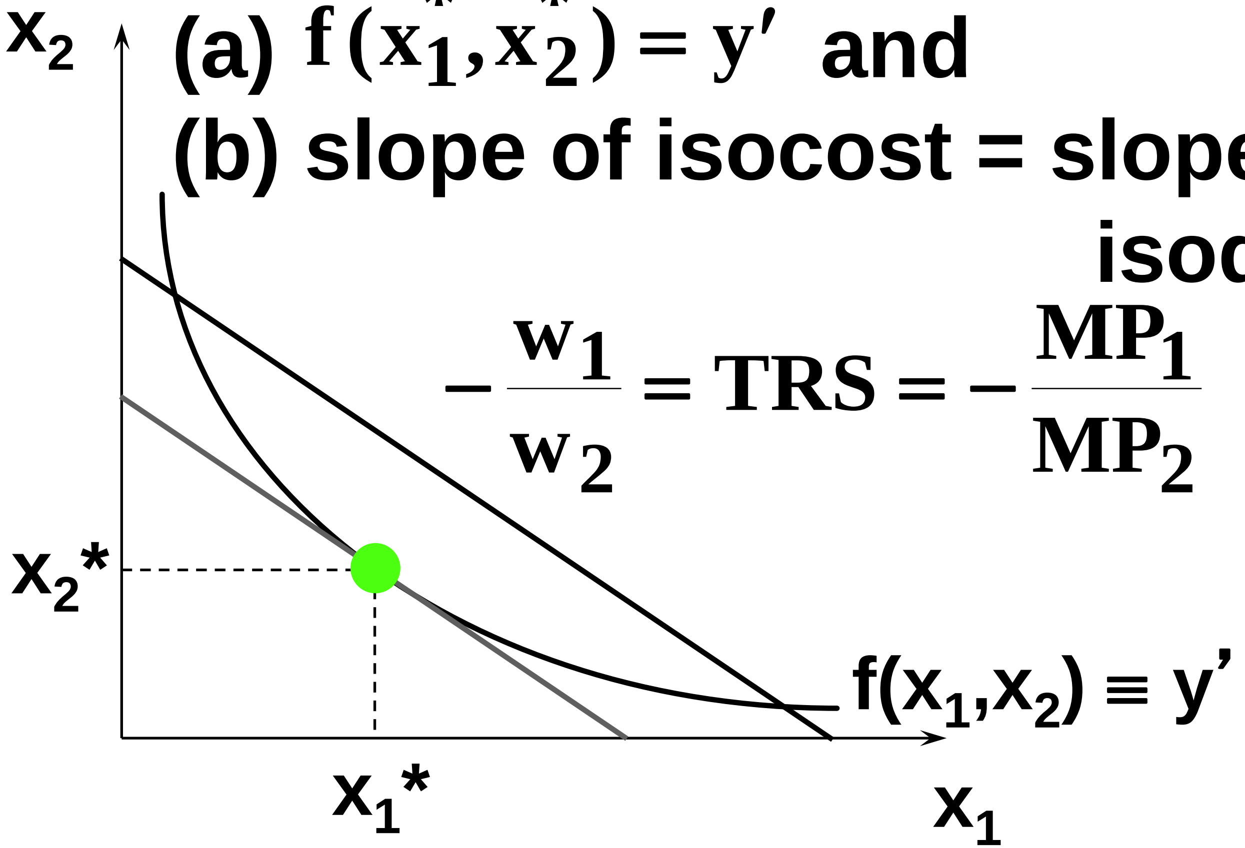
The Cost-Minimization Problem

At an interior cost-min input bundle:

(a) $f(x_1^*, x_2^*) = y'$ **and**

(b) slope of isocost = slope of isoquant; i.e.

$$-\frac{w_1}{w_2} = TRS = -\frac{MP_1}{MP_2} \text{ at } (x_1^*, x_2^*).$$



A Cobb-Douglas Example of Cost Minimization

- **A firm's Cobb-Douglas production function is** $\mathbf{y = f(x_1, x_2) = x_1^{1/3} x_2^{2/3}}$.
- **Input prices are w_1 and w_2.**
- **What are the firm's conditional input demand functions?**

A Cobb-Douglas Example of Cost Minimization

At the input bundle (x_1^*, x_2^*) which minimizes the cost of producing y output units:

(a)

$$y = (x_1^*)^{1/3}(x_2^*)^{2/3} \quad \textbf{and}$$

(b)

$$-\frac{w_1}{w_2} = -\frac{\partial y / \partial x_1}{\partial y / \partial x_2} = -\frac{(1/3)(x_1^*)^{-2/3}(x_2^*)^{2/3}}{(2/3)(x_1^*)^{1/3}(x_2^*)^{-1/3}}$$

$$= -\frac{x_2^*}{2x_1^*}.$$

A Cobb-Douglas Example of Cost Minimization

(a) $y = (x_1^*)^{1/3}(x_2^*)^{2/3}$ **(b)** $\frac{w_1}{w_2} = \frac{x_2^*}{2x_1^*}.$

A Cobb-Douglas Example of Cost Minimization

(a) $y = (x_1^*)^{1/3}(x_2^*)^{2/3}$ **(b)** $\frac{w_1}{w_2} = \frac{x_2^*}{2x_1^*}.$

From (b), $x_2^* = \frac{2w_1}{w_2}x_1^*.$

A Cobb-Douglas Example of Cost Minimization

(a) $y = (x_1^*)^{1/3}(x_2^*)^{2/3}$ **(b)** $\frac{w_1}{w_2} = \frac{x_2^*}{2x_1^*}.$

From (b), $x_2^* = \frac{2w_1}{w_2}x_1^*.$

Now substitute into (a) to get

$$y = (x_1^*)^{1/3}\left(\frac{2w_1}{w_2}x_1^*\right)^{2/3}$$

A Cobb-Douglas Example of Cost Minimization

(a) $y = (x_1^*)^{1/3}(x_2^*)^{2/3}$ **(b)** $\frac{w_1}{w_2} = \frac{x_2^*}{2x_1^*}.$

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Now substitute into (a) to get

$$y = (x_1^*)^{1/3}\left(\frac{2w_1}{w_2}x_1^*\right)^{2/3} = \left(\frac{2w_1}{w_2}\right)^{2/3} x_1^*.$$

A Cobb-Douglas Example of Cost Minimization

(a) $y = (x_1^*)^{1/3}(x_2^*)^{2/3}$ (b) $\dfrac{w_1}{w_2} = \dfrac{x_2^*}{2x_1^*}$.

From (b), $x_2^* = \dfrac{2w_1}{w_2} x_1^*$.

Now substitute into (a) to get

$$y = (x_1^*)^{1/3}\left(\frac{2w_1}{w_2} x_1^*\right)^{2/3} = \left(\frac{2w_1}{w_2}\right)^{2/3} x_1^*.$$

So $x_1^* = \left(\dfrac{w_2}{2w_1}\right)^{2/3} y$ is the firm's conditional demand for input 1.

A Cobb-Douglas Example of Cost Minimization

Since $x_2^* = \frac{2w_1}{w_2} x_1^*$ **and** $x_1^* = \left(\frac{w_2}{2w_1}\right)^{2/3} y$

$$x_2^* = \frac{2w_1}{w_2}\left(\frac{w_2}{2w_1}\right)^{2/3} y = \left(\frac{2w_1}{w_2}\right)^{1/3} y$$

is the firm's conditional demand for input 2.

A Cobb-Douglas Example of Cost Minimization

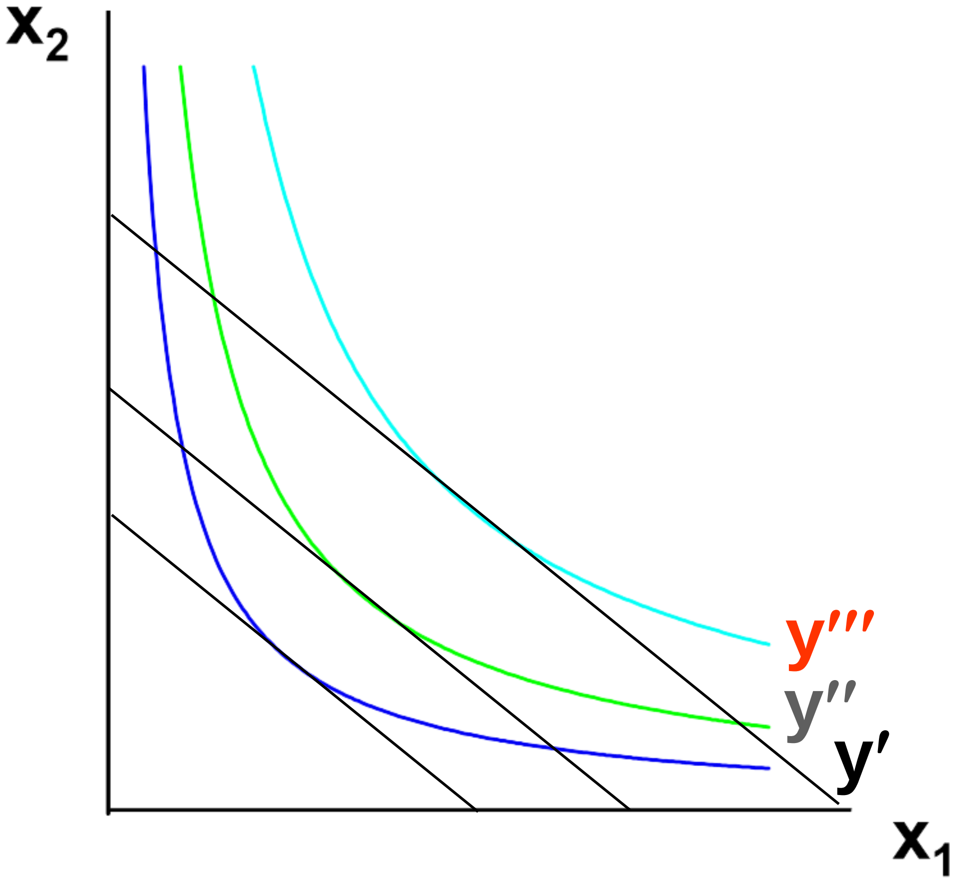
So the cheapest input bundle yielding y output units is

$$\left(x_1^*(w_1, w_2, y), x_2^*(w_1, w_2, y)\right)$$

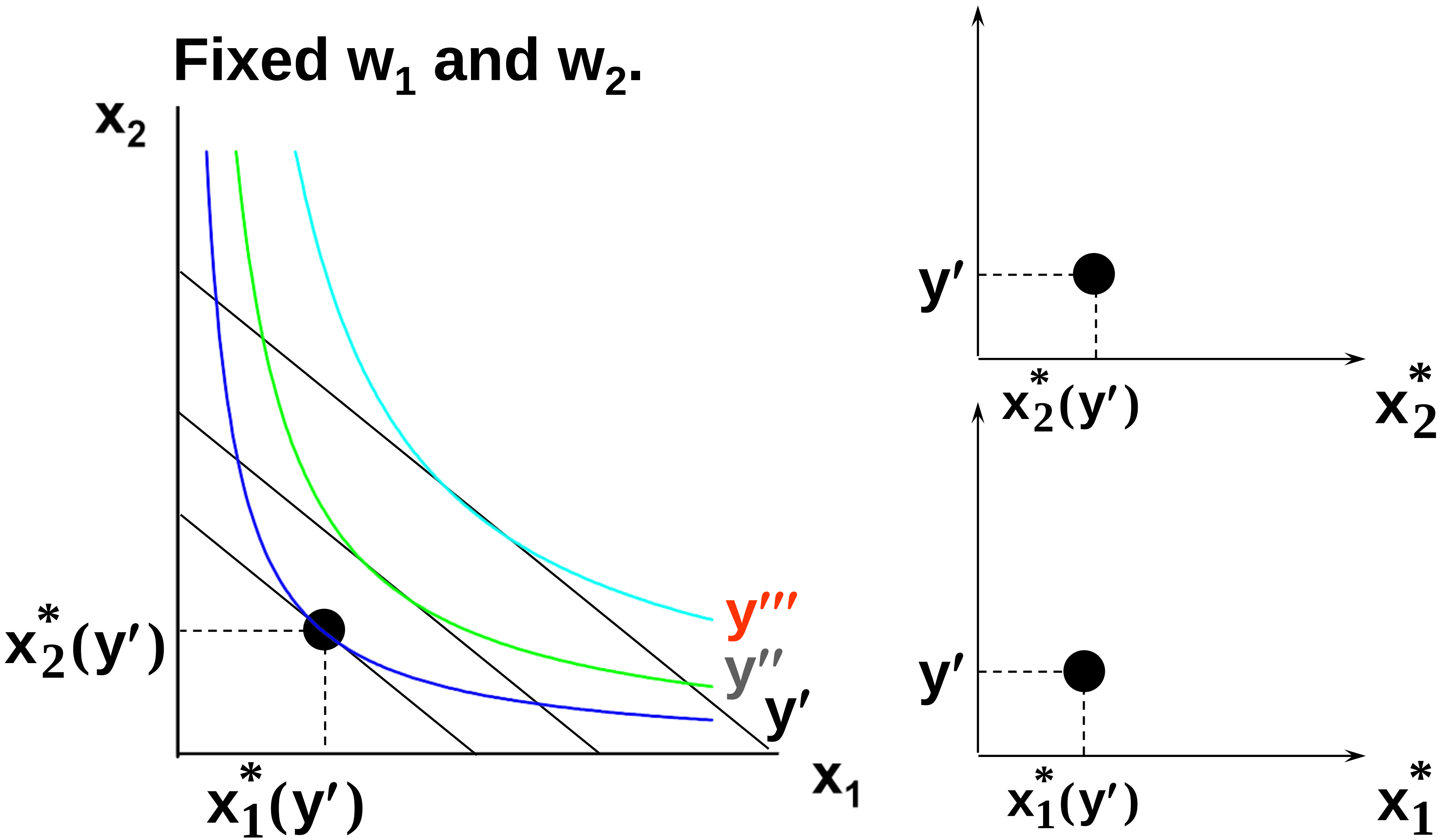
$$= \left(\left(\frac{w_2}{2w_1}\right)^{2/3} y, \left(\frac{2w_1}{w_2}\right)^{1/3} y\right).$$

Conditional Input Demand Curves

Fixed w_1 and w_2.

