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Ολοκληρώματα ρητών συναρτήσεων

Β περίπτωση

$\frac{f(x)}{g(x)}$ 1) Βαθμός $f(x)$ < Βαθμός παρονομαστή
2) $g(x)$ έχει πολλαπλές ρίζες ή απλές και πολλαπλές

Παραδείγματα

1) $$I=\int \frac{1}{(x-2)^2}dx$$

$\boxed{\omega=x-2} \Rightarrow d\omega=d(x-2)=(x-2)'dx=dx \Rightarrow \boxed{d\omega=dx}$

$$I = \int \frac{1}{\omega^2}\, d\omega = \int \omega^{-2}\, d\omega = \frac{\omega^{-2+1}}{-2+1} + c = -\frac{1}{x-2} + c^{2}$$

$c \in \mathbb{R}$

2)

$$I = \int \frac{x-1}{(x-2)^2}\, dx$$

$$\frac{x-1}{(x-2)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2} = \frac{A(x-2)+B}{(x-2)^2}$$

$$\Rightarrow \frac{x-1}{(x-2)^2} = \frac{Ax-2A+B}{(x-2)^2} \Rightarrow x-1 = Ax-2A+B$$

$$\Rightarrow \begin{cases} A = 1 \\ -2A+B = -1 \end{cases} \Rightarrow \begin{cases} A = 1 \\ -2+B = -1 \end{cases} \Rightarrow \begin{cases} A = 1 \\ B = 1 \end{cases}$$

$$\frac{x-1}{(x-2)^2} = \frac{1}{x-2} + \frac{1}{(x-2)^2}$$

$$\int \frac{x-1}{(x-2)^2}\,dx = \int \frac{1}{x-2}\,dx + \int \frac{1}{(x-2)^2} = \ln|x-2| - \frac{1}{x-2} + c,$$

$c \in \mathbb{R}$

$$\frac{x-1}{(x-2)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2} = \frac{A(x-2)^2 + B(x-2)}{(x-2)(x-2)^2}$$

$$= \frac{A(x-2)+B}{(x-2)^2} \Rightarrow x-1 = A(x-2)+B$$

$$\Rightarrow \ldots$$

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$$I=\int \frac{x^2+x+1}{(x+2)(x+3)^2}\,dx$$

$$\frac{x^2+x+1}{(x+2)(x+3)^2}=\frac{A}{x+2}+\frac{B}{x+3}+\frac{\Gamma}{(x+3)^2}\Rightarrow$$

$$\Rightarrow \frac{x^2+x+1}{(x+2)(x+3)^2}=\frac{A(x+3)^2+B(x+2)(x+3)+\Gamma(x+2)}{(x+2)(x+3)^2}$$

$$\Rightarrow x^2+x+1=A(x+3)^2+B(x+2)(x+3)+\Gamma(x+2)$$

$x=-3$ $\quad (-3)^2+(-3)+1=\Gamma(-3+2)\Rightarrow 7=-\Gamma\Rightarrow \boxed{\Gamma=-7}$

$x=-2$ $\quad (-2)^2+(-2)+1=A(-2+3)^2=A\Rightarrow \boxed{A=3}$

$x=0$ $\quad 1=9A+6B+2\Gamma=27+6B-14\Rightarrow \boxed{B=-2}$

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$$\int \frac{x^2+x+1}{(x+2)(x+3)^2}dx = \int \frac{3}{x+2}dx + \int \frac{-2}{x+3}dx + \int \frac{-7}{(x+3)^2}dx$$

$$= 3\left(\ln|x+2| + c_1\right) - 2\left(\ln|x+3| + c_2\right) - 7\left(-\frac{1}{x+3} + c_3\right)$$

$$= 3\ln|x+2| - 2\ln|x+3| + \frac{7}{x+3} - 7c_3 + 3c_1 - 2c_2$$

$$= 3\ln|x+2| - 2\ln|x+3| + \frac{7}{x+3} + C,$$

$$C = 3c_1 - 2c_2 - 7c_3, \quad c_1, c_2, c_3 \in \mathbb{R}, \quad C \in \mathbb{R}$$

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$$\int \frac{1}{(x-2)^2(x+2)^2}\,dx$$

$$\frac{1}{(x-2)^2(x+2)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{\Gamma}{x+2} + \frac{\Delta}{(x+2)^2}$$

$$\Rightarrow \frac{1}{(x-2)^2(x+2)^2} = \frac{A(x-2)(x+2)^2 + B(x+2)^2 + \Gamma(x+2)(x-2)^2 + \Delta(x-2)^2}{(x-2)^2(x+2)^2}$$

$$\Rightarrow 1 = A(x-2)(x+2)^2 + B(x+2)^2 + \Gamma(x+2)(x-2)^2 + \Delta(x-2)^2$$

