CHAPTER 18

DUALITY FOR THE HOUSEHOLD: THEORY AND APPLICATIONS

JEFFREY T. LA FRANCE
University of California, Berkeley

0. INTRODUCTION

The purpose of this chapter is to present a concise summary of the current status of microeconomic theory of consumer choice. In one form or another, this theory underpins all economic analyses of food demand. Many modern treatments employ household production theory, which combines the economic theory of the firm with that of the consumer. A detailed discussion of the naïve neoclassical model of consumer choice is contained in Barten and Böhm (1982), while Deaton (1986) addresses the econometric issues of naïve demand analysis. Similarly, Nadiri (1982) presents an in depth survey of neoclassical production theory, while Jorgenson (1986) deals with econometric modeling for producer behavior. The focus here is on a synthesis that incorporates consumer preferences, household production activities, quality attributes of foods and other goods, and produced nonmarket household commodities. An outline of the underlying individual components essential for this synthesized framework are presented first, followed by a general abstract model of consumer choice in a static framework. This framework is extended and refined to accommodate consumer choice problems in intertemporal environments. In this framework, expectations for future prices, incomes, and asset returns, durable goods in household consumption, and naïve and rational habit formation are analyzed. Many new results are contained in this chapter which can not be found elsewhere in the literature either in the same form or level of generality. However, they all have been derived using straightforward applications of previously existing analytical tools.

The chapter is organized in the following way. Section 1 briefly summarizes the neoclassical theory of consumer choice. Section 2 develops the theory of household production, combines and relates this theory to the neoclassical model and important special cases, including hedonic price functions, Gorman's and Lancaster's characteristics model of quality, and the Fisher-Shell repackaging model. Section 3 discusses dynamic models of consumer choice, with an emphasis on the role of individuals' expectations about their future economic environment, including models of perfect foresight and myopic, adaptive, quasi-rational, and rational expectations. Section 4 presents the economic theory of intertemporal choice in a household production framework with durable stocks. Some of these stocks may be unobservable "consumption habits" and naïve and rational habit formation models are interpreted in this context. The final section summarizes the main results. The primary emphasis throughout the chapter is to develop and analyze a valid duality in each of the generalizations to the neoclassical model of consumer choice.

1. NEOCLASSICAL DEMAND THEORY

Neoclassical consumer choice theory begins with the set of bundles of consumer goods that can be selected by a consuming household, X, a subset of a separable topological space. In this section, we take X to be a subset of a finite dimensional Euclidean space. Associated with the set X is a binary preference relation, $\succsim$. The notation "$x \succsim y$" means the consumption bundle x is at least as preferred as y. The relation $\succsim$ is endowed with properties that ensure that consumer choices are logically consistent. The following is a standard representation:

(i) *reflexivity*; $\forall\, x \in X,\ x \succsim x$;

(ii) *transitivity*; $\forall\ x, y, z \in X,\ x \succsim y\ and\ y \succsim z \Rightarrow x \succsim z$;

(iii) *completeness*; $\forall\ x, y \in X,\ either\ \ x \succsim y\ \ or\ \ y \succsim x$;

(iv) *closure*; $\forall\, x \in X$, *the sets* $\{x^1 \in X : x^1 \succsim x\}$ *and* $\{x^1 \in X : x \succsim x^1\}$ *are closed.*

These properties imply that $\succsim$ is a complete ordering on X and that there exists a continuous utility function, u: X $\rightarrow$ $\mathbb{R}$, such that $\forall\, x,\ x^1 \in X,\ u(x) \geq u(x^1)$ if and only if $x \succsim x^1$ (Bowen (1968); Debreu (1954, 1959, 1964); Eilenberg (1941); Rader (1963)).

The consumer's decision problem is to choose a bundle of market goods from the set X that is maximal for $\succsim$, given market prices, $p \in \mathbb{R}^n_+$, and income, $m \in \mathbb{R}_+$. Given properties i-iv, this can be represented as $\sup\{u(x)\colon\ x \in X,\ p'x \le m\}$.[1] Let $\succ$ denote the binary relation "strictly preferred to", so that $x \succ y$ means $x \succsim y$ and not $y \succsim x$. Then the solution to the consumer's utility maximization problem is unique under the following additional conditions:

(v) *nonsatiation*; $\nexists\ x \in X \ni x \succ x^1\ \forall\ x^1 \in X$;

(vi) *strict convexity*; $x \succsim x^1$ *and* $t \in (0,1) \Rightarrow (tx + (1-t)x^1) \succ x^1$;

(vii) *survival*; $\inf\{p'x : x \in X\} < m$; *and*

(viii) *X is convex and bounded from below by* $\mathbf{0}$, *i.e.*, $x \in X \Rightarrow x_i \ge 0\ \forall\ i$.

In addition to continuity, properties i-viii imply that the utility function is strictly quasiconcave (Arrow and Enthoven (1961)), the utility-maximizing demand set is nonempty and a singleton, and the budget constraint is satisfied with equality at the optimal choice for the consumption bundle. The utility maximizing quantities demanded, $x = h(p,m)$, are known as the Marshallian ordinary demand functions. Marshallian demands are positive-valued and have the following properties:

(M.1) $0°$ *homogeneity in* (p,m); $h(p,m) \equiv h(tp,tm)\ \forall\ t \ge 0$;

(M.2) *adding up*; $p'h(p,m) \equiv m$; *and*

(M.3) *symmetry and negativity; the matrix of substitution effects,*

$$\boldsymbol{S} \equiv \left[\frac{\partial h(p,m)}{\partial p'} + \frac{\partial h(p,m)}{\partial m} h(p,m)'\right],$$

is symmetric and negative semidefinite, provided that $\boldsymbol{S}$ *exists and is continuous.*

The maximum level of utility given prices p and income m, $v(p,m) \equiv u[h(p,m)]$, is the indirect utility function. Under i-viii, the indirect utility function has the following properties:

[1] The notation "sup" denotes the supremum, or least upper bound, of the objective function on the associated set. Since the utility function is continuous, if the set X is closed and bounded from below and $p \gg \mathbf{0}$, then we can replace "sup" with "max". Similarly, the notation "inf" denotes the infimum, or greatest lower bound, of the objective function on the choice set. If the choice set is compact (closed and bounded) and the objective function is continuous, then we can replace "inf" with "min".

