Παραγώγιση - κανόνας της αλυσίδας

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$$y(x) = f(g(x))$$

$$\frac{dy}{dx} = \frac{df}{dg}\frac{dg}{dx}$$

3. $y(x) = e^{x^2}$ $\quad f(x) = e^x$ $\quad g(x) = x^2$

$$\frac{dy}{dx} = \frac{de^{x^2}}{dx^2}\frac{dx^2}{dx} = e^{x^2}2x = 2xe^{x^2}$$

$$y(x) = e^{x^2}$$

$$\frac{dy(x)}{dx} = e^{x^2}\cdot(x^2)' = 2xe^{x^2}$$

4. $y(x) = ημ(2x+3)$ $\quad f(x) = ημx \quad g(x) = 2x+3$

$y(x) = f(g(x)) \qquad \dfrac{dy}{dx} = \dfrac{df}{dg}\dfrac{dg}{dx} = \dfrac{d\,ημ(2x+3)}{d(2x+3)}\dfrac{d(2x+3)}{dx} =$

$= συν(2x+3)\cdot 2 = 2συν(2x+3)$

5. $y(x) = ημ(συνx)$

$y'(x) = \dfrac{dy(x)}{dx} = συν(συνx)\cdot(συνx)' = -ημx\,συν(συνx)$

6. $y(x) = εφ(ημx)$, $y'(x) = \dfrac{1}{συν^2(ημx)}\cdot(ημx)' = \dfrac{συνx}{συν^2(ημx)}$

7. $y(x)=\sqrt{\eta\mu x}\ ,\ y'(x)=\dfrac{1}{2\sqrt{\eta\mu x}}\cdot(\eta\mu x)'=\dfrac{\sigma\upsilon\nu x}{2\sqrt{\eta\mu x}}$

8. $y(x)=\dfrac{1}{\sigma\upsilon\nu x}\ ,\ y'(x)=\left[(\sigma\upsilon\nu x)^{-1}\right]'=-(\sigma\upsilon\nu x)^{-1-1}\cdot(\sigma\upsilon\nu x)'$

$=-(\sigma\upsilon\nu x)^{-2}(-\eta\mu x)=\dfrac{\eta\mu x}{(\sigma\upsilon\nu x)^2}$

9. $y(x)=e^{\frac{1}{x}}\ ,\ y'(x)=e^{\frac{1}{x}}\cdot\left(\dfrac{1}{x}\right)'=-\dfrac{1}{x^2}e^{\frac{1}{x}}$

10. $\dfrac{1}{\sqrt{x-1}\ \eta\mu x}=y(x),\ y'(x)=\dfrac{1'\sqrt{x-1}\ \eta\mu x-1\left(\sqrt{x-1}\ \eta\mu x\right)'}{\left(\sqrt{x-1}\ \eta\mu x\right)^2}=-\dfrac{(\sqrt{x-1})'\eta\mu x+\sqrt{x-1}\,(\eta\mu x)'}{(x-1)(\eta\mu x)^2}$

$$= -\frac{\frac{1}{2\sqrt{x-1}}(x-1)'\eta\mu x + \sqrt{x-1}\,\sigma\upsilon\nu x}{(x-1)(\eta\mu x)^2} = -\frac{\eta\mu x + 2(x-1)\sigma\upsilon\nu x}{2\sqrt{x-1}\,(x-1)(\eta\mu x)^2}$$

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Τον κανόνα της αλυσίδας μπορούμε να τον εφαρμόσουμε και επανειλημμένα

Παραδείγματα

1. $y(x) = [\eta\mu(3x+2)]^2, \quad y'(x) = 2\,\eta\mu(3x+2)\cdot[\eta\mu(3x+2)]'$

$= 2\eta\mu(3x+2)\cdot\sigma\upsilon\nu(3x+2)\cdot(3x+2)'$

$= 6\eta\mu(3x+2)\sigma\upsilon\nu(3x+2)$

2. $y(x) = \text{εφ}(\text{συν}(x^2))$, $y'(x) = \dfrac{1}{\text{συν}^2[\text{συν}(x^2)]} \cdot [\text{συν}(x^2)]'$ 5

$$= -\frac{1}{\text{συν}^2[\text{συν}(x^2)]} \cdot \text{ημ}(x^2) \cdot (x^2)' = -\frac{2x\,\text{ημ}(x^2)}{\text{συν}^2[\text{συν}(x^2)]}$$