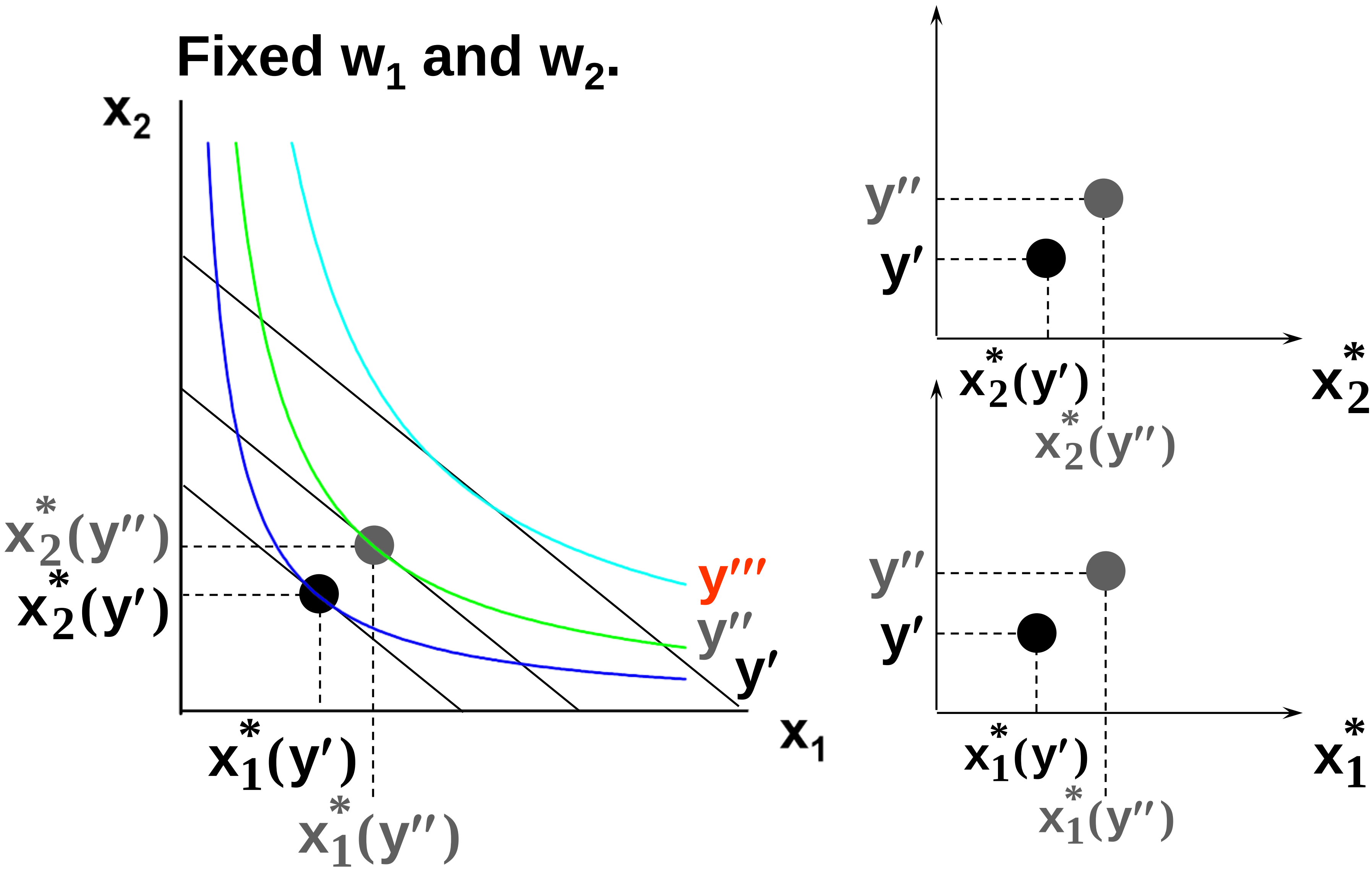


Conditional Input Demand Curves

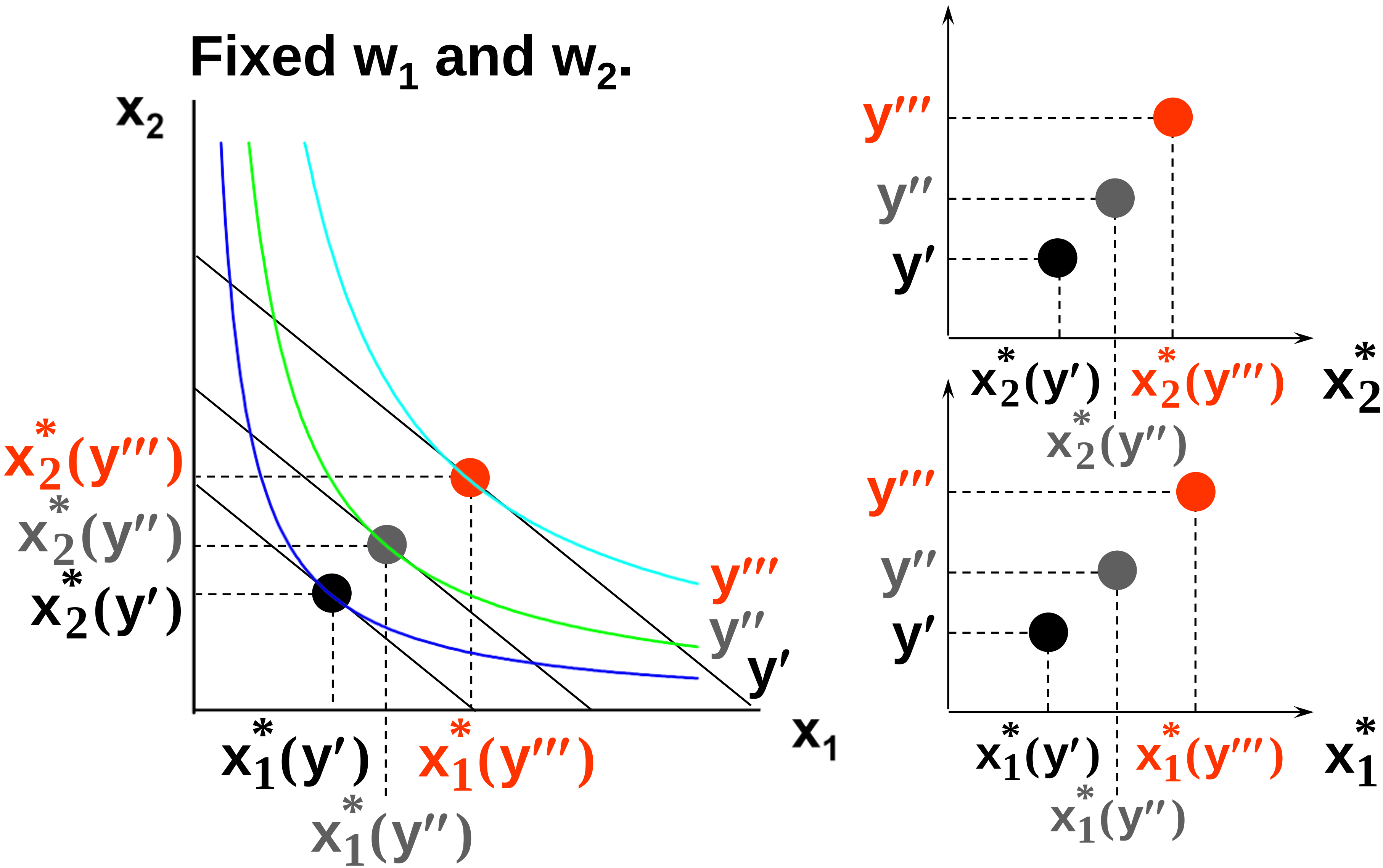


Conditional Input Demand Curves

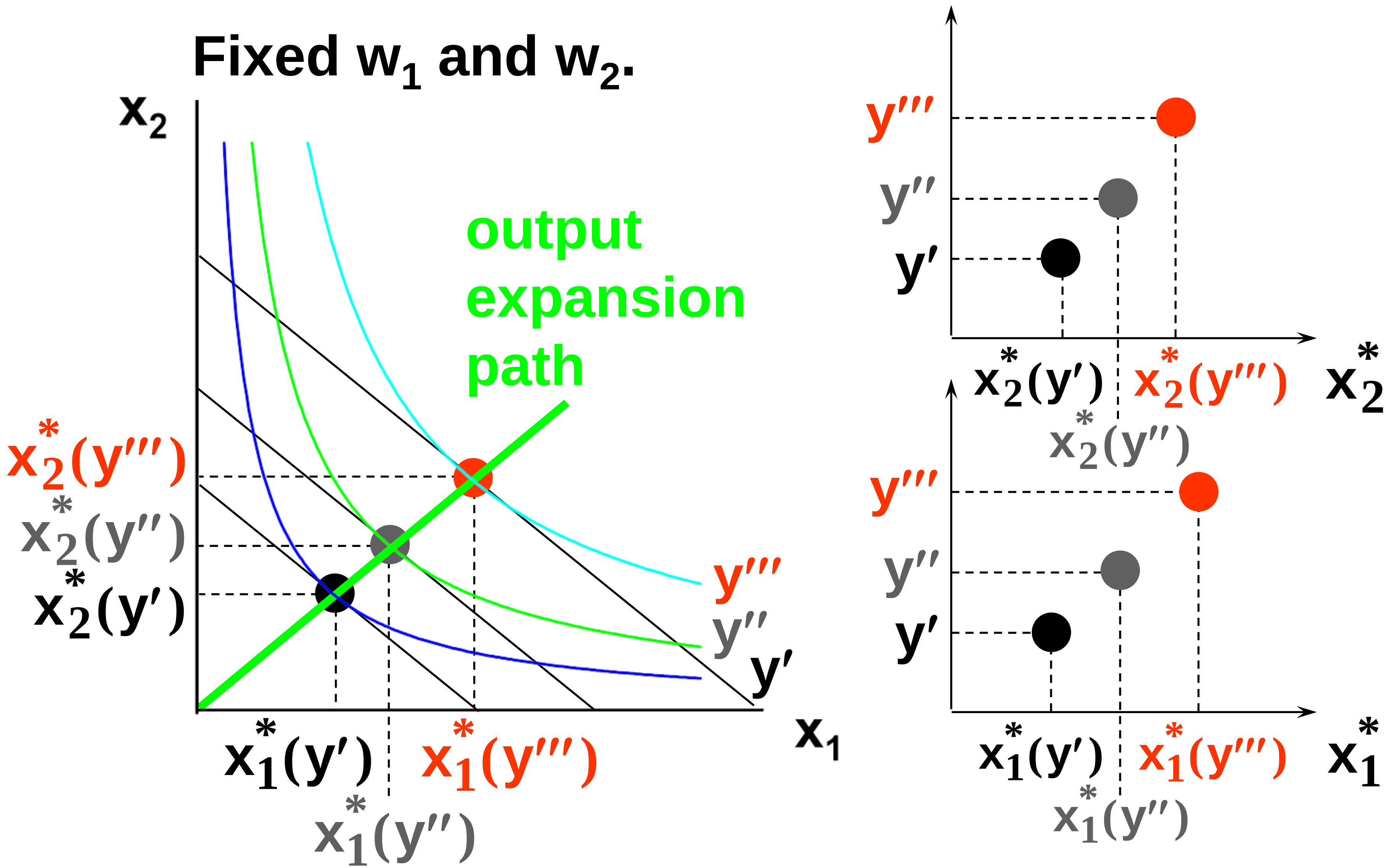
Fixed w_1 and w_2.