■ $x=2 \Rightarrow B = \dots$ ■ $x=-2 \Rightarrow \Delta = \dots$

■ $x=0 \Rightarrow * A + * B = *$ ■ $x=1 \Rightarrow * A + * B = *$

Γ Περίπτωση

$\frac{f(x)}{g(x)}$ 1) Βαθμός $f(x)$ < Βαθμός $g(x)$

2) $g(x)$ να μην έχει πραγματικές ρίζες, δηλ. να έχει μιγαδικές, ή να έχει και πραγματικές και μιγαδικές

Παραδείγματα

1) $\int \frac{1}{x^2-x+1}\,dx$

$\Delta = (-1)^2 - 4 \cdot 1 = 1 - 4 = -3 < 0$

$1 = 2 \cdot \frac{1}{2}$

$x^2 - x + 1 = x^2 - 2\frac{1}{2}x + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1 = \left(x - \frac{1}{2}\right)^2 - \frac{1}{4} + \frac{4}{4} = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4}$

$= \left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$

$$\int \frac{1}{x^2-x+1}dx = \int \frac{1}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dx = \int \frac{1}{\left(\frac{\sqrt{3}}{2}\right)^2\left(1+\frac{\left(x-\frac{1}{2}\right)^2}{\left(\frac{\sqrt{3}}{2}\right)^2}\right)}dx$$

$$= \frac{1}{\left(\frac{\sqrt{3}}{2}\right)^2}\int \frac{1}{1+\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)^2}dx = \frac{4}{3}\int \frac{1}{1+\left(\frac{\frac{2x-1}{2}}{\frac{\sqrt{3}}{2}}\right)^2}dx$$

$$= \frac{4}{3}\int \frac{1}{1+\left(\frac{2x-1}{\sqrt{3}}\right)^2}dx$$

Θέτω $\omega = \frac{2x-1}{\sqrt{3}}$ $\Rightarrow d\omega = d\left(\frac{2x-1}{\sqrt{3}}\right)$

$$\Rightarrow d\omega = \left(\frac{2x-1}{\sqrt{3}}\right)'dx = \frac{2}{\sqrt{3}}dx \Rightarrow dx = \frac{\sqrt{3}}{2}d\omega$$

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Επομένως $\displaystyle\int \frac{1}{x^2-x+1}\,dx = \frac{4}{3}\int \frac{1}{1+\omega^2}\,\frac{\sqrt{3}}{2}\,d\omega$

$\displaystyle = \frac{4}{3}\,\frac{\sqrt{3}}{2}\int \frac{1}{1+\omega^2}\,d\omega = \frac{2}{\sqrt{3}}\left(\text{τοξεφ}\,\omega + c\right)$

$\displaystyle = \frac{2}{\sqrt{3}}\,\text{τοξεφ}\,\frac{2x-1}{\sqrt{3}} + \frac{2}{\sqrt{3}}\,c = \frac{2}{\sqrt{3}}\,\text{τοξεφ}\,\frac{2x-1}{\sqrt{3}}$

$+\,C_1,\quad C_1 = \frac{2}{\sqrt{3}}\,c,\ c\in\mathbb{R},\ C_1\in\mathbb{R}.$

$x^2 - x + 1 \qquad \rho_{1,2} = \frac{-(-1) \pm \sqrt{\Delta}}{2} = \frac{1 \pm \sqrt{-3}}{2} = \frac{1 \pm \sqrt{-1}\sqrt{3}}{2}$ ($\sqrt{-1} = i$!!)

$$\Rightarrow \rho_{1,2} = \frac{1 \pm i\sqrt{3}}{2}$$

2)

$$I = \int \frac{x+2}{x^2 - x + 1}\, dx$$

$$\frac{x+2}{x^2 - x + 1} = \frac{1}{2} \frac{2(x+2)}{x^2 - x + 1} = \frac{1}{2} \frac{2x+4}{x^2-x+1} = \frac{1}{2} \frac{(2x-1)+5}{x^2-x+1}$$

$(x^2 - x + 1)' = 2x - 1$

$$= \frac{1}{2} \frac{2x-1}{x^2-x+1} + \frac{1}{2} \frac{5}{x^2-x+1}$$

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$$I = \frac{1}{2}\int \frac{2x-1}{x^2-x+1}dx + \frac{1}{2}\int \frac{5}{x^2-x+1}dx$$

Βασικό ολοκλήρωμα $\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$

$$\Rightarrow I = \frac{1}{2}\left(\ln|x^2-x+1| + c_1\right) + \frac{5}{2}\int \frac{1}{x^2-x+1}dx$$

$\Delta < 0$

$x^2-x+1>0 \ \forall x \in \mathbb{R}$

$$\Rightarrow I = \frac{1}{2}\ln(x^2-x+1) + \frac{5}{2}\frac{2}{\sqrt{3}} \text{τοξεφ}\frac{2x-1}{\sqrt{3}} + \underbrace{\frac{1}{2}c_1 + \frac{5}{2}c_2}_{C}$$

$$\Rightarrow I = \frac{1}{2}\ln(x^2-x+1) + \frac{5}{\sqrt{3}}\text{τοξεφ}\frac{2x-1}{\sqrt{3}} + C, \quad c_1, c_2 \in \mathbb{R}, C \in \mathbb{R}$$