Ολοκληρώματα ρητών συναρτήσεων

Πρώτη περίπτωση ①

$\dfrac{f(x)}{g(x)}$ Βαθ $f(x) <$ Βαθ $g(x)$

$g(x)$ έχει απλές ρίζες

Παραδείγματα

1) $\displaystyle\int \frac{1}{x-3}\,dx = \ln|x-3| + c$

2) $\displaystyle\int \frac{1}{x^2-5x+6}\,dx$

$\Delta = (5)^2 - 4\cdot 6 = 25 - 24 = 1 > 0$

$\rho_{1,2} = \dfrac{-(-5)\pm 1}{2} = \dfrac{5\pm 1}{2}$ $\rho_1 = 2$ $\rho_2 = 3$

$x^2 - 5x + 6 = (x-2)^1(x-3)^1$

$$x^2-5x+6=(x-2)(x-3)$$

ΑΝΑΛΥΣΗ ΣΕ ΑΠΛΑ ΚΛΑΣΜΑΤΑ (2)

$$\frac{1}{x^2-5x+6}=\boxed{\frac{1}{(x-2)(x-3)}}=\frac{A}{x-2}+\frac{B}{x-3}\Rightarrow$$

$$\frac{1}{(x-2)(x-3)}=\frac{A(x-3)+B(x-2)}{(x-2)(x-3)}\Rightarrow\frac{1}{(x-2)(x-3)}=\frac{(A+B)x-3A-2B}{(x-2)(x-3)}$$

$$\Rightarrow 1=(A+B)x-3A-2B\Rightarrow\begin{cases}A+B=0\\-3A-2B=1\end{cases}\Rightarrow\begin{cases}A=-B\\-3A-2B=1\end{cases}$$

$$\Rightarrow\begin{cases}A=-B\\3B-2B=1\end{cases}\Rightarrow\begin{cases}A=-1\\B=1\end{cases}$$

$$\int \frac{1}{x^2-5x+6}\,dx = \int\left(\frac{-1}{x-2} + \frac{1}{x-3}\right)dx =$$

$$= \int -\left(\frac{1}{x-2}\right)dx + \int \frac{1}{x-3}\,dx = -\int \frac{1}{x-2}\,dx + \int \frac{1}{x-3}\,dx$$

$$= -\ln|x-2| - C_1 + \ln|x-3| + C_2 = \ln|x-3| - \ln|x-2| + C_2 - C_1$$

$$= \ln\left|\frac{x-3}{x-2}\right| + C, \quad C = C_2 - C_1, \; C_1, C_2 \in \mathbb{R}, \; C \in \mathbb{R}.$$

$$\boxed{\int x^k\,dx = \frac{x^{k+1}}{k+1} + C, \quad k \neq -1}$$

3) Υπολογίστε το $\displaystyle\int \frac{x^2-x-1}{(x-1)(x-2)(x+3)}\,dx$

ΑΝΑΛΥΣΗ ΣΕ ΑΠΛΑ ΚΛΑΣΜΑΤΑ

$$\frac{x^2-x-1}{(x-1)(x-2)(x+3)} = \frac{A_1}{x-1} + \frac{A_2}{x-2} + \frac{A_3}{x+3} = \frac{A_1(x-2)(x+3)+A_2(x-1)(x+3)+A_3(x-1)(x-2)}{(x-1)(x-2)(x+3)}$$

$$\Rightarrow x^2-x-1 = A_1(x-2)(x+3)+A_2(x-1)(x+3)+A_3(x-1)(x-2) \quad \forall x\in\mathbb{R}$$

$x=2$ Εχω $2^2-2-1=1=A_2(2-1)(2+3) \Rightarrow 1=5A_2 \Rightarrow \boxed{A_2=\frac{1}{5}}$

$x=1$ Εχω $1^2-1-1=-1=A_1(1-2)(1+3)=-4A_1 \Rightarrow \boxed{A_1=\frac{1}{4}}$

$x=-3$ Εχω $(-3)^2-(-3)-1=11=A_3(-3-1)(-3-2)=20A_3 \Rightarrow \boxed{A_3=11/20}$

$$\int \frac{x^2-x-1}{(x-1)(x-2)(x+3)}\,dx = \int \left(\frac{1/4}{x-1} + \frac{1/5}{x-2} + \frac{11/20}{x+3} \right) dx$$ 5

$$= \frac{1}{4}\int \frac{1}{x-1}\,dx + \frac{1}{5}\int \frac{1}{x-2}\,dx + \frac{11}{20}\int \frac{1}{x+3}\,dx$$

$$= \frac{1}{4}\ln|x-1| + \frac{1}{5}\ln|x-2| + \frac{11}{20}\ln|x+3| + C, \quad C \in \mathbb{R}$$

$$C = \underbrace{\frac{1}{4}c_1 + \frac{1}{5}c_2 + \frac{11}{20}c_3}_{= C}$$

4) Υπολογίστε το $\int \frac{x+1}{(x+3)(x^2-5x+6)}\,dx$ (6)

$$x^2-5x+6=(x-2)(x-3)$$

$$\frac{x+1}{(x+3)(x^2-5x+6)}=\frac{x+1}{(x+3)(x-2)(x-3)}=\frac{A_1}{x+3}+\frac{A_2}{x-2}+\frac{A_3}{x-3}$$

$$\Rightarrow x+1=A_1(x-2)(x-3)+A_2(x+3)(x-3)+A_3(x+3)(x-2)$$

- $x=2$
- $x=3$
- $x=-3$

$\Rightarrow \ldots$ Άσκηση στο σπίτι

B περίπτωση

$\int \frac{f(x)}{g(x)}dx$ Βαθ $f(x)$ < Βαθ $g(x)$

Το $g(x)$ έχει ρίζες πολλαπλές πραγματικές ή και απλές και πολλαπλές πραγματικές ρίζες