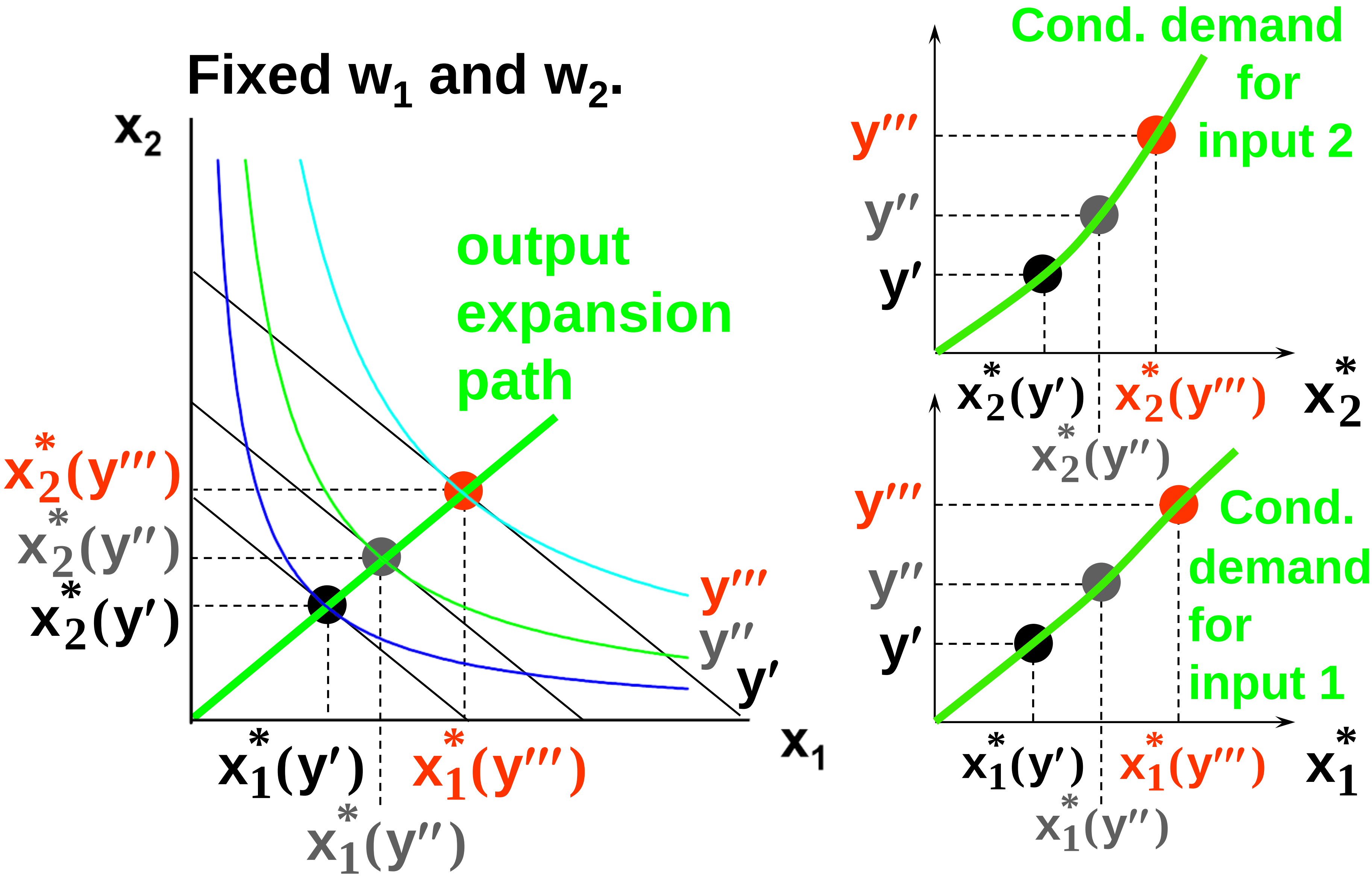
Conditional Input Demand Curves



Conditional Input Demand Curves



Conditional Input Demand Curves



A Cobb-Douglas Example of Cost Minimization

For the production function

$$y = f(x_1, x_2) = x_1^{1/3} x_2^{2/3}$$

the cheapest input bundle yielding y output units is

$$\left(x_1^*(w_1, w_2, y), x_2^*(w_1, w_2, y)\right)$$

$$= \left(\left(\frac{w_2}{2w_1} \right)^{2/3} y, \left(\frac{2w_1}{w_2} \right)^{1/3} y \right).$$

A Cobb-Douglas Example of Cost Minimization

So the firm's total cost function is

$$\mathbf{c(w_1, w_2, y) = w_1 x_1^*(w_1, w_2, y) + w_2 x_2^*(w_1, w_2, y)}$$

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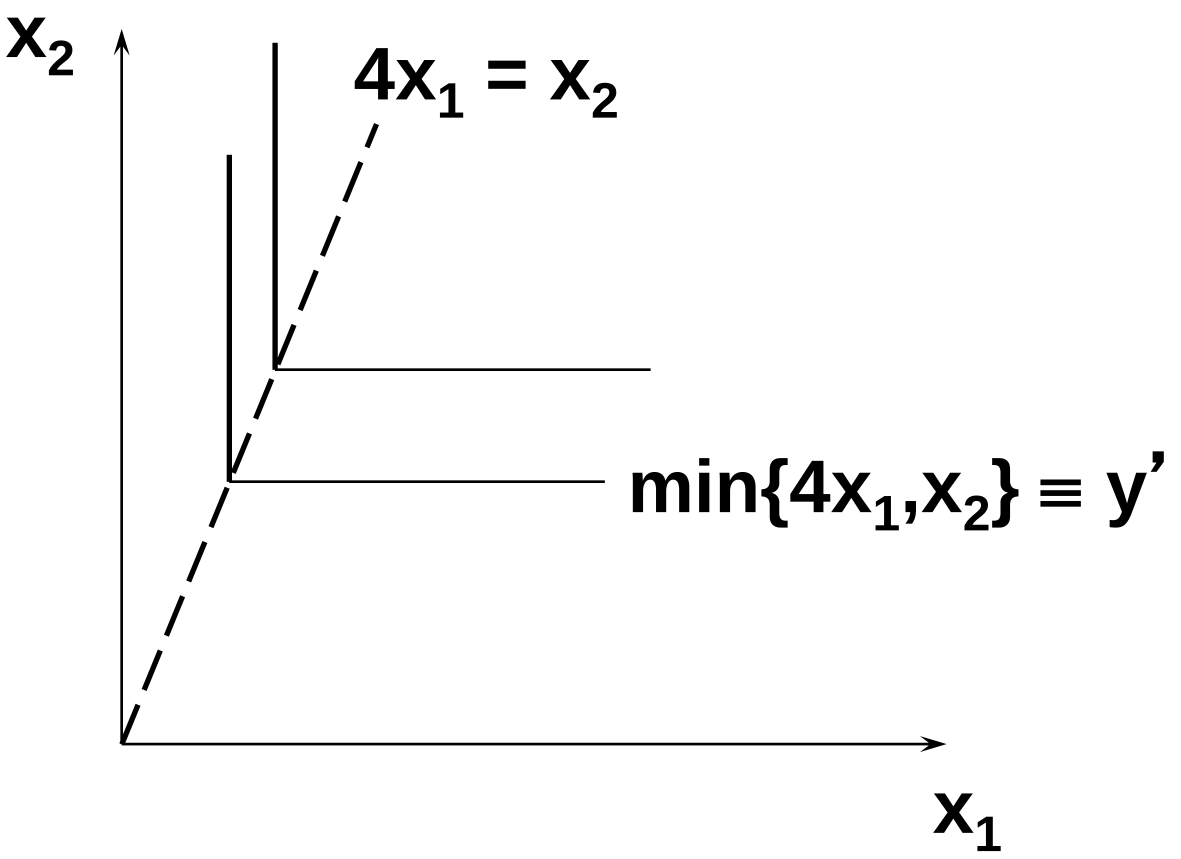
$$= 3\left(\frac{w_1 w_2^2}{4}\right)^{1/3} y.$$

A Perfect Complements Example of Cost Minimization

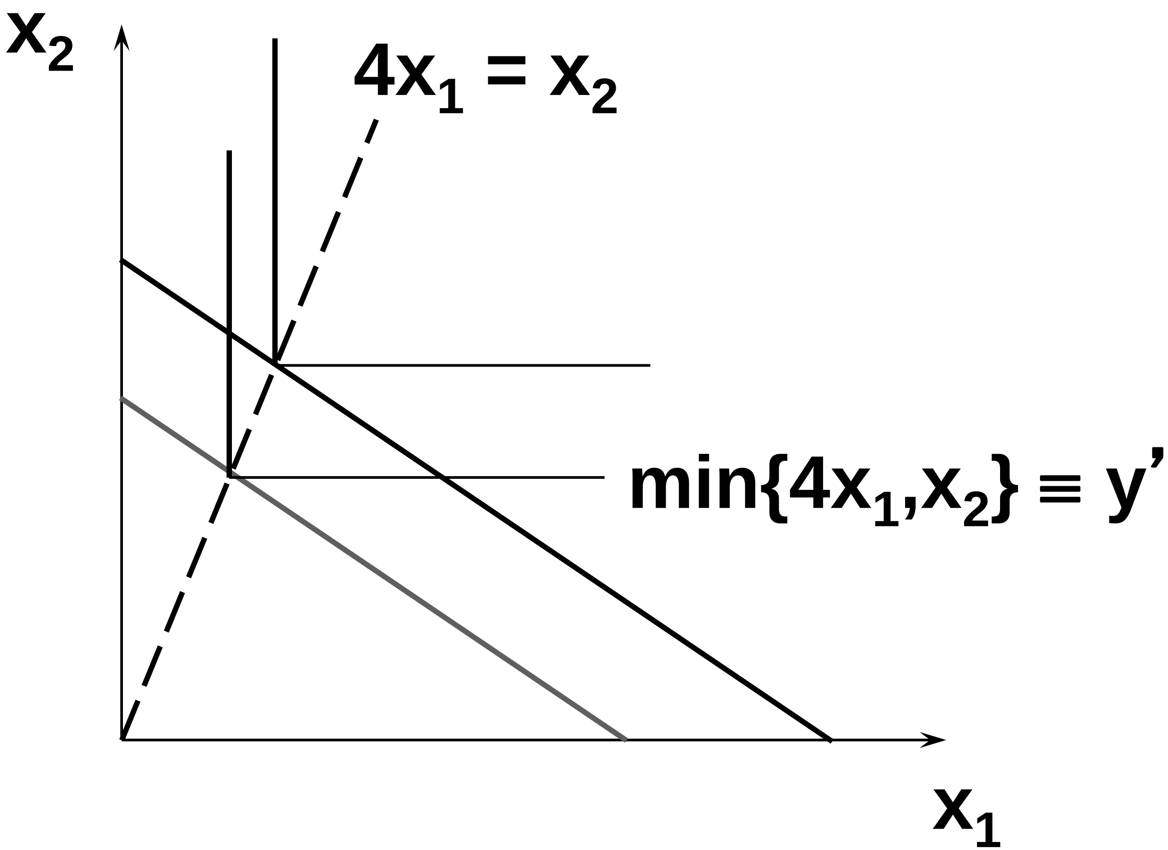
- The firm's production function is
$$y = \min\{4x_1, x_2\}.$$

- Input prices w_1 and w_2 are given.
- What are the firm's conditional demands for inputs 1 and 2?
- What is the firm's total cost function?