(V.1) *continuous in* (p, m);

(V.2) *decreasing and strictly quasiconvex in* p;

(V.3) *increasing in* m;

(V.4) $0°$ *homogeneous in* (p, m);

(V.5) *Roy's identity,*

$$h(p,m) \equiv -\left(\frac{\partial v(p,m)/\partial p}{\partial v(p,m)/\partial m}\right),$$

provided the right-hand side is well-defined.

Dual to the utility maximization problem is the problem of minimizing the expenditure necessary to obtain a fixed level of utility, u, given market prices p, $\inf\{p'x \colon x \in X,\ u(x) \geq u\}$. The expenditure minimizing demands, $g(p, u)$, are known as the Hicksian compensated demand functions. Hicksian demands are positive valued and have the following properties:

(H.1) $0°$ *homogeneous in* p;

(H.2) *the Slutsky equations*;

$$\left[\frac{\partial g(p,u)}{\partial p'}\right] \equiv \left[\frac{\partial h(p,e(p,u))}{\partial p'} + \frac{\partial h(p,e(p,u))}{\partial m} h(p,e(p,u))'\right]$$

is symmetric and negative semidefinite, provided the derivatives exist and are continuous.

The expenditure function, $e(p,u) \equiv p'g(p,u)$, has the following properties:

(E.1) *continuous in* (p, u);

(E.3) *increasing,* $1°$ *homogeneous, and concave in* p;

(E.4) *increasing in u; and*

(E.5) *Shephard's Lemma,*

$$g(p,u) \equiv \frac{\partial e(p,u)}{\partial p},$$

provided the derivatives on the right exist.

A large body of theoretical and empirical literature exists for the neoclassical model of consumer choice. Much of this literature is based on the observation that $e(p, u)$ and $v(p, m)$ are inverse functions with respect to their n+1[st] arguments, yielding, inter alia, the following set of identities:

(1.1) $e[p, v(p,m)] \equiv m$;

(1.2) $v[p, e(p,u)] \equiv u$;

(1.3) $g(p,u) \equiv h[p, e(p,u)]$; *and*

(1.4) $h(p,m) \equiv g[p, v(p,m)]$.

However, the neoclassical model has few empirical implications, embodied in the sign and symmetry of the substitution effects due to changes in the market prices of the goods x, and leaves all variances in consumption behavior not explained by prices and income to differences in tastes and preferences. The neoclassical model also is entirely static. Finally, the neoclassical model does not readily accommodate technological change, the introduction of new goods in the market, or changes in the quality or characteristics of the goods that are available. These considerations led to extensions of the neoclassical model of consumer choice. Among these extensions, the most widely employed is the theory of household production. The seminal references are Becker (1965), Gorman (1956), and Lancaster (1966, 1971). Household production theory integrates the neoclassical theory of the consumer with that of the firm. The theory of the firm relates to that part of household decision making that is concerned with the efficient use of market goods, household time, and capital as inputs in the production of utility-yielding non-market commodities. The model posits that market goods and household time are combined via production processes analogous to the production functions of the theory of the firm to produce various commodities from which utility is obtained directly. This approach advances traditional consumer choice theory by permitting analyses of issues such as the number of family members in the work force, time as a constraining factor in household consumption choices, quality changes among goods, durable goods in consumption, and consumer reactions to the introduction of new goods in the market place.

2. THE THEORY OF HOUSEHOLD PRODUCTION

In this section, we present a model of consumer choice that is sufficiently general and rich to account for many of the concerns summarized above. We first need some preliminary definitions and notation. Let $x \in \mathbb{R}_+^n$ denote market goods and time used by the household, let $b \in \mathbb{R}^s$ be a vector of parameters associated with the market goods, objectively measured and quantifiable by all economic agents, and let $z \in \mathbb{R}^m$ be a vector of utility bearing commodities or service flows desired by the household and produced from *x*. We assume that there is a household production relationship for each household relating *x* to *z*, and this relationship depends explicitly on the parameters, *b*. For given *b*, let $T(b) \subset \mathbb{R}^{m+n}$ denote a joint production possibilities set and let $y = [x' \ z']' \in T(b)$ denote a feasible vector of goods and commodities. For each possible *b*, the properties of $T(b)$ associated with a well-defined joint production function are (Rockafellar (1970); Jorgenson and Lau(1974)):

(i) *origin;* $\mathbf{0} \in T(b)$;

(ii) *bounded;* $\forall\ i,\ y \in T(b)\ \textit{and}\ |y_j| < \infty\ \forall\, j \neq i \Rightarrow |y_i| < \infty$,

(iii) *closure;* $y^n \in T(b)\ \forall\ n\ \textit{and}\ y^n \to y \Rightarrow y \in T(b)$;

(iv) *convexity;* $y, y^1 \in T(b)\ \textit{and}\ t \in [0,1] \Rightarrow ty + (1-t)y^1 \in T(b)$;

(v) *monotonicity;* $\exists\ i \ni y \in T(b),\ y_j' = y_j\ \ \forall\ \ j \neq i,\ \textit{and}\ \ y_i' \leq y_i \Rightarrow y' \in T(b)$.

Given i-v, we define the production function by

(2.1) $-F(y_{\sim i}, b) = \sup\{y_i :\ y \in T(b)\}$,

where $y_{\sim i}$ is the subvector of elements excluding the i^{th} with *i* chosen to satisfy v. Then $\forall\ y \in T(b)$, we have $y_i + F\left(y_{\sim i}, b\right) \leq 0$. For a given value of *b*, the function $F(\cdot,\, b)$ is closed (lower semicontinuous), proper ($F(y_{\sim i}, b) < +\infty$ for at least one $y_{\sim i}$ and $F(y_{\sim i}, b) > -\infty\ \forall\ y_{\sim i}$), and convex. Monotonicity in at least one element of *y* (i.e., free disposal of y_i) is equivalent to applicability of the implicit function theorem to the transformation $G(y,b) = 0$, which defines the boundary of $T(b)$, to obtain the form (2.1). Free disposal of all elements of *y* implies monotonicity of $G(\cdot, b)$ in *y*.