3. $y(x) = \text{ημ}(\text{συν}(\text{ημ}x))$, $y'(x) = \text{συν}(\text{συν}(\text{ημ}x)) \cdot (\text{συν}(\text{ημ}x))' = \text{συν}(\text{συν}(\text{ημ}x)) \cdot (-\text{ημ}(\text{ημ}x)) \cdot (\text{ημ}x)' =$

$$= -\text{συν}x \cdot \text{συν}(\text{συν}(\text{ημ}x)) \cdot \text{ημ}(\text{ημ}x)$$

4. $y(x) = 5\sqrt{e^{\sqrt{x}}}$, $y'(x) = 5\dfrac{1}{2\sqrt{e^{\sqrt{x}}}}\left(e^{\sqrt{x}}\right)' = 6$

$$= \frac{5}{2\sqrt{e^{\sqrt{x}}}}\, e^{\sqrt{x}} \cdot \left(\sqrt{x}\right)' = \frac{5e^{\sqrt{x}}}{4\sqrt{e^{\sqrt{x}}}\,\sqrt{x}}$$

$$\left(\sqrt{x}\right)' = \frac{1}{2\sqrt{x}}$$

Παραγώγιση συναρτήσεων της μορφής $f(x)^{g(x)}$, $f(x)>0$ 7

Υπενθύμιση

1. $\alpha = e^{\ln \alpha}, \alpha>0$ $\qquad f(x) = e^{\ln f(x)}, f(x)>0$

2. $\ln \alpha^{b} = b \ln \alpha, \alpha>0$ $\qquad \ln f(x)^{g(x)} = g(x) \ln f(x), f(x)>0$

$(x^2)' = 2x, \quad (2^x)' = ?$

$(2^x)'$

$2^x > 0 \ \forall x, \quad 2^x = e^{\ln 2^x} = e^{x \ln 2} \Rightarrow (2^x)' = (e^{x \ln 2})' \Rightarrow$

$\Rightarrow (2^x)' = e^{x \ln 2} (x \ln 2)' = 2^x \ln 2 \Rightarrow (2^x)' = 2^x \ln 2$

B τρόπος

$y = 2^x \Rightarrow \ln y = \ln 2^x \Rightarrow \ln y = x\ln 2 \Rightarrow$

$\boxed{y = y(x) = 2^x}$ $\Rightarrow (\ln y)' = (x\ln 2)' \Rightarrow \dfrac{y'}{y} = \ln 2 \Rightarrow y' = y\ln 2$

$\Rightarrow y' = 2^x \ln 2$

2. $y(x) = 2^{\varepsilon\varphi x}$

$2^{\varepsilon\varphi x} = e^{\ln 2^{\varepsilon\varphi x}} = e^{\varepsilon\varphi x \ln 2} \Rightarrow (2^{\varepsilon\varphi x})' = (e^{\varepsilon\varphi x \ln 2})' \Rightarrow$

$\Rightarrow (2^{\varepsilon\varphi x})' = e^{\varepsilon\varphi x \ln 2}(\varepsilon\varphi x \ln 2)' = 2^{\varepsilon\varphi x} \dfrac{\ln 2}{\sigma\upsilon\nu^2 x} = \ln 2 \dfrac{2^{\varepsilon\varphi x}}{\sigma\upsilon\nu^2 x}$

B τρόπος

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$$y=2^{\varepsilon\varphi x}\Rightarrow \ln y=\ln 2^{\varepsilon\varphi x}=\varepsilon\varphi x\ln 2\Rightarrow(\ln y)'=(\varepsilon\varphi x\ln 2)'\Rightarrow$$

$$\Rightarrow\frac{y'}{y}=\frac{\ln 2}{\sigma\upsilon\nu^2 x}\Rightarrow y'=y\frac{\ln 2}{\sigma\upsilon\nu^2 x}=2^{\varepsilon\varphi x}\frac{\ln 2}{\sigma\upsilon\nu^2 x}$$

3.

$$\cancel{y}(x)=x^x,\ x>0$$

$$x^x=e^{\ln x^x}=e^{x\ln x}\Rightarrow(x^x)'=(e^{x\ln x})'=e^{x\ln x}(x\ln x)'$$

$$\Rightarrow(x^x)'=x^x\left(x'\ln x+x\frac{1}{x}\right)=x^x(\ln x+1)\Rightarrow$$

$$\Rightarrow\boxed{(x^x)'=x^x(\ln x+1)}$$

B τρόπος

$$y = x^x \Rightarrow \ln y = \ln x^x = x\ln x \Rightarrow (\ln y)' = (x\ln x)' \Rightarrow$$

$$\Rightarrow \frac{y'}{y} = (\ln x + 1) \Rightarrow y' = y(\ln x + 1) \Rightarrow y' = x^x(\ln x + 1)$$