A Perfect Complements Example of Cost Minimization



A Perfect Complements Example of Cost Minimization



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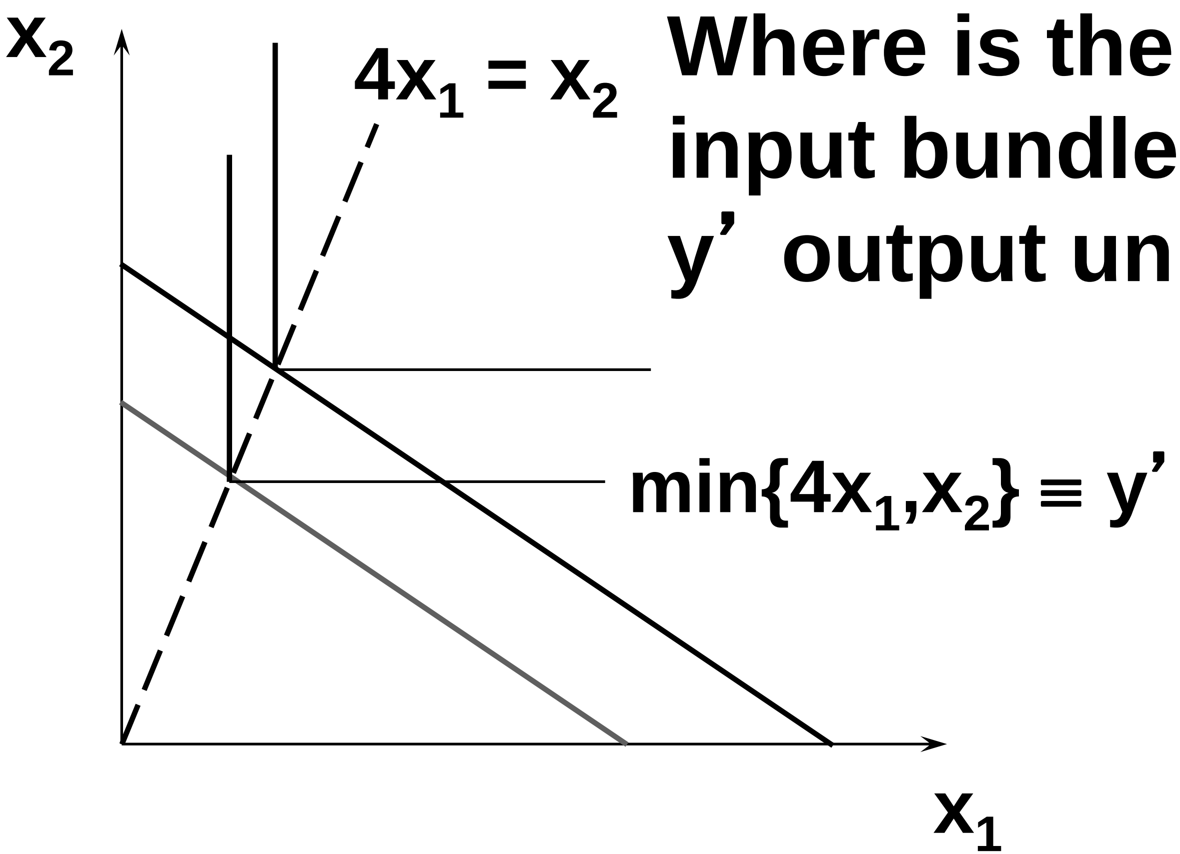
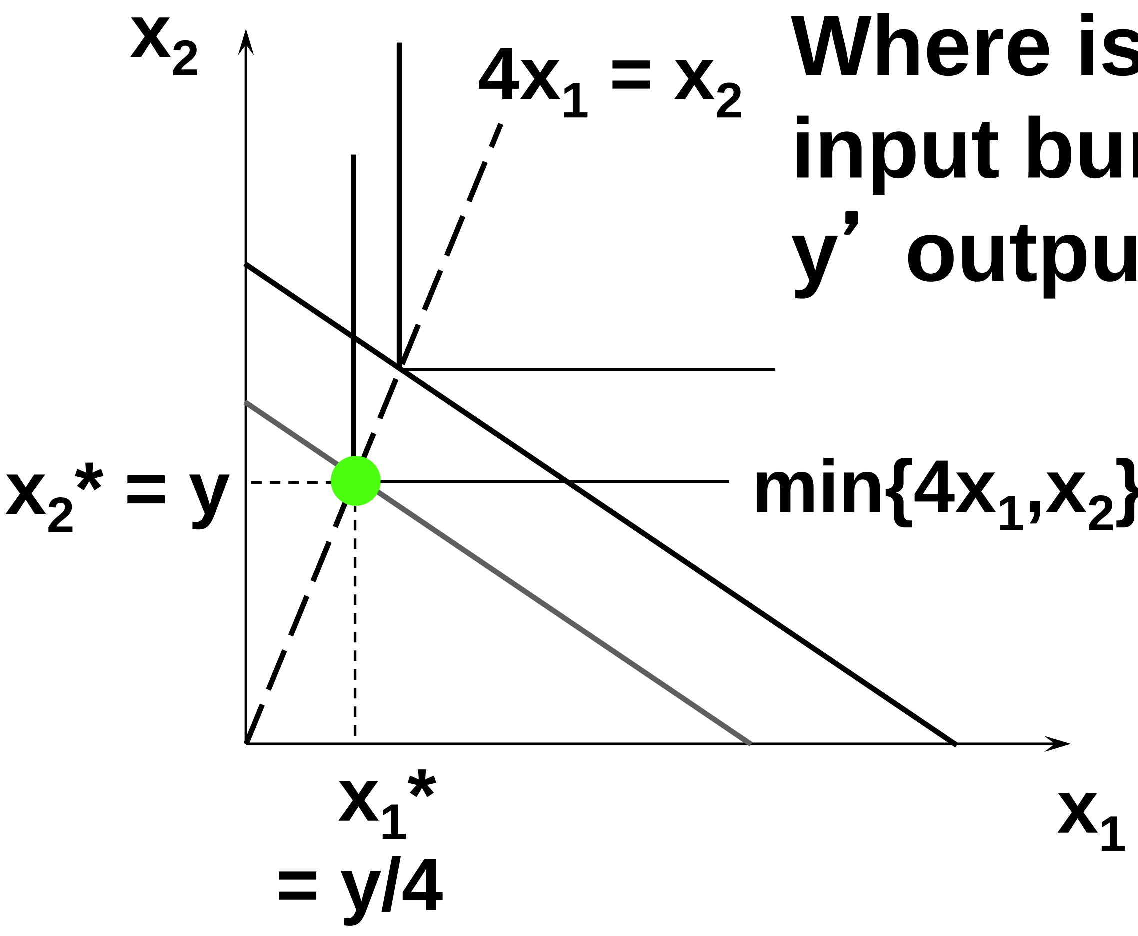


Where is the least costly input bundle yielding y' output units?

A Perfect Complements Example of Cost Minimization



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The firm's production function is

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and the conditional input demands are

$$x_1^*(w_1, w_2, y) = \frac{y}{4} \quad \text{and} \quad x_2^*(w_1, w_2, y) = y.$$

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$$+ w_2 x_2^*(w_1, w_2, y)$$

$$= w_1 \frac{y}{4} + w_2 y = \left(\frac{w_1}{4} + w_2\right) y.$$

Average Total Production Costs

- For positive output levels y, a firm's average total cost of producing y units is

$$AC(w_1, w_2, y) = \frac{c(w_1, w_2, y)}{y}.$$

Returns-to-Scale and Av. Total Costs

- **The returns-to-scale properties of a firm's technology determine how average production costs change with output level.**
- **Our firm is presently producing y' output units.**
- **How does the firm's average production cost change if it instead produces 2y' units of output?**

Constant Returns-to-Scale and Average Total Costs

- **If a firm's technology exhibits constant returns-to-scale then doubling its output level from y' to 2y' requires doubling all input levels.**

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Constant Returns-to-Scale and Average Total Costs

- **If a firm's technology exhibits constant returns-to-scale then doubling its output level from y' to 2y' requires doubling all input levels.**
- **Total production cost doubles.**
- **Average production cost does not change.**

Decreasing Returns-to-Scale and Average Total Costs

- **If a firm's technology exhibits decreasing returns-to-scale then doubling its output level from y' to 2y' requires more than doubling all input levels.**

Decreasing Returns-to-Scale and Average Total Costs

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Increasing Returns-to-Scale and Average Total Costs

- **If a firm's technology exhibits increasing returns-to-scale then doubling its output level from y' to 2y' requires less than doubling all input levels.**

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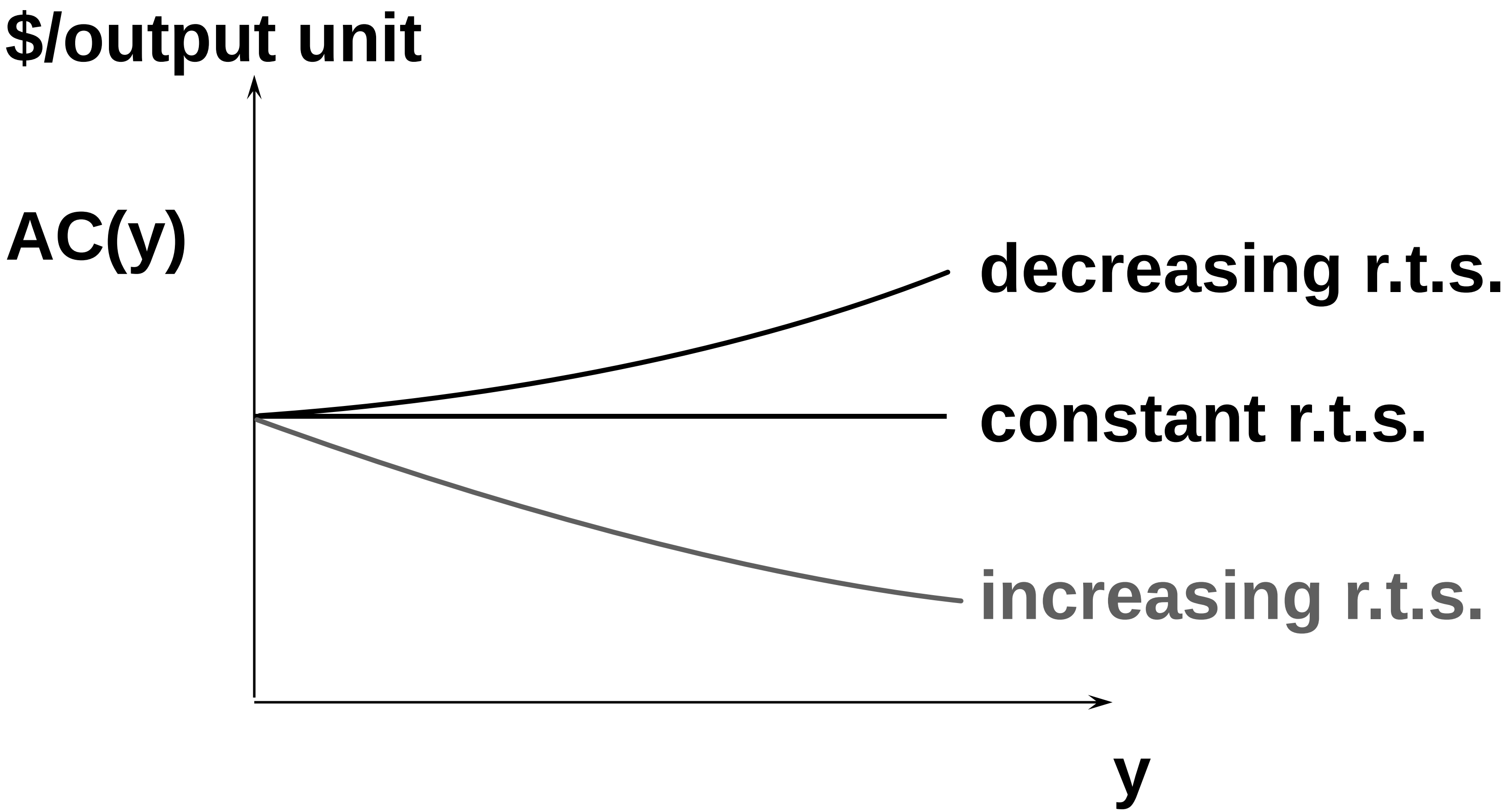
Returns-to-Scale and Av. Total Costs

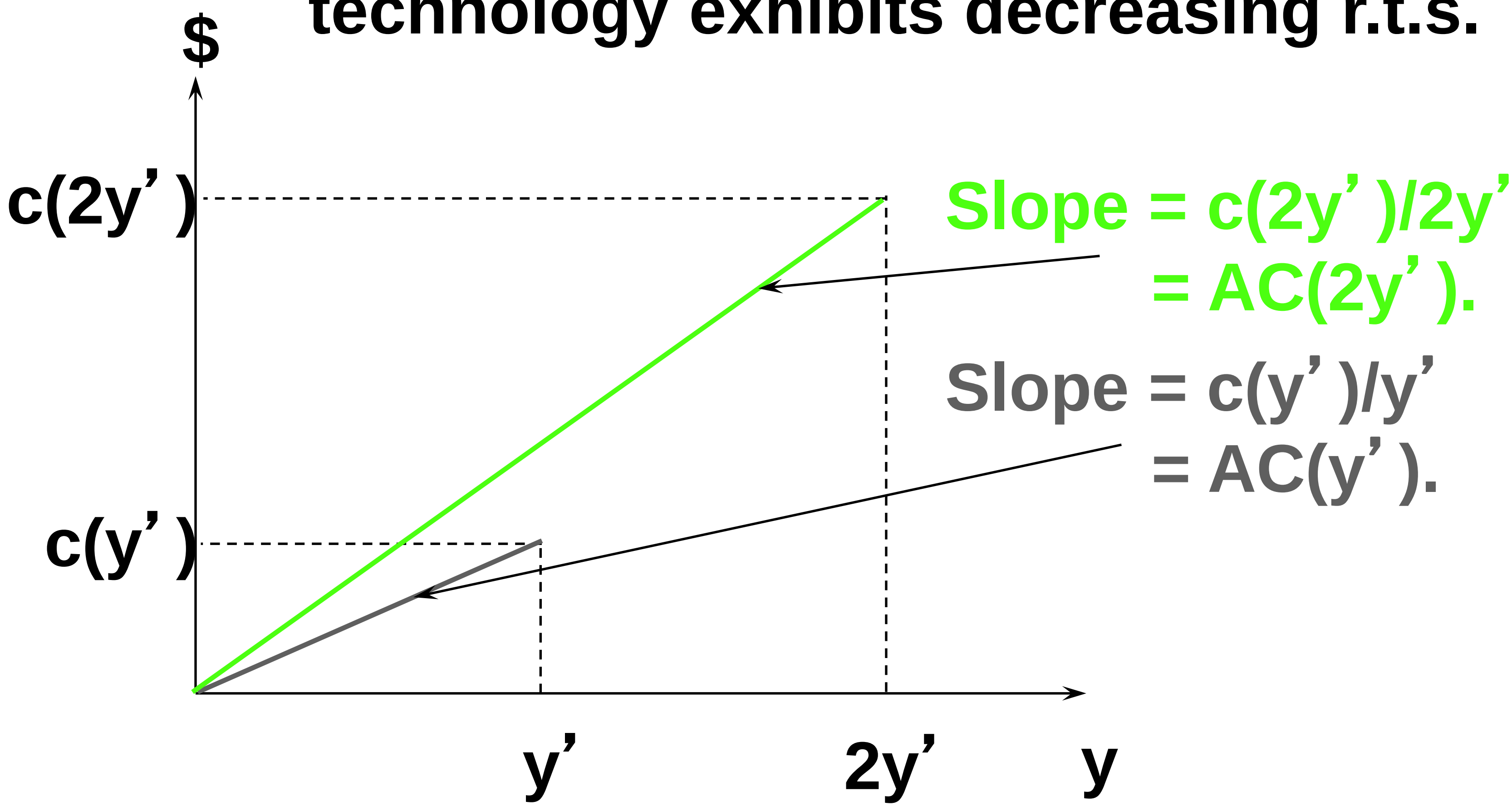


Returns-to-Scale and Total Costs

- **What does this imply for the shapes of total cost functions?**

Returns-to-Scale and Total Costs

Av. cost increases with y if the firm's technology exhibits decreasing r.t.s.