We define the *epigraph* of $F(\cdot,\, b)$ to be the set

(2.2) $$T^*(b) = \{y \in \mathbb{R}^{m+n} :\ -y_i \geq F(y_{\sim i}, b)\}.$$

It follows immediately from the definition of $F(\cdot,\, b)$ that $T^*(b) \equiv T(b)$. Therefore, since a convex function is defined by its epigraph — equivalently, a closed convex function is the pointwise supremum of all affine functions that are majorized by it (Rockafellar (1970), Theorem 12.1), while a closed convex set is the intersection of all of the closed half spaces defined by its supporting hyperplanes — the properties of $F(\cdot,\, b)$ imply the properties of $T(b)$, and conversely.

Hence, let the goods/commodities/qualities efficient transformation frontier be defined by the implicit function $G(x,z,b) = 0$. We interpret $G(\cdot)$ to be a joint household production function with inputs x, outputs z, and parameters b. $G(\cdot,\, b)$ is convex in $(x,\, z)$, increasing in z, decreasing in x, and without loss in generality, strictly increasing in z_1. For given b, the feasible goods/commodities production possibilities set is defined in terms of $G(\cdot,\, b)$ by

(2.3) $$T(b) = \{(x,z) \in \mathbb{R}^r \times \mathbb{R}^m \colon\; G(x,z,b) \le 0\}\ .$$

For fixed b, $T(b)$ is non-empty, closed, and convex; $G(\mathbf{0},\mathbf{0},b) = 0$; and $G(x,z,b) = 0$ and $z \gg \mathbf{0}$ imply that $x \ge \mathbf{0}$, where $x \ge \mathbf{0}$ means $x_j \ge 0\ \ \forall\ j$ and $x \ne \mathbf{0}$.

We assume that the correspondence $T(b)$ is continuous over the set of parameter vectors, $B \subset \mathbb{R}^s$, and that boundedness and closure of $T(b)$ hold throughout B. These conditions ensure that $G(x,z,\cdot)$ is continuous in $b \in B$, which can be demonstrated in the following way. Define $F(x,z,b)$ by

(2.4) $$-F(x,z,b) = \sup\{z_1 \colon (x,z) \in T(b)\}\ .$$

Let $G(x,z,b) = z_1 + F(x,z,b)$. If $(x^n, z^n) \in T(b^n)\ \forall\ n$ and $(x^n, z^n, b^n) \to (x,z,b)$ then $(x,z) \in T(b)$ by the continuity of $T(\cdot)$ in b. Uniqueness of the supremum implies $z_1 \le -F(z,x,b)$. On the other hand, $z_1^n = -F(x^n, z^n, b^n)\ \forall\ n$ implies that $z_1 \ge -F(x,z,b)$. Continuity of $G(x,z,\cdot)$ follows immediately. This means simply that the boundary of the feasible set $T(b)$ is connected and contained in $T(b)$, and that small changes in b do not induce large changes in the boundary of $T(b)$. Therefore the greatest possible output of z_1 does not change much either.

The above conditions on the set $T(b)$ are standard in the general theory of the firm and do not exclude cases where the commodity vector z includes some or all of the market goods. For example, if $m = n$ and $z_i \equiv x_i \ \forall \ i$, then the model reduces to the neoclassical framework where market goods are the desired commodities from which utility is derived directly. More generally, for any pair i and j such that $z_i \equiv x_j$, we can simply incorporate the function $z_i - x_j$ into the definition of the transformation function $G(\cdot)$.

An important aspect of this model setup is the manner in which time allocation tacitly enters the decision problem. If t_0 is the vector of labor times supplied to the market and w the vector of market wage rates received for this labor, then the budget constraint has the form $p_x' \, x_{\sim t} \leq y + w't_0$, where y is non-labor income. Let t denote the vector of household times used in the production of nonmarket commodities. Then $x = [x_{\sim t}', t']'$ and the joint household production function tacitly depends upon t. Finally, if T is the vector of time endowments, then $T - t - t_0$ is a vector of leisure times and is tacitly included as part of the vector z. The vector of time constraints take the form $t + t_0 = T$ (Becker (1965); Deaton and Muellbauer (1980); Pollak and Wachter (1975)). Substituting $t_0 = T - t$ into the budget constraint gives $p_x' \, x_{\sim t} \leq w'(T - t)$, or equivalently, $p_x' \, x_{\sim t} + w't \leq y + w'T \equiv m$, where m is the household's *full income* (Becker (1965)).

2.1 Static Consumer Choice Theory with Household Production

In addition to the joint transformation function relating goods, commodities, and qualities, we assume that there exists a continuous, quasiconcave utility function defined over the space of commodities, $u(z)$, such that $u(z) \geq u(z^1)$ if and only if $z \succsim z^1$, where $\succsim$ is the binary preference relation defined over commodities. The consumer's decision problem is taken to be to seek a combination of market goods and household time, x, that will produce the vector of commodities, z, that maximizes utility, $u(z)$, subject to a budget constraint, $p'x \leq m$. Here p is an n-vector of market prices, defined by $p = (p_{x_{\sim t}}', w')'$ and $m = y + w'T$ is the household's full income. In addition to the standard budget constraint, the choice problem is subject to the constraint that the vector of commodities produced from the market goods and household time is feasible, $G(x,z,b) \le 0$. This problem is related to the neoclassical consumer choice problem by the following theorem, which permits the translation of the utility function defined over produced nonmarket commodities to an induced utility function defined over market goods and time.