Παραδείγματα

1) $\int \frac{1}{x^2-6x+9}dx$

$\Delta = (-6)^2 - 4\cdot 9 = 0$

$x^2-6x+9 = (x-3)(x-3) = (x-3)^2$

$$\int \frac{1}{x^2-6x+9}dx = \int \frac{1}{(x-3)^2}dx$$

$$\int \frac{1}{(x-3)^2}\,dx$$

Θέτω $\boxed{\omega = x-3} \Rightarrow d\omega = d(x-3) = (x-3)'\,dx = dx \Rightarrow$

$\Rightarrow \boxed{d\omega = dx}$

$$\int \frac{1}{(x-3)^2}\,dx = \int \frac{1}{\omega^2}\,d\omega = \int \omega^{-2}\,d\omega = \frac{\omega^{-2+1}}{-2+1} + c$$

$$= \frac{\omega^{-1}}{-1} + c = -\frac{1}{\omega} + c = -\frac{1}{x-3} + c$$

$$\int \frac{1}{x^2-6x+9}\,dx = -\frac{1}{x-3}+c$$

2) $$\int \frac{1}{(x^2-6x+9)^{10}}\,dx = \int \frac{1}{\left((x-3)^2\right)^{10}}\,dx = \int \frac{1}{(x-3)^{20}}\,dx$$

Απο την άσκηση 1

Θέτω $x-3=\omega \Rightarrow d\omega = dx$

$$\int \frac{1}{(x-3)^{20}}\,dx = \int \frac{1}{\omega^{20}}\,d\omega = \frac{\omega^{-20+1}}{-20+1}+c = -\frac{1}{19}\frac{1}{\omega^{19}}+c = \frac{-1}{19}\frac{1}{(x-3)^{19}}+c$$

3) Υπολογίστε το $\displaystyle\int \frac{x+1}{(x+2)(x^2-6x+9)}\,dx$ 10

$\Delta=(-6)^2-4\cdot 9=0 \qquad x^2-6x+9=(x-3)^2$

$$\frac{x+1}{(x+2)(x^2-6x+9)}=\frac{x+1}{(x+2)(x-3)^2}=\frac{A_1}{x+2}+\frac{A_2}{x-3}+\frac{A_3}{(x-3)^2}$$

$$\Rightarrow \frac{x+1}{(x+2)(x-3)^2}=\frac{A_1(x-3)^2+A_2(x+2)(x-3)+A_3(x+2)}{(x+2)(x-3)^2}$$

$$\Rightarrow x+1 = A_1(x-3)^2 + A_2(x+2)(x-3) + A_3(x+2) \quad \forall x \in \mathbb{R}$$

$x=3$ Έχω: $3+1 = A_3 \cdot 5 \Rightarrow \boxed{A_3 = \frac{4}{5}}$

$x=-2$ Έχω: $-2+1 = A_1(-2-3)^2 = 25A_1 \Rightarrow \boxed{A_1 = -\frac{1}{25}}$

$x=-1$ Έχω: $0 = A_1(-1-3)^2 + A_2(-1+2)(-1-3) + A_3(-1+2)$

$$\Rightarrow 0 = 16A_1 - 4A_2 + A_3 \Rightarrow$$

$$\Rightarrow 4A_2 = 16A_1 + A_3 \Rightarrow 4A_2 = -\frac{16}{25} + \frac{4}{5} = \frac{4}{25}$$

$$\Rightarrow \boxed{A_2 = \frac{1}{25}}$$

$$\int \frac{x+1}{(x+2)(x^2-6x+9)}\,dx = \int \frac{x+1}{(x+2)(x-3)^2}\,dx =$$ 12

$$= \int \left(\frac{-\frac{1}{25}}{x+2} + \frac{\frac{1}{25}}{x-3} + \frac{\frac{4}{5}}{(x-3)^2} \right) dx =$$

$$= -\frac{1}{25}\int \frac{1}{x+2}\,dx + \frac{1}{25}\int \frac{1}{x-3}\,dx + \frac{4}{5}\int \frac{1}{(x-3)^2}\,dx$$

$$= -\frac{1}{25}\left(\ln|x+2| + C_1\right) + \frac{1}{25}\left(\ln|x-3| + C_2\right) + \frac{4}{5}\left(\frac{-1}{x-3} + C_3\right) =$$

$$= -\frac{1}{25}\ln|x+2| + \frac{1}{25}\ln|x-3| - \frac{4}{5}\frac{1}{x-3} - \frac{1}{25}c_1 + \frac{1}{25}c_2 + \frac{4}{5}c_3$$ 13

$$= -\frac{1}{25}\ln|x+2| + \frac{1}{25}\ln|x-3| - \frac{4}{5}\frac{1}{x-3} + C,$$

$$C = -\frac{1}{25}c_1 + \frac{1}{25}c_2 + \frac{4}{5}c_3, \quad c_1, c_2, c_3 \in \mathbb{R}, \quad C \in \mathbb{R}$$