Returns-to-Scale and Total Costs

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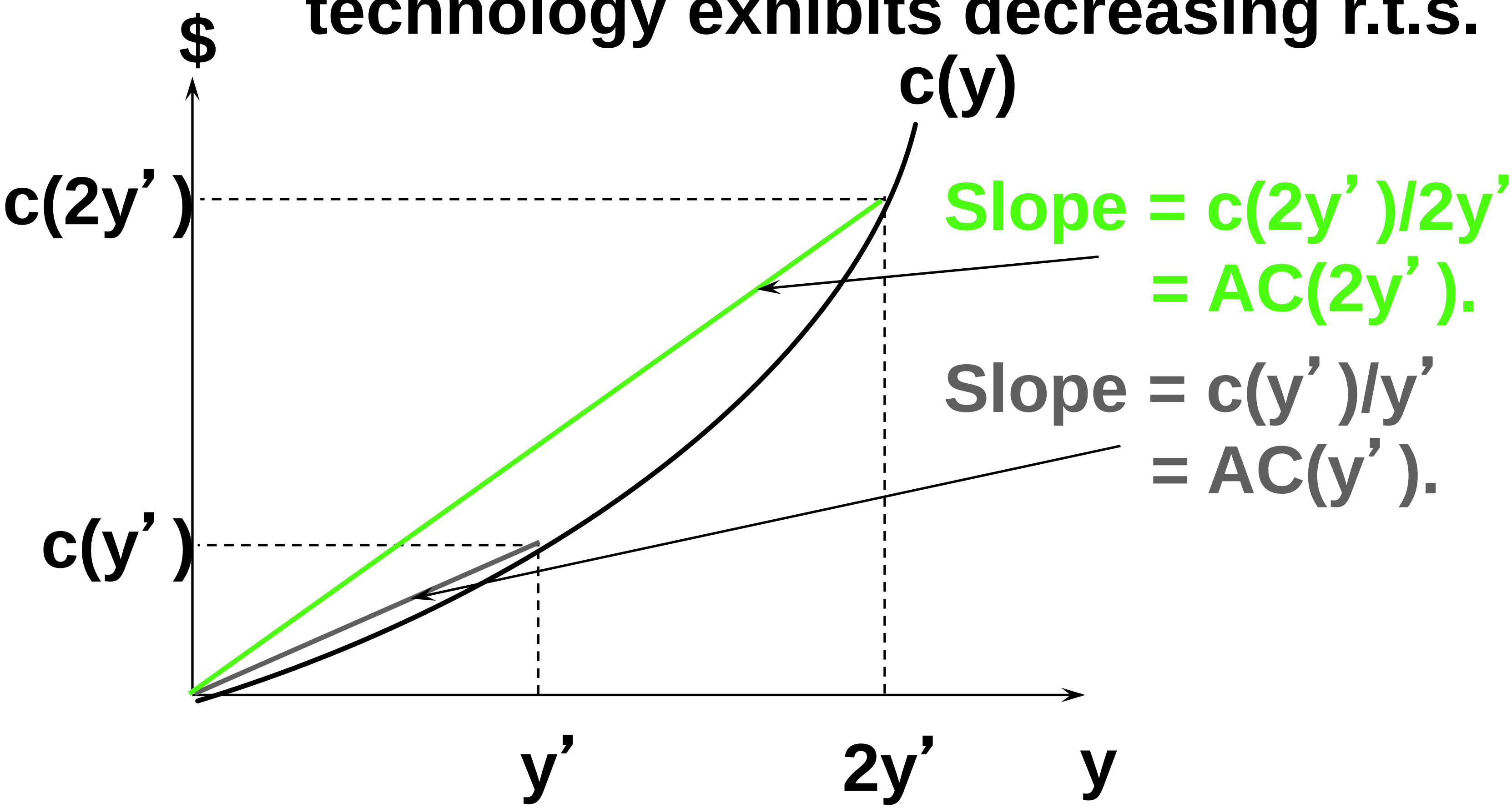

Returns-to-Scale and Total Costs

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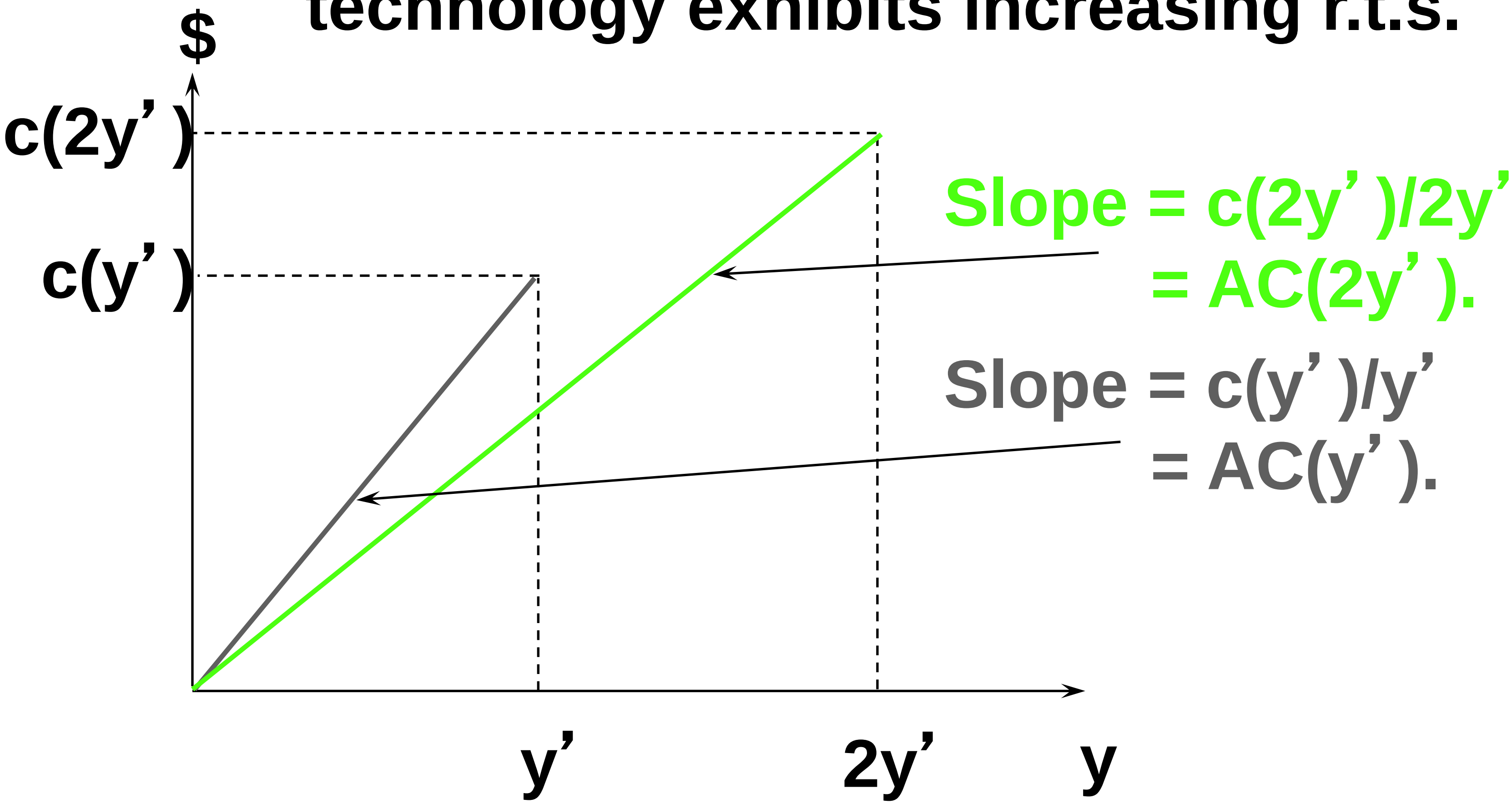

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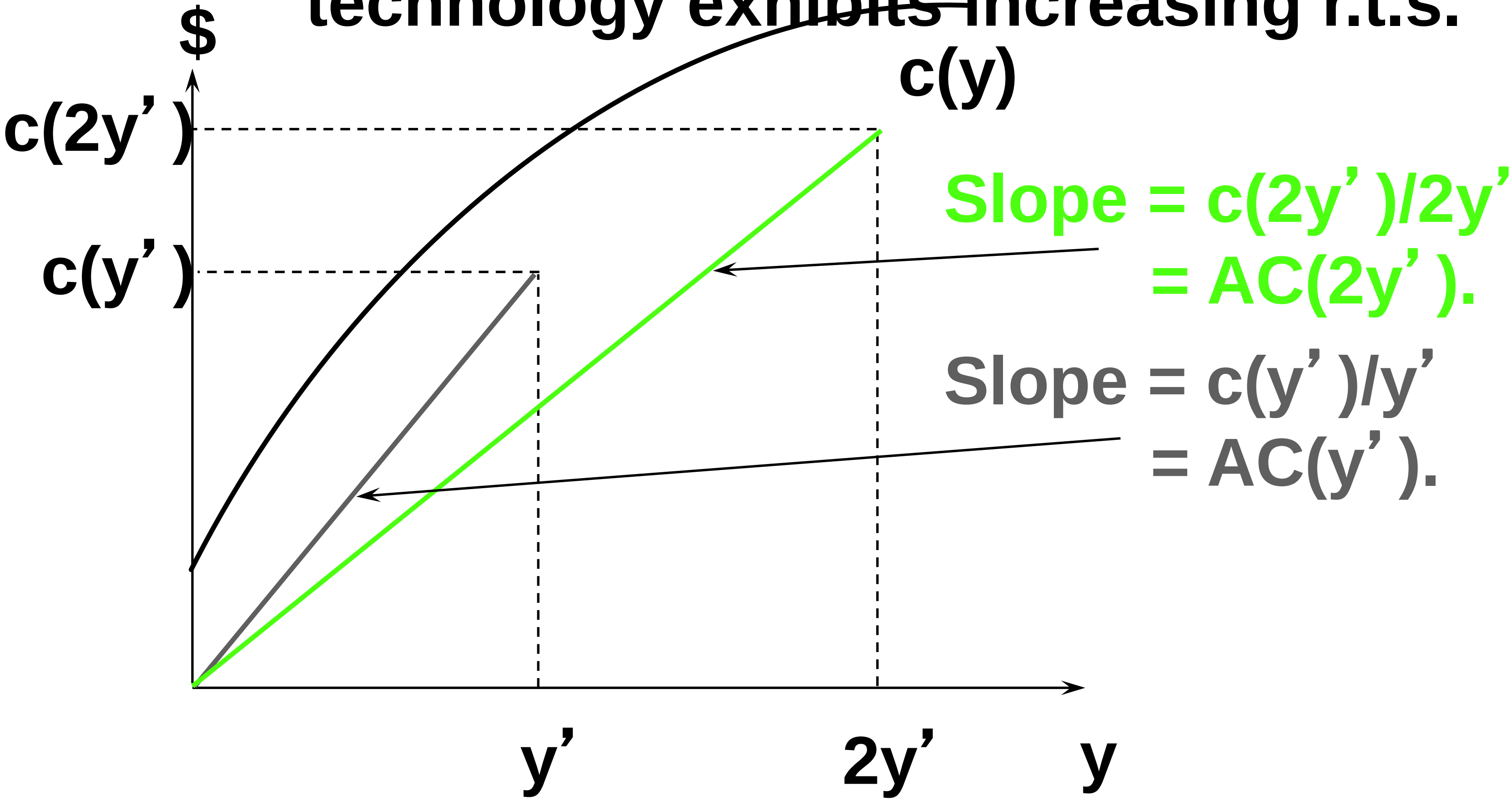
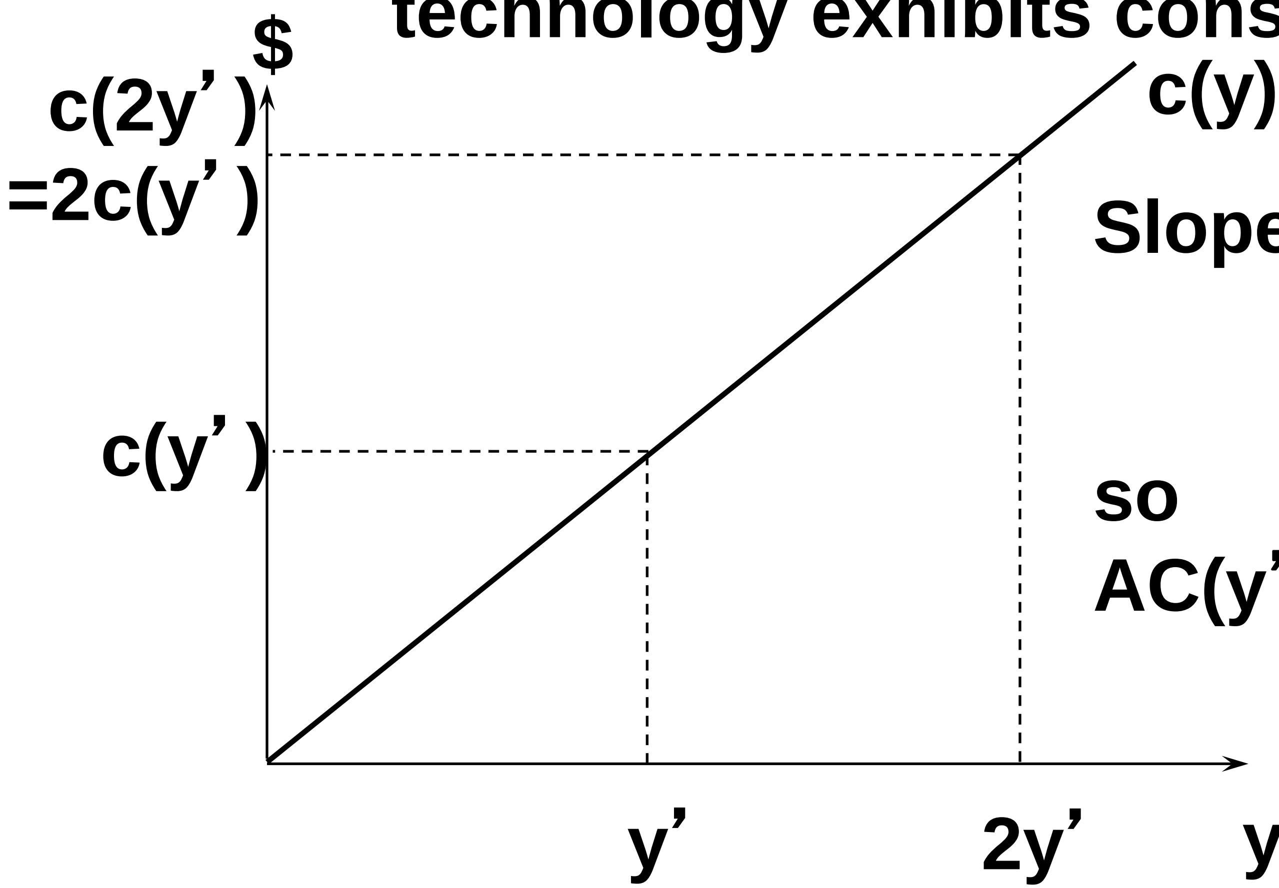