> ***Theorem 1:*** *For each* $(x,z) \in T(b)^{\mathrm{o}}$, *the interior of* $T(b)$, *suppose that the set of feasible commodity bundles,* $W(x,b) = \{z \in \mathbb{R}^m\colon G(x,z,b) \le 0\}$, *has a non-empty interior. Then the induced utility function on the space of market goods,* $u^*\colon \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}$, *defined by*
>
> $$u^*(x,b) \equiv \sup\{u(z)\colon z \in W(x,b)\},$$
>
> *is strictly quasiconcave in x.*

PROOF: Fix two feasible goods vectors, x and x^1 such that $u^*(x,b) = k$ and $u^*(x^1,b) \ge k$ for some real number k. Define $x^2 = tx + (1-t)x^1$ for some $t \in (0,1)$. Pick vectors z, z^1 and z^2 to satisfy $G(z,x,b) \le 0$, $G(z^1,x^1,b) \le 0$, $G(z^2,x^2,b) \le 0$, $u(z) = u^*(x,b)$, $u(z^1) = u^*(x^1,b)$, and $u(z^2) = u^*(x^2,b)$. The strict convexity of $G(\cdot,\, b)$ implies that $G[tz + (1-t)z^1, x^1, b] < 0$; hence $tz + (1-t)z^1$ is a feasible commodity bundle. On the other hand, strict quasiconcavity of $u(\cdot)$ implies that $u[tz+(1-t)z^1] > u(z)$. Consequently, $u^*(x^2,b) = u(z^2) \ge u[tz + (1-t)z^1] > u(z) = k$. ■

This result permits the household production model to be translated into the standard neoclassical model of consumer choice. The consumer choice problem of maximizing $u(z)$ subject to the constraints $p'x \le m$ and $G(x,z,b) \le 0$ thus can be represented in terms of the simpler problem of choosing x to maximize $u^*(x,b)$ subject to $p'x \le m$. Quasiconcavity of $u^*(\cdot,b)$ implies that the resulting demands for the goods x have the standard properties of neoclassical demand functions. However, these demands also relay information regarding both preferences and consumption technology (Pollak and Wachter (1975); Barnett (1977)).

Under some general regularity conditions on u and G, the following theorem describes the basic properties of the function $u^*(x,b)$ and the nature of the dual information on preferences and household consumption technology relayed by it. With little loss in generality, we restrict our attention to the case of strictly positive commodity consumption bundles.

Theorem 2: *Suppose that $u(\cdot)$ is strictly quasiconcave and continuously differentiable, $G(\cdot)$ is strictly convex in z and continuously differentiable in (x,z,b), $W(x,b)$ has a non-empty interior, $u(\cdot)$ is nonsatiated on $T(b)$, and the optimal commodity vector satisfies $z(x, b) \gg \mathbf{0}$. Then $u^*(\cdot)$ has the following properties:*

(2.a) $u^*(\cdot, b)$ *is increasing in* x;

(2.b) $u^*(\cdot)$ *is continuously differentiable in* (x, b);

(2.c) $\partial u^*/\partial x = -(\partial u/\partial z_1)\cdot(\partial G/\partial x)/(\partial G/\partial z_1)$;

(2.d) $\partial u^*/\partial b = -(\partial u/\partial z_1)\cdot(\partial G/\partial b)/(\partial G/\partial z_1)$;

(2.e) $\operatorname{sgn}(\partial u^*/\partial b_k) = -\operatorname{sgn}(\partial G/\partial b_k)\ \forall\ k$;

(2.f) *The preference map defined by* $u^*(\cdot)$ *is invariant to increasing transformations of* $u(\cdot)$ *and* $G(\cdot)$.

PROOF: Let the Lagrangean for the constrained maximization problem be

$$L = u(z) - \mu\, G(x,z,b),$$

where $\mu \geq 0$ is a Lagrange multiplier. Strict quasiconcavity of $u(\cdot)$ and strict convexity of $G(\cdot)$ in z imply that the necessary and sufficient conditions for an interior constrained maximum are

$$\partial L/\partial z = \partial u/\partial z - \mu\partial G/\partial z = \mathbf{0},$$

$$\partial L/\partial \mu = -G(x,z,b) = 0,$$

where the second condition follows from nonsatiation of u throughout $T(b)$. Substituting the solution functions, $z(x, b)$, into the first order conditions generates identities in a neighborhood of the point (x, b). Define the indirect objective function by $u^*(x,b) \equiv u[z(x,b)]$ and the constraint identity by $G[x,z(x,b),b] \equiv 0$. Substituting these and the optimal solution function for the Lagrange multiplier, $\mu(x, b)$, into the Lagrangean gives $u^*(x,b) \equiv L(x,b)$. Differentiating with respect to x then gives

$$\partial u^*/\partial x = (\partial u/\partial z - \mu\partial G/\partial z)\,\partial z/\partial x - (\partial\mu/\partial x)G - \mu(\partial G/\partial x).$$

The first two terms on the right-hand side vanish and the Lagrange multiplier can be written as

$$\mu = (\partial u/\partial z_1)/(\partial G/\partial z_1) > 0.$$

Combining the right-hand-side expressions gives,

$$\partial u^* / \partial x = -(\partial u / \partial z_1) \cdot (\partial G / \partial x) / (\partial G / \partial z_1).$$

A completely analogous argument applies to $\partial u^* / \partial b$, verifying (2.a) - (2.e).

To demonstrate (2.f), let $w: \mathbb{R} \to \mathbb{R}$ and $v: \mathbb{R} \to \mathbb{R}$ be arbitrary strictly increasing, continuously differentiable functions. Define the set

$$W^*(x,b) \equiv \{z \in \mathbb{R}^m : v[G(x,z,b)] \le v(0)\}.$$

Then $W^*(x,b) \equiv W(x,b)$; that is, $z \in W(x,b)$ if and only if $z \in W^*(x,b)$ and $u(z)$ achieves a maximum in $W^*(x, b)$ at the same point as it does in $W(x, b)$. Similarly, $w[u(z)]$ achieves a maximum on either $W(x, b)$ or $W^*(x, b)$ if and only if $u(z)$ does. Finally, let

$$L^* = w[u(z)] + \mu^* \{v(0) - v[G(x,z,b)]\}.$$

The first-order conditions for an interior constrained maximum now are

$$\partial L^* / \partial z = w' \, \partial u / \partial z - \mu^* v' \, \partial G / \partial z = 0,$$

$$\partial L^* / \partial \mu^* = v(0) - v[G(x,z,b)] = 0.$$

Since $v^{-1}[v(0)] = 0$, the latter is equivalent to $G(x,z,b) = 0$. The former can be written as

$$(w' \partial u / \partial z_i) / (w' \partial u / \partial z_1) = (\mu^* v' \, \partial G / \partial z_i) / (\mu^* v' \partial G / \partial z_1) \ \forall \ i,$$

which after canceling $\left(w' / w'\right)$ and $\left(\mu^* v' / \mu^* v'\right)$ is equivalent to the conditions implied by

$$\partial u / \partial z - \mu \, \partial G / \partial z = 0. \qquad \blacksquare$$