A τρόπος $\quad x^x = e^{\ln x^x} = e^{x\ln x} \Rightarrow (x^x)' = (e^{x\ln x})' = \ldots$

B τρόπος $\quad y(x) = x^x \Rightarrow \ln y(x) = \ln x^x = x\ln x \Rightarrow$

$$\Rightarrow (\ln y(x))' = (x\ln x)' \Rightarrow$$

$$\Rightarrow \frac{y'(x)}{y(x)} = (x\ln x)' \Rightarrow \ldots$$

4. $y(x) = x^{x^x}$ (11)

Α τρόπος

$$x^{x^x} = e^{\ln x^{\boxed{x^x}}} = e^{x^x \ln x} \Rightarrow \left(x^{x^x}\right)' = \left(e^{x^x \ln x}\right)' \Rightarrow$$

$$\Rightarrow \left(x^{x^x}\right)' = \boxed{e^{x^x \ln x}}\left(x^x \ln x\right)' = x^{x^x}\left(\underline{(x^x)'}\ln x + x^x \frac{1}{x}\right)$$

Από την προηγούμενη άσκηση: $(x^x)' = x^x(\ln x + 1)$

Αν/στω:

$$\underline{\left(x^{x^x}\right)'} = \underline{x^{x^x}}\left(x^x(\ln x + 1)\ln x + x^{x-1}\right)$$

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Β τρόπος

$$y(x) = x^{x^x} \Rightarrow \ln y(x) = \ln x^{x^x} = x^x \ln x \Rightarrow (\ln y(x))' = (x^x \ln x)' \Rightarrow$$

$$\Rightarrow \frac{y'(x)}{y(x)} = \left((x^x)' \ln x + x^x (\ln x)'\right) \Rightarrow$$

$$\Rightarrow y'(x) = y(x)\left(x^x(\ln x + 1)\ln x + x^x \frac{1}{x}\right) \Rightarrow$$

$$\Rightarrow y'(x) = x^{x^x}\left(x^x(\ln x + 1)\ln x + x^{x-1}\right)$$

5.

$(ημx)^{\frac{1}{x}}$

$$(ημx)^{\frac{1}{x}} = e^{\ln(ημx)^{\frac{1}{x}}} = e^{\frac{1}{x}\ln(ημx)} \Rightarrow$$

$$\left((ημx)^{\frac{1}{x}}\right)' = \left(e^{\frac{1}{x}\ln(ημx)}\right)' = e^{\frac{1}{x}\ln(ημx)}\left(\frac{1}{x}\ln(ημx)\right)'$$

$$\Rightarrow \left[(ημx)^{\frac{1}{x}}\right]' = (ημx)^{\frac{1}{x}}\left(\left(\frac{1}{x}\right)'\ln(ημx) + \frac{1}{x}\left(\ln(ημx)\right)'\right) \Rightarrow$$

$$\Rightarrow \left[(ημx)^{\frac{1}{x}}\right]' = (ημx)^{\frac{1}{x}}\left(-\frac{1}{x^2}\ln(ημx) + \frac{1}{x}\,\frac{1}{ημx}\,συνx\right)$$

Β τρόπος Άσκηση Στο σπίτι $f(x)^{g(x)}$, $f(x)>0$

Α τρόπος

$$f(x)^{g(x)} = e^{\ln f(x)^{g(x)}} = e^{g(x)\ln f(x)} \Rightarrow \left(f(x)^{g(x)}\right)' =$$

$$= \left(e^{g(x)\ln f(x)}\right)' \Rightarrow \left(f(x)^{g(x)}\right)' = e^{g(x)\ln f(x)}\left(g(x)\ln f(x)\right)'$$

$$\Rightarrow \boxed{\left(f(x)^{g(x)}\right)' = f(x)^{g(x)}\left(g'(x)\ln f(x) + g(x)\frac{1}{f(x)}f'(x)\right)}$$

B Τρόπος

$$y(x) = f(x)^{g(x)} \Rightarrow \ln y(x) = \ln f(x)^{g(x)} = g(x)\ln f(x) \Rightarrow$$

$$\Rightarrow \left(\ln y(x)\right)' = \left(g(x)\ln f(x)\right)' \Rightarrow \frac{y'(x)}{y(x)} = \left(g(x)\ln f(x)\right)'$$

$$\Rightarrow \boxed{y'(x) = f(x)^{g(x)}\left(g'(x)\ln f(x) + g(x)\frac{f'(x)}{f(x)}\right)}$$