Returns-to-Scale and Total Costs

Av. cost is constant when the firm's technology exhibits constant r.t.s.



Slope = c(2y')/2y'
= 2c(y')/2y'
= c(y')/y'
so
AC(y') = AC(2y').

Short-Run & Long-Run Total Costs

- **In the long-run a firm can vary all of its input levels.**
- **Consider a firm that cannot change its input 2 level from x_2' units.**
- **How does the short-run total cost of producing y output units compare to the long-run total cost of producing y units of output?**

Short-Run & Long-Run Total Costs

- **The long-run cost-minimization problem is** $\min_{x_1, x_2 \geq 0} w_1 x_1 + w_2 x_2$

  subject to $f(x_1, x_2) = y.$

- **The short-run cost-minimization problem is** $\min_{x_1 \geq 0} w_1 x_1 + w_2 x_2'$

  subject to $f(x_1, x_2') = y.$

Short-Run & Long-Run Total Costs

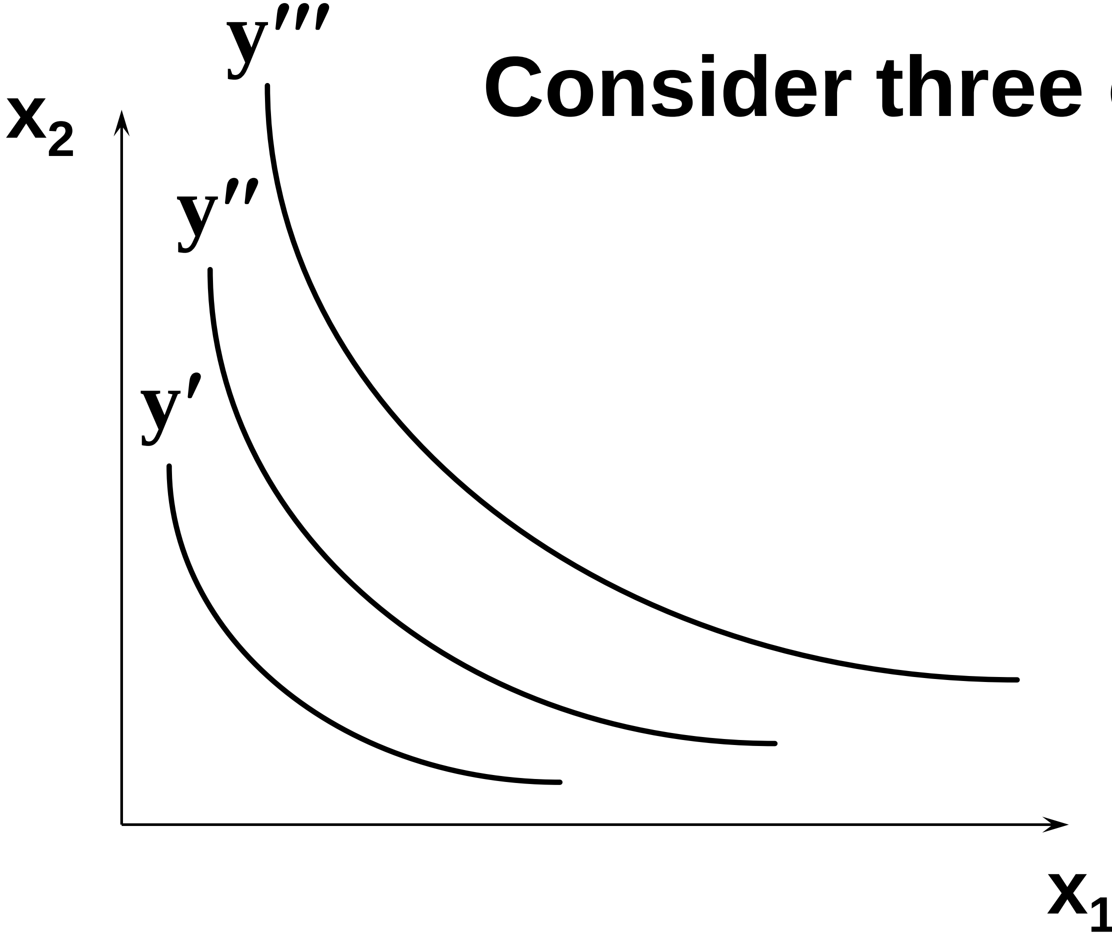
- **The short-run cost-min. problem is the long-run problem subject to the extra constraint that $x_2 = x_2'$.**
- **If the long-run choice for x_2 was x_2' then the extra constraint $x_2 = x_2'$ is not really a constraint at all and so the long-run and short-run total costs of producing y output units are the same.**

Short-Run & Long-Run Total Costs

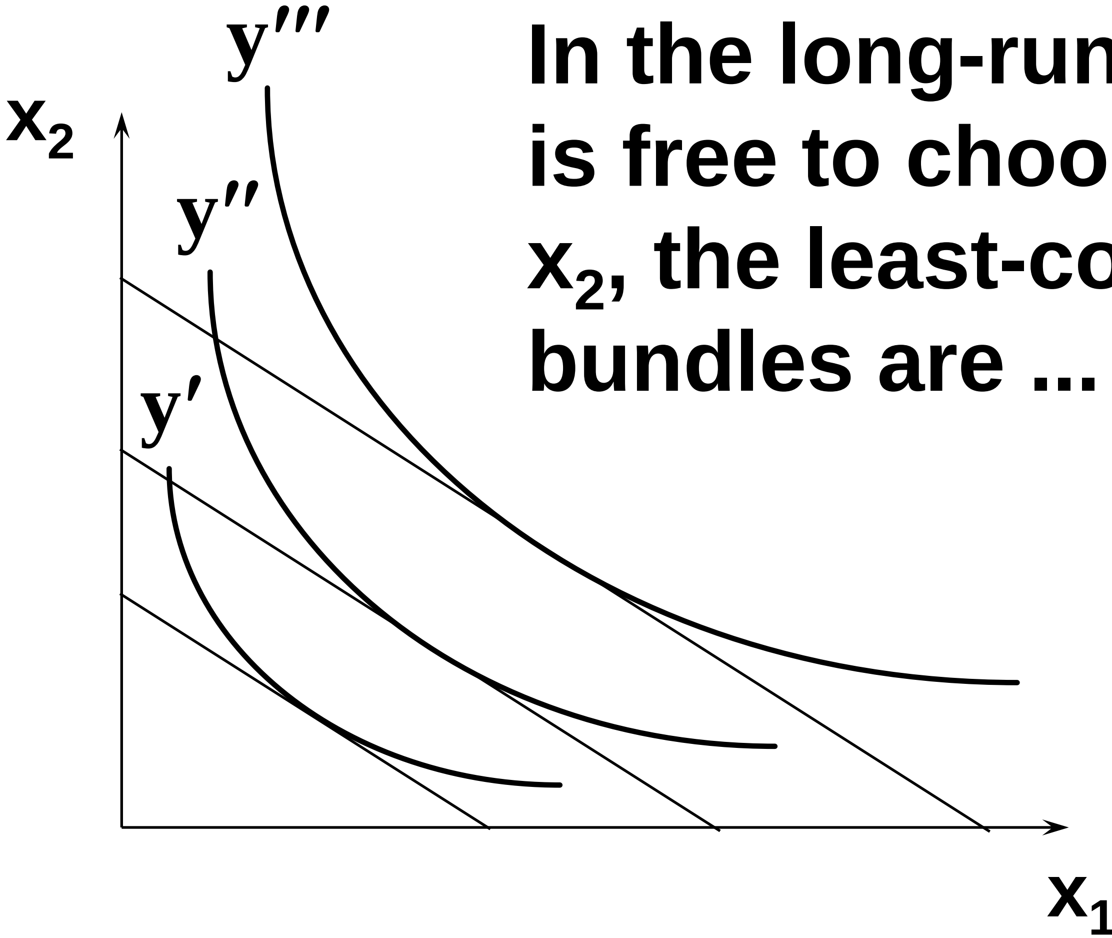
- **The short-run cost-min. problem is therefore the long-run problem subject to the extra constraint that $x_2 = x_2''$.**
- **But, if the long-run choice for $x_2 \neq x_2''$ then the extra constraint $x_2 = x_2''$ prevents the firm in this short-run from achieving its long-run production cost, causing the short-run total cost to exceed the long-run total cost of producing y output units.**

Short-Run & Long-Run Total Costs

Consider three output levels.