Several additional properties of the consumer choice model that uses $u^*(x, b)$ as its starting point can be derived from the duality theory of the neoclassical model. Consider the problem,

(2.1.1) $$v^*(p,m,b) \equiv \sup\{u^*(x,b) : p'x \le m, \ x \in \mathbb{R}^n_+\}.$$

This generates ordinary market demands, $x^* = h^*(p,m,b)$, Lagrange multiplier, $\lambda^*(p,m,b)$, and indirect utility function, $v^*(p,m,b) \equiv u^*[h^*(p,m,b),b]$. Then consider an artificial two-stage formulation of the problem,

(2.1.2) $$\begin{aligned} v(p,m,b) &\equiv \sup\{u(z) : p'x \le m, G(x,z,b) \le 0\} \\ &\equiv \sup\left\{\sup\{u(z) : G(x,z,b) \le 0\} : p'x \le m\right\}. \end{aligned}$$

This yields ordinary market demands, $x = h(p,m,b)$, nonmarket commodity demands, $z = f(p,m,b)$, Lagrange multiplier, $\lambda(p,m,b)$, and indirect utility function,

$v(p,m,b) \equiv u[f(p,m,b)]$. By the uniqueness of the supremum, we have

(2.1.3) $$v(p,m,b) \equiv v^*(p,m,b).$$

This simple observation leads directly to the following result, which for brevity is stated in terms of $v(\cdot)$.

> ***Theorem 3.*** *Under the conditions of Theorem 2, $v(p,m,b)$ is continuously differentiable, homogeneous of degree zero in (p,m), increasing in m, quasiconvex and decreasing in p, and*
>
> *(3.a)* $\partial v/\partial m = \lambda(p,m,b)$*;*
>
> *(3.b)* $h(p,m,b) \equiv -(\partial v/\partial p)/(\partial v/\partial m)$ *so long as the right-hand-side is well-defined;*
>
> *(3.c)* $\partial v/\partial b \equiv -\lambda\, \partial G/\partial b \equiv \partial u^*/\partial b$*;*
>
> *(3.d)* $G[h(p,m,b), f(p,m,b), b] \equiv 0$ *; and*
>
> *(3.e)* $p'h(p,m,b) \equiv m$ *.*

PROOF: Continuous differentiability of $v(\cdot)$ follows from the continuous differentiability of $u(\cdot)$ and $G(\cdot)$, strict quasiconcavity of $u(\cdot)$, and strict convexity of $G(\cdot, b)$. Zero degree homogeneity follows from the fact that the budget set does not change if we multiply both sides by a positive constant, $p'x \le m$ if and only if $tp'x \le tm \;\forall\; t > 0$. For monotonicity, if $p \ge p^{\circ}$, then $B = \{x: p'x \le m\} \subset B^{\circ} = \{x: (p^{\circ})'x \le m\}$, and the maximum of $u^*(x, b)$ over B° can be no less than the maximum over B. Hence $v(\cdot, m, b)$ is decreasing in p. The proof of monotonicity in m is of the same nature.

For quasiconvexity in p, fix p and p^1 such that $v(p,m,b) \le k$ and $v(p^1,m,b) \le k$ for some real number k. Define $p^2 = tp + (1-t)p^1$ for $t \in [0,1]$. If $tp'x + (1-t)(p^1)'x \le m$ then $p'x \le m$, $(p^1)'x \le m$, or both. Otherwise, $p'x > m \Rightarrow tp'x > tm$ and $(p^1)'x > m \Rightarrow (1-t)(p^1)'x > (1-t)m$. Summing the last inequalities in each of case implies $tp'x + (1-t)(p^1)'x > m$, which is a contradiction. Therefore,

$$v(p^2,m,b) = \sup\left\{u^*(x,b)\colon (\mathrm{p}^2)'x \le m\right\} \le \max\left\{v(p,m,b), v(p^1,m,b)\right\} \le k .$$

Properties a - c are the result of the envelope theorem, d and e follow from the nonsatiation of $u(\cdot)$. ■

This theorem simply states that there will be one and only one maximum level of utility, and the associated choice functions are invariant to the manner in which we choose to view the consumer choice problem, provided of course that the problem is well-behaved. In addition, it shows how the joint preference and household production technology information contained in $u^*(\cdot)$ is transmitted to the indirect utility function $v(\cdot)$ and hence the demand functions $h(\cdot)$.

Analogous to the neoclassical consumer choice model, dual to the utility maximization problem is an expenditure minimization problem,

(2.1.4) $$e(p,u,b) \equiv \inf\{p'x : u^*(x,b) \geq u,\ x \in \mathbb{R}^n_+\} .$$

This generates Hicksian compensated demands, $x = g(p,u,b)$, Lagrange multiplier, $\mu(p,u,b)$, and expenditure function, $e(p,u,b) \equiv p'g(p,u,b)$. Our next result relates the indirect utility function to the expenditure function.

Theorem 4: *Under the conditions of theorem 2, the expenditure function is increasing, 1° homogeneous, and concave in p; increasing in u; continuously differentiable in (p, u, b); and,*

(4.a) $\partial e(p,u,b)/\partial p \equiv g(p,u,b)$;

(4.b) $\partial e(p,u,b)/\partial u \equiv \mu(p,u,b)$;

(4.c) $\partial e(p,u,b)/\partial b \equiv -\mu(p,u,b) \cdot \partial u^*(g(p,u,b),b)/\partial b$;

(4.d) $e[p,v(p,m,b),b] \equiv m$;

(4.e) $v[p,e(p,u,b),b] \equiv u$;

(4.f) $g[p,v(p,m,b),b] \equiv h(p,m,b)$;

(4.g) $h[p,e(p,u,b),b] \equiv g(p,u,b)$;

(4.h) $\mu(p,u,b) \equiv \lambda[p,e(p,u,b),b]^{-1}$;

(4.i) $\lambda(p,m,b) \equiv \mu[p,v(p,m,b),b]^{-1}$.

PROOF: With the exception of c, these are all straightforward duality results proved in the same manner as in the neoclassical utility theory model. Property c follows from the envelope theorem. ■

Thus, the static consumer choice model with household production inherits all of the properties of the indirect utility function and expenditure function from the neoclassical model. The relationship between the Marshallian ordinary demands and Hicksian compensated demands with respect to the parameters b are summarized in the next theorem.

> ***Theorem 5:*** *The Marshallian demand functions are positive-valued, continuous and 0° homogeneous in (p, m), and so long as the associated derivatives exist and are continuous,*
>
> *(5.a)* $$\frac{\partial g(p,u,b)}{\partial b'} \equiv \frac{\partial h(p,e(p,u,b),b)}{\partial b'} + \frac{\partial h(p,e(p,u,b),b)}{\partial m} \cdot \frac{\partial e(p,u,b)}{\partial b'}, \text{ and}$$
>
> *(5.b)* $$p'\,\partial h(p,m,b)/\partial b' \equiv \mathbf{0}'.$$

PROOF: These results follow from differentiating 4.g and 3.e with respect to *b*. ■

2.2 Hedonic Price Functions

An alternative approach to the consumer choice problem with a household production function as an added constraint is obtained by focusing on the cost of obtaining a given vector of commodities from market goods and household time given market prices. This perspective is called the method of *hedonic price functions* (Lucas (1975); Muellbauer (1974); Rosen (1974)). To develop this approach in the current model framework, we require a version of the Shephard-Uzawa-McFadden duality theorem for cost functions.

> ***Theorem 6:*** *For a feasible commodity vector z, the cost function,*
>
> $$c(p,z,b) \equiv \inf\{p'x\colon G(x,z,b) \le 0,\ x \in \mathbb{R}^n_+\},$$
>
> *is continuous in (p, z, b), 1° homogeneous, increasing and concave in p, increasing and convex in z. When the inputs requirements set*
>
> $$X(z,b) = \{x \in \mathbb{R}^n_+ : G(x,z,b) \le 0\}$$
>
> *is strictly convex and* $\partial c(p,z,b)/\partial p$ *exists, c(p, z, b) obeys Shephard's Lemma (Shephard (1953)),*
>
> $$G[\partial c(p,z,b)/\partial p, z, b] \equiv 0.$$
>
> *When G is differentiable in z the ratio of marginal costs of two commodities is equal to the marginal rate of transformation between them,*
>
> $$(\partial c/\partial z_i)/(\partial c/\partial z_{i'}) = (\partial G/\partial z_i)/(\partial G/\partial z_{i'})\ \forall\ i, i'.$$

PROOF: Convexity of $T(b)$ implies convexity of $X(z, b)$. To see this, let x^0 and x^1 be any two points such that $G(x^0,z,b) \le 0$ and $G(x^1,z,b) \le 0$. Define $x^2 = tx^0 + (1-t)x^1$ for

$t \in [0,1]$. Then $G(x^2, z, b) \leq 0$, since $z = tz + (1-t)z$. Similarly, closure of $T(b)$ implies closure of $X(x, b)$. To see this, let $\{z^k\}$ and $\{x^k\}$ be any two sequences such that $z^k = z \;\forall\; k$, $x^k \to x$, and $G(x^k, z^k, b) \leq 0 \;\forall\; k$. Then closure of $T(b)$ implies that $G(x,z,b) \leq 0$; hence $X(z, b)$ is closed. Decreasing monotonicity of G in x implies free disposal, since $x \geq x^0$ and $G(x^0, z, b) = 0 \Rightarrow G(x,z,b) \leq 0$. The nonemptiness of $T(b)$ and the feasibility of z imply $X(z,b) \neq \varnothing$. Theorems 1 and 2 of Uzawa (1964) and Lemma 1 of McFadden (1973) follow from this set of hypotheses, proving continuity, monotonicity, 1° homogeneity, and concavity in p.

To prove convexity in z, fix two feasible outputs z^0 and z^1, and define $z^2 = tz^0 + (1-t)z^1$ for $t \in [0,1]$. Let $p'x^0 = c(p, z^0, b)$ and $p'x^1 = c(p, z^1, b)$. Then $p'x^2 \equiv c(p, z^2, b) \leq p'x \;\forall\; x \in X(z^2, b)$. In particular, let $x = tx^0 + (1-t)x^1$. Convexity of $T(b) \Rightarrow (x, z^2) \in T(b)$. Therefore, since x is feasible but not necessarily optimal for z^2,

$$c\,(p, z^2, b) \leq p'x = tp'x^0 + (1-t)\,p'x^1 = tc(p, z^0, b) + (1-t)\,c(p, z^1, b)\,.$$

When $G(\cdot, z, b)$ is strictly convex in x, so that the input requirements sets are strictly convex, Shephard's Lemma is obtained from the *primal-dual* function

$$\phi\,(p, x, z, b) \equiv p'x - c\,(p, z, b)\,.$$

$\phi(\cdot)$ is non-negative for all $x \in X(z,b)$ and attains a minimum at $x = x(p,z,b)$, the cost-minimizing bundle of market goods. If $\partial c/\partial p$ exists, minimizing $\phi(\cdot)$ with respect to p requires

$$\partial\phi\,(p, x, z, b)/\partial p \equiv x - \partial c\,(p, z, b)/\partial p \equiv 0\,. \qquad \blacksquare$$