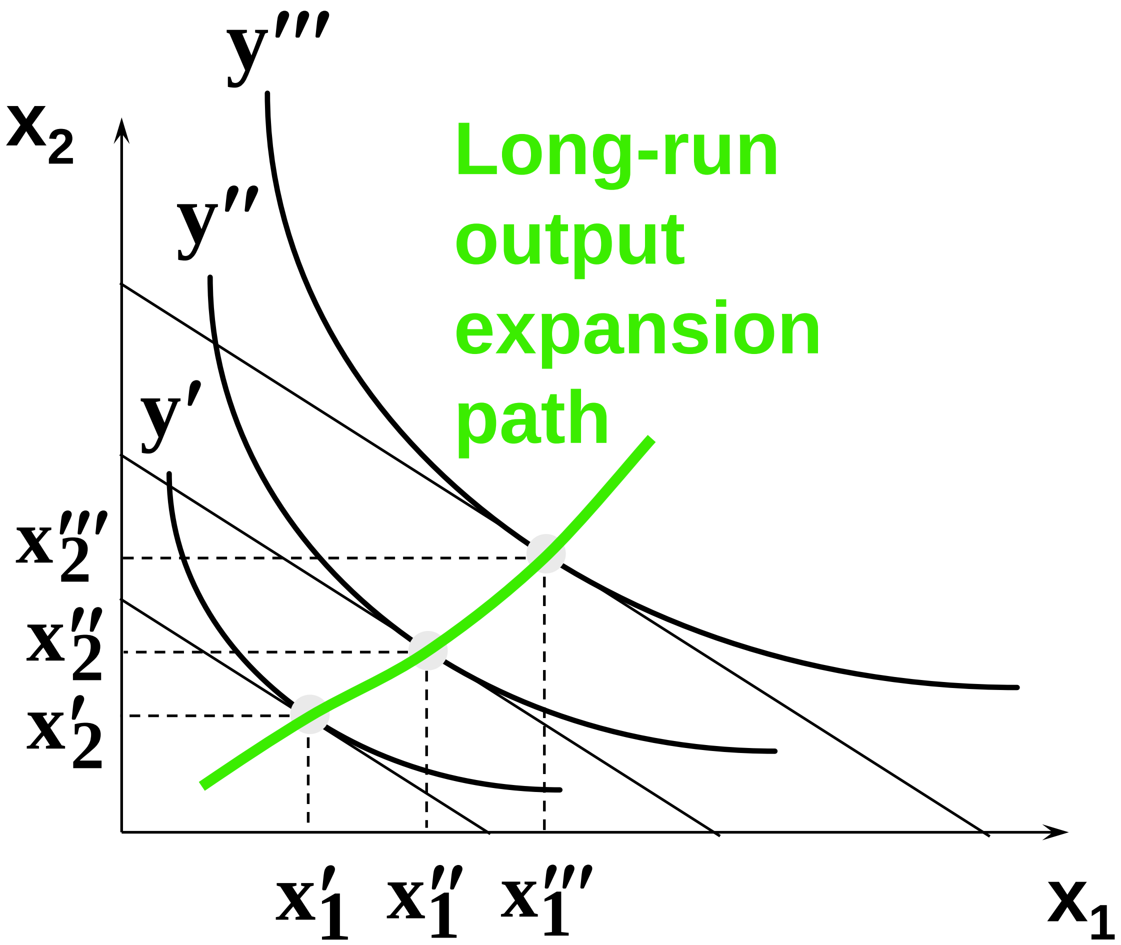


Short-Run & Long-Run Total Costs

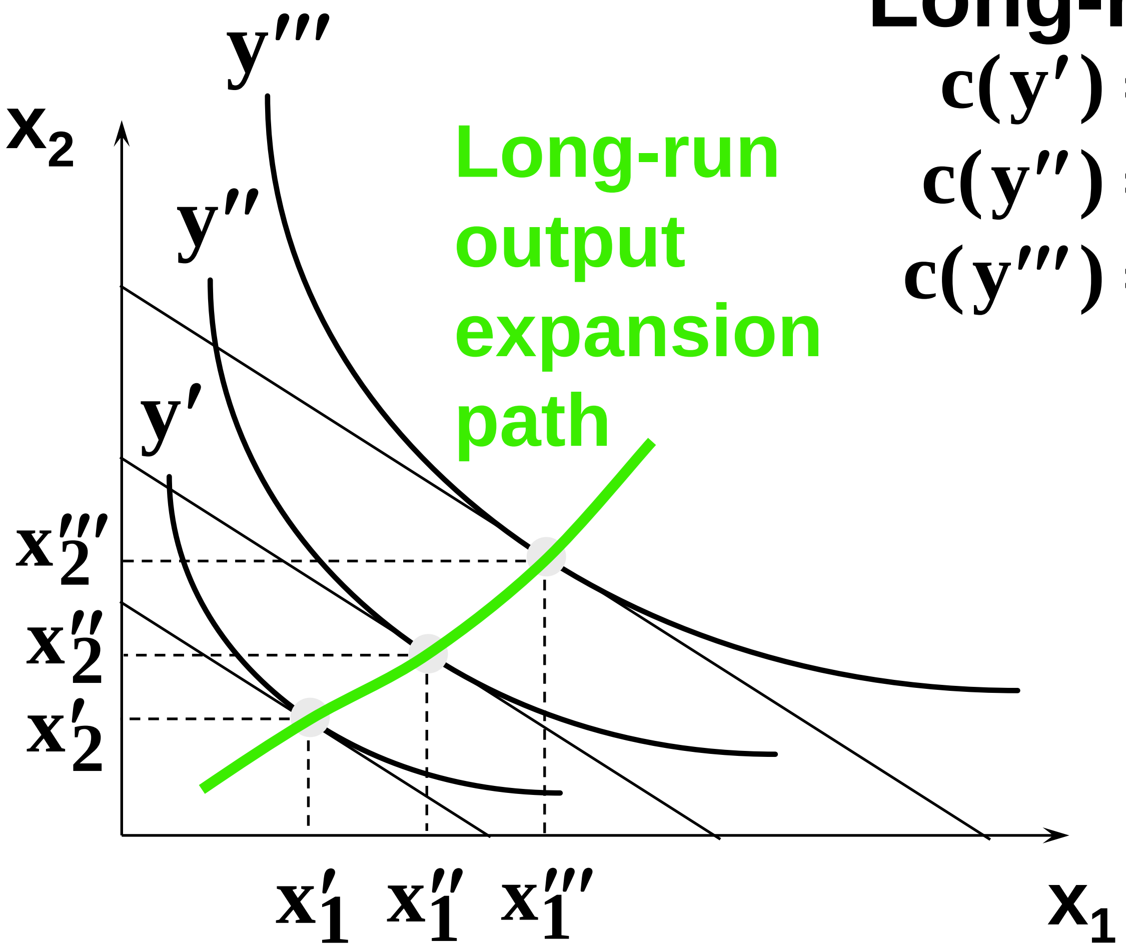


In the long-run when the firm is free to choose both x_1 and x_2, the least-costly input bundles are ...

Short-Run & Long-Run Total Costs



Short-Run & Long-Run Total Costs



Long-run costs are:

$$c(y') = w_1 x_1' + w_2 x_2'$$

$$c(y'') = w_1 x_1'' + w_2 x_2''$$

$$c(y''') = w_1 x_1''' + w_2 x_2'''$$

Short-Run & Long-Run Total Costs

- **Now suppose the firm becomes subject to the short-run constraint that $x_2 = x_2''$.**

Short-Run & Long-Run Total Costs

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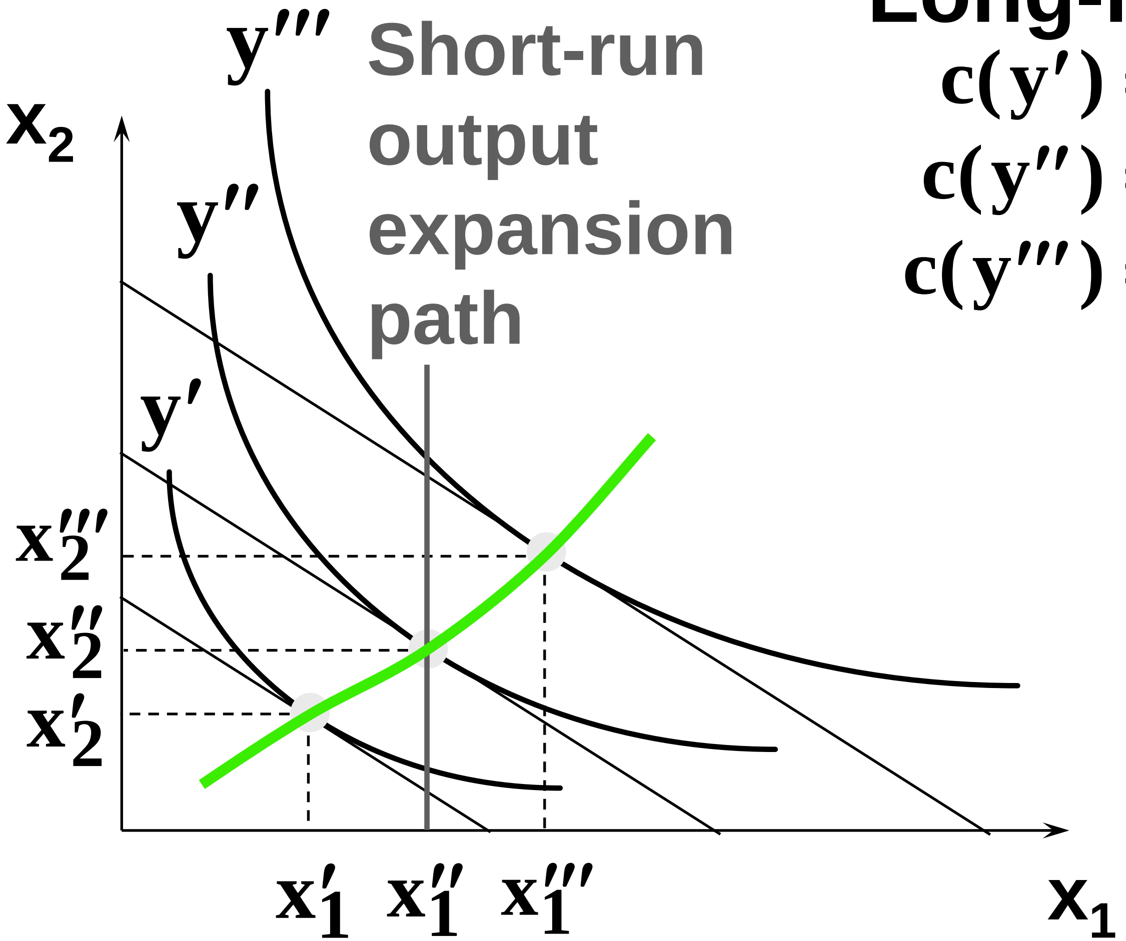
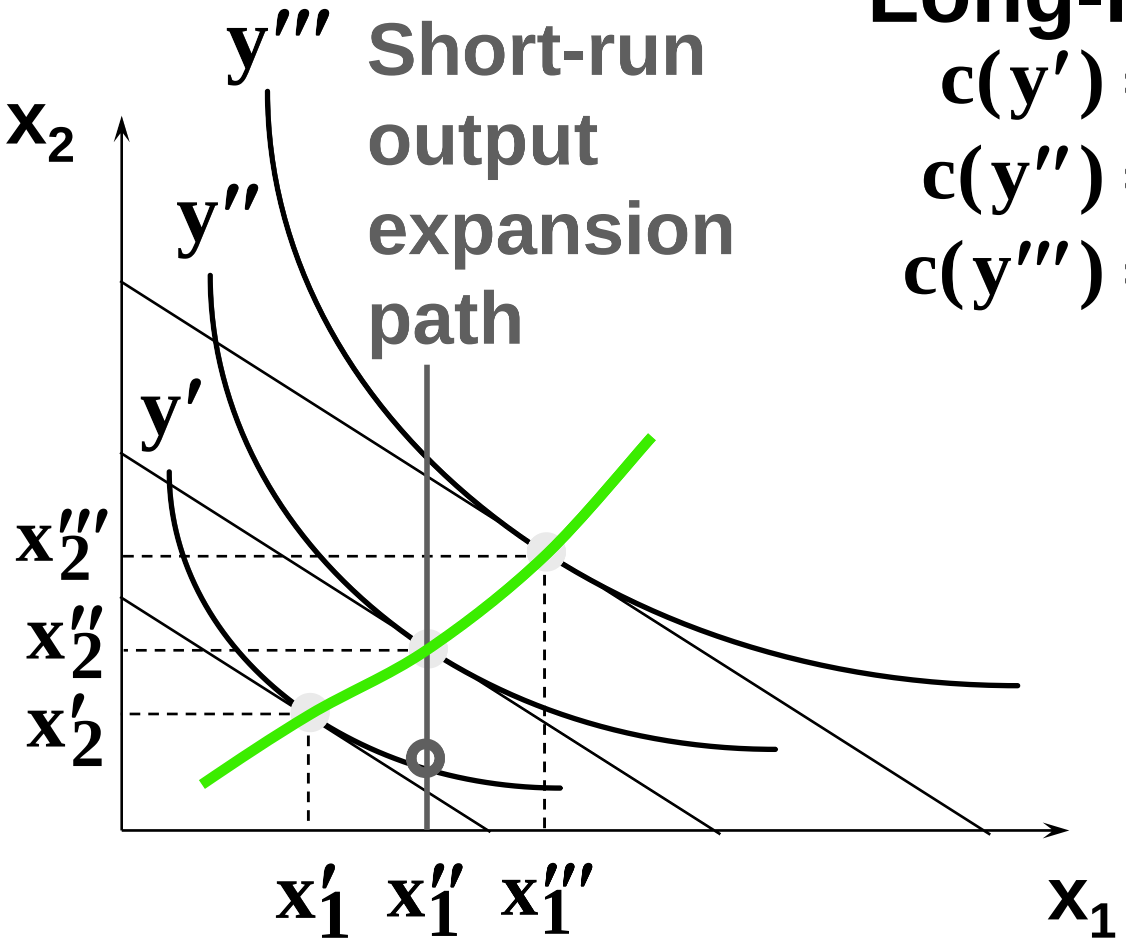


Short-Run & Long-Run Total Costs



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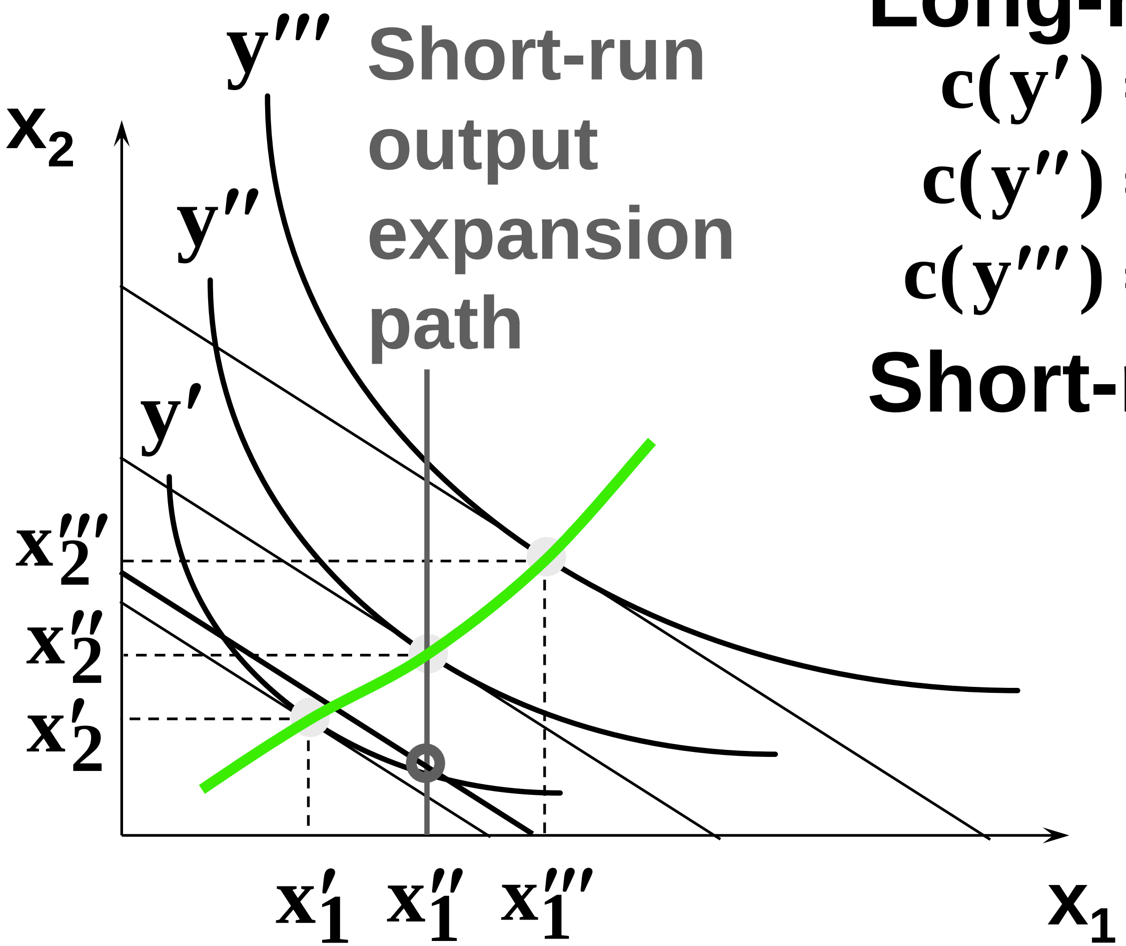

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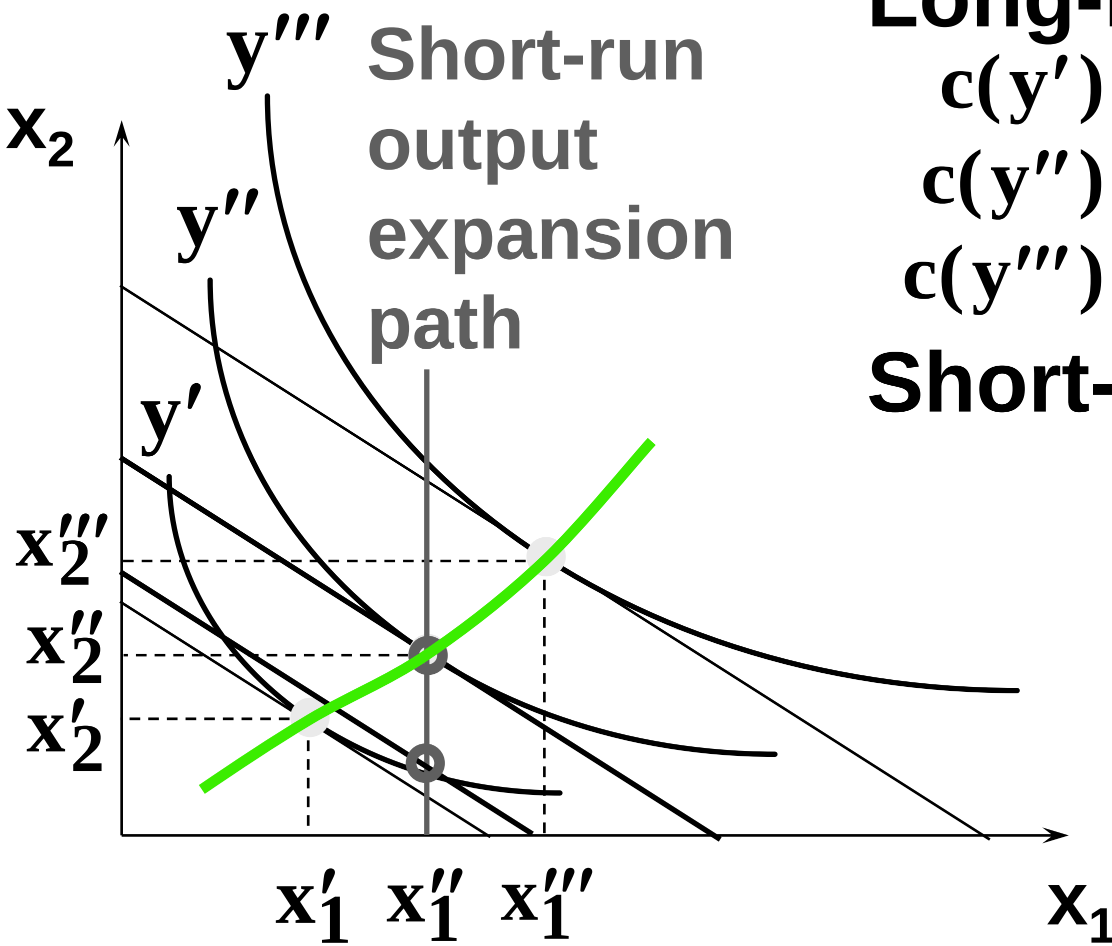
Short-run costs are:

$$c_s(y') > c(y')$$

Short-Run & Long-Run Total Costs



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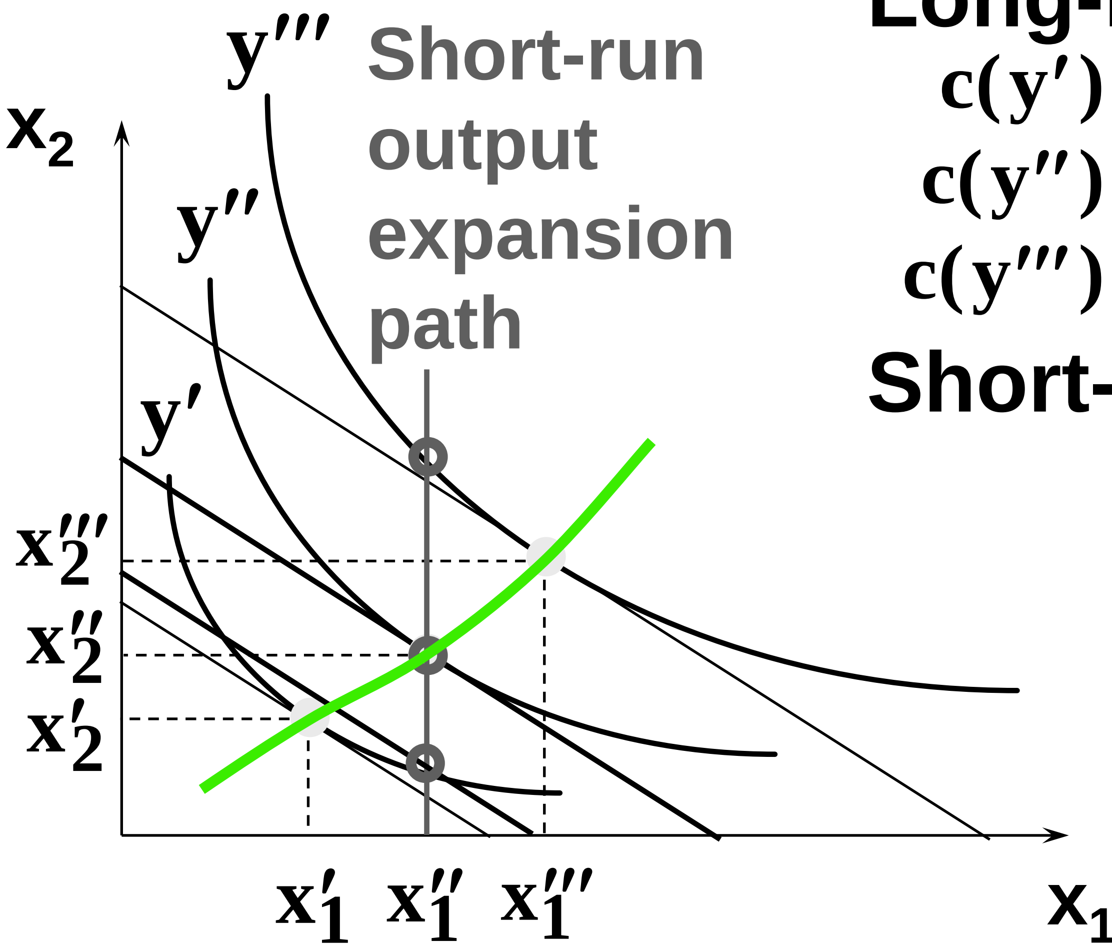
Short-Run & Long-Run Total Costs



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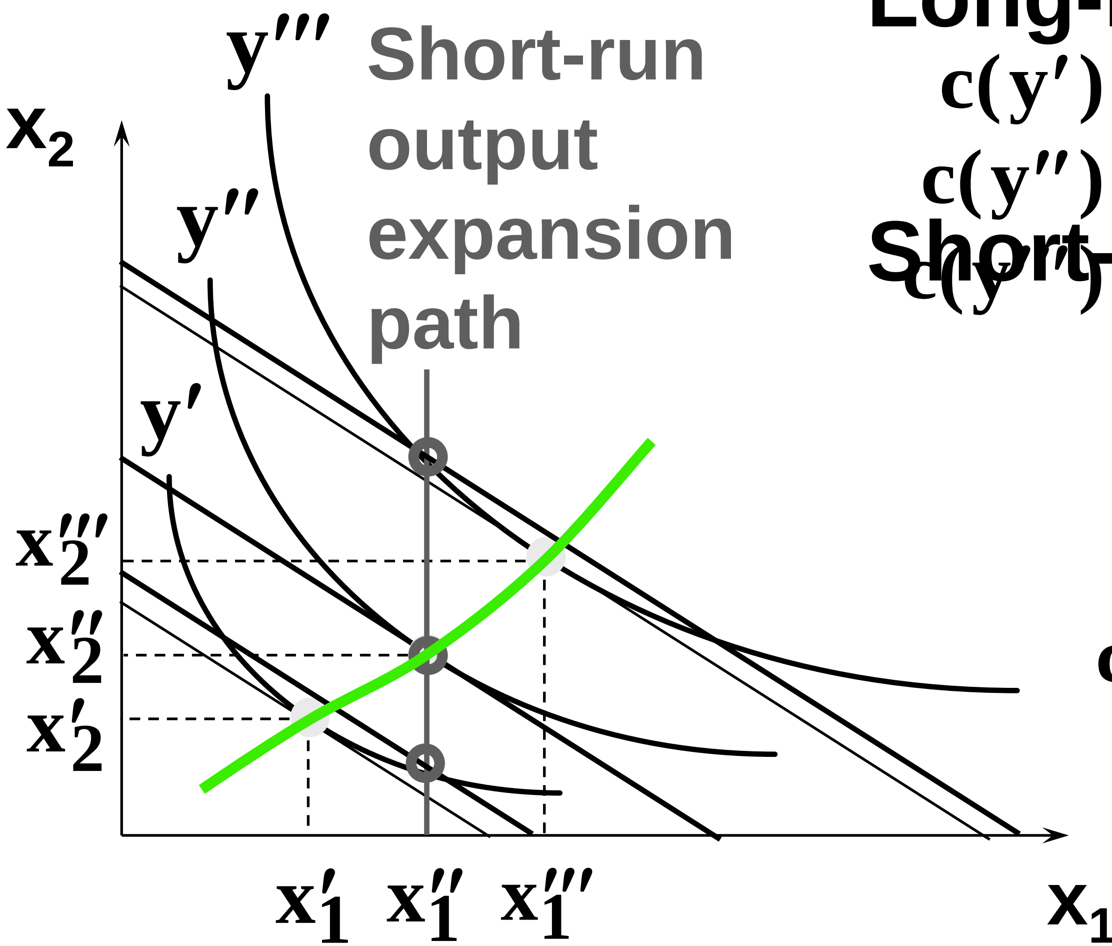

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Short-Run & Long-Run Total Costs

- **Short-run total cost exceeds long-run total cost except for the output level where the short-run input level restriction is the long-run input level choice.**
- **This says that the long-run total cost curve always has one point in common with any particular short-run total cost curve.**

Short-Run & Long-Run Total Costs

A short-run total cost curve always has one point in common with the long-run total cost curve, and is elsewhere higher than the long-run total cost curve.