An additional property of the cost function is that if $G(\cdot, b)$ exhibits constant returns to scale with respect to x and z, so that $G\,(x, z, b) = 0 \Rightarrow G\,(tx, tz, b) = 0 \;\forall\; t \geq 0$, then $c(p, z, b)$ is 1° homogeneous in z (Hall (1973)). By Euler's theorem we then have

(2.2.1) $$c\,(p, z, b) \equiv \frac{\partial c(p, z, b)}{\partial z} ©z\,.$$

The *hedonic price* for the i th commodity is the marginal cost of its production, $\rho_i(p, z, b) \equiv \partial c(p, z, b)/\partial z_i$. Under constant returns to scale, then, we can represent the consumer's choice problem as

(2.2.2) $$\sup\{u(z)\!: \rho' z \leq m,\; z \in \mathbb{R}^m_+\}\,.$$

Now, suppose that we define the implicit commodity prices conditionally on the optimal level of commodity consumption, $z^* = f(p, m, b)$, and then solve the household's choice problem given $\rho^* = \rho(p, z^*, b)$. These conditional shadow prices ρ^* define the hyperplane that separates the projection of the production possibility set onto the m-dimensional commodity subspace from the upper contour set of the utility function. Under constant returns to scale, the commodity demand vector, $z^* = z(\rho^*, m, b)$, possesses all of the properties of neoclassical demand functions with respect to (ρ^*, m) (Barnett (1977)). That is, taking ρ^* as a vector of constants associated only with the separating hyperplane, the functions $z(\rho^*, m, b)$ are homogeneous of degree zero in (ρ^*, m), satisfy the adding up condition, $\rho^{*\prime} z(\rho^*, m, b) \equiv m$, and obey the Slutsky sign and symmetry conditions with respect to ρ^* and income m, i.e., $\partial z/\partial \rho^{*\prime} + (\partial z/\partial m) z'$ is symmetric and negative semidefinite.

The unconditional commodity shadow prices are defined by

$$\rho(p, m, b) \equiv \rho^*(p, f(p, m, b), b), \tag{2.2.3}$$

so that we have the identity

$$z(\rho(p, m, b), m, b) \equiv f(p, m, b). \tag{2.2.4}$$

Therefore, the hedonic prices for the non-market commodities and the commodity demands are necessarily simultaneously determined. Consequently, simply estimating either $\rho^* = \rho(p, z^*, b)$ or $z^* = z(\rho^*, m, b)$ with standard techniques leads to biased and inconsistent empirical results (Pollak and Wachter (1975)). Moreover, without constant returns to scale,

$$m \equiv p' h(p, m, b) \equiv c[p, f(p, m, b), b] \neq \rho(p, m, b)' f(p, m, b), \tag{2.2.5}$$

and the above results for $z(\rho^*, m, b)$ no longer hold. Even with constant returns to scale, the hedonic price functions relay information regarding both consumer preferences and the household production function.

The simultaneity between the hedonic price functions and the commodity demand functions is overcome when the household production function displays both constant returns to scale and nonjoint production. When both of these conditions are satisfied, the joint cost function takes the additively separable form (Hall (1973); Muellbauer (1974); Samuelson (1966)),

$$c\,(p,z,b) = \tilde{c}(p,b)'z\,, \tag{2.2.6}$$

where $\tilde{c}_i(p,b)$ is the cost of producing a unit of the i^{th} commodity. In this case, $\rho(p,z,b) \equiv \tilde{c}(p,b)$ is independent of z, so that constant returns to scale and nonjoint production imply a linear budget constraint in commodities space. As long as the Jacobian matrix for the hedonic price functions is of full rank, that is, $\min(m,n) = \text{rank}[\partial\tilde{c}/\partial p']$, the shadow price functions can be locally inverted to give price functions of the form $p = \psi(\rho,b)$. This is the form in which nearly all empirical analyses of hedonic price functions have been undertaken, with the commodity shadow prices ρ estimated as constants in a linear or log-linear regression equation.

However, Pollak and Wachter point out that nonjointness is a restrictive assumption since it implies that the time spent in household production activities cannot yield utility except in terms of the amount of leisure time that is reduced by these activities. Moreover, we expect *a priori* that different households have different consumption technologies, and hence, different implicit price equations even if we are willing to impose constant returns to scale and nonjoint production. This leads to different unit cost functions for the commodity outputs and different hedonic price relationships for different households.

Additionally, in any model of equilibrium price relationships, demand and supply conditions combine in the marketplace to create market clearing prices (Rosen (1974); Lucas (1975)). The implicit prices for quality calculated from an hedonic price equation therefore represent the marginal conditions equating supply and demand, mapping observed market quantities, prices, and qualities into a single point in the space of producer costs and consumer preferences. Consequently, these relationships yield information about consumer preferences only in the sense of market equilibrium conditions. They do not contain any information regarding the direction or size of quantity or price changes that are likely to result from changes in the quality levels contained in market goods.

2.3 Special Cases

The Gorman/Lancaster model of product characteristics arises when the household production function can be decomposed into the linear system

(2.3.1) $$z_i = \sum_{j=1}^{n} b_{ij} x_j,\ i = 1,\ldots,m$$

where $s = m{\cdot}n$. The utility function over goods and qualities is then of the form

(2.3.2) $$u^*(x,b) = u\left(\sum_{j=1}^{n} b_{1j}x_j, \sum_{j=1}^{n} b_{2j}x_j, \cdots, \sum_{j=1}^{n} b_{mj}x_j\right).$$

The Muth/Becker/Michael model results when $G(x,z,b) = 0$ is nonjoint, so that each commodity is produced by an individual production function of the form

(2.3.3) $$z_i = f_i(x_{(i)}, b_{(i)}),\ i = 1,\ldots,m\ ,$$

where $x = \sum_{i=1}^{m} x_{(i)}$ and $b = [b'_{(1)}\, b'_{(2)} \cdots b'_{(m)}]'$. In this case, the utility function for goods has the form

(2.3.4) $$u^*(x,b) = u\left(f_1(x_{(1)}, b_{(1)}), \cdots, f_m(x_{(m)}, b_{(m)})\right).$$

The additional property of constant returns to scale, advocated by Muth as merely a question of definitions, generates demand-side hedonic price equations discussed above.

Many applications of the hedonic price function model of quality employ translation and scaling methods (Pollak and Wales (1981)) to produce a utility function defined over goods and qualities in the scaled form

(2.3.5) $$u^*(x,b) = u(\varphi_1(b_{(1)})x_1, \cdots, \varphi_n(b_{(n)})x_n)\ ,$$

or in the translated form

(2.3.6) $$u^*(x,b) = u(x_1 + \varphi_1(b_{(1)}), \cdots, x_n + \varphi_n(b_{(n)}))\ ,$$

or a combination of the two. In the former case, the ordinary demand functions take the form

(2.3.7) $$x_i = h^i\left(p_1/\varphi_1(b_{(1)}), \cdots, p_n/\varphi_n(b_{(n)}), m\right)\Big/\varphi_i(b_{(i)})\ ,$$

which is commonly called the Fisher-Shell repackaging model (Fisher and Shell (1971)). In the latter case, the ordinary demand functions take the form

(2.3.8) $$x_i = h^i\left(p, m + \sum_{i=1}^{m} p_i\varphi_i(b_{(i)})\right) - \varphi_i(b_{(i)})\ ,$$

(Hanemann (1980); Pollak and Wales (1981)). In both cases, one implication is that preferences are weakly separable in the partition $\{(x_1, b_{(1)}), \cdots, (x_n, b_{(n)})\}$. However, this will not be the case if goods are scaled by an $n{\times}n$ matrix, say $\mathbf{A}(b) = [\varphi_{ij}(b)]$, that is nonsingular and has nonzero off-diagonal elements. Then the Marshallian ordinary demands take the form (Samuelson (1948), p. 137),

$$x = \mathbf{A}(b)^{-1} h(\mathbf{A}(b)^{-1} p, m) . \tag{2.3.9}$$

Note that the linear Gorman/Lancaster characteristics model is a special case of (2.3.9).

3. Intertemporal Models of Consumer Choice

Many studies of food consumption use time series data. The static neoclassical model of consumer choice has been extended to accommodate the analysis of household decisions over time. In this section, we discuss models of intertemporal consumer choice that combine the structures of the previous section with the dynamic nature of household decision problems. Initially, we consider a model that mirrors the static neoclassical theory of consumer choice through additively separable preferences across points in time. The consuming household's process of expectations formation for future values of economic factors such as prices and income plays an important role in this model. Intertemporal models of household production are considered next, including the purchase, use, depreciation, maintenance, and replacement of durable household goods. Some durable stocks might be interpreted as habits in consumption. One thread of intertemporal consumer choice theory deals with naïve and rational models of habit formation.

In continuous time, the simplest form of the neoclassical intertemporal consumer choice model considers a consuming household which chooses the time path of consumption for the vector of goods, $x(t) \geq \mathbf{0}$, to[2]

$$\text{maximize } U = \int_0^T u(x(t)) e^{-\rho t} dt \tag{3.1}$$

subject to the intertemporal budget constraint,

$$dM(t)/dt = -p(t)' x(t) e^{-rt}, \;\; M(0) = M_0, \;\; M(t) \geq 0 \;\; \forall \;\; t \in [0, T], \tag{3.2}$$

---

[2] The assumption of a constant discount rate for consumers over the life cycle is probably too strong, but it simplifies matters considerably, and plays only a minor role in the developments that follow. The assumption of a fixed, known, and finite planning horizon also is almost certainly too strong, as is the lack of risk and uncertainty in the model. However, time and space preclude a detailed analysis of these issues here.

where $u(\cdot)$ is the instantaneous flow of utility from consumption, ρ is the consuming household's rate of time discount, or impatience, r is the real market discount rate, at which the consumer can freely borrow or lend, M_0 is the household's initial wealth plus the discounted present value of its full income stream, and $p(t)$ is the vector of market prices for the goods $x(t)$ in period t.[3] We assume throughout that $u(\cdot)$ is twice continuously differentiable, strictly increasing ($\partial u(x)/\partial x \gg 0$) and strongly concave ($\partial^2 u(x)/\partial x \partial x'$ is negative definite) $\forall\ x \in \mathbb{R}^n_+$.[4]

3.1 Perfect Foresight

In this subsection, we assume that the consuming household has complete information regarding all past, present, and future prices, incomes, and other relevant economic variables. The Hamiltonian for this problem can be written as

$$H = u(x(t))e^{-\rho t} - \lambda(t)p(t)'x(t)e^{-rt}\,, \tag{3.1.1}$$

where $\lambda(t)$ is the shadow price, or costate variable for the equation of motion for household wealth. The first-order necessary and sufficient conditions for the maximum principle are

$$\partial H/\partial x = e^{-\rho t}\partial u/\partial x - e^{-rt}\lambda p \le 0,\quad x \ge 0,\quad x'\partial H/\partial x = 0\ \forall\ t \in [0,T], \tag{3.1.2}$$

$$\partial H/\partial M = 0 = -\dot{\lambda}\,, \tag{3.1.3}$$

$$\partial H/\partial \lambda = -p'xe^{-rt} = \dot{M},\quad M(0) = M_0,\quad M \ge 0\ \forall\ t \in [0,T]. \tag{3.1.4}$$

---

[3] We rely primarily on the continuous time maximum principle to study the structure of solutions to this problem under different hypotheses about expectations formed by the household for future prices, and so forth. However, we freely make use of some results from dynamic programming and the calculus of variations to fully develop the duality of this choice problem.

[4] Additive separability, twice continuous differentiability, and quasiconcavity of U imply that $u(\cdot)$ is concave. This can be proven for discrete time with a finite planning horizon by a simple extension of the arguments in Gorman (1970). This argument then can be extended further to continuous time by passing to the limit via increasingly small time increments and by appealing to continuity of the Hessian matrix. It also can be shown that concavity of $u(\cdot)$ is necessary for the existence of an optimal consumption path. Strong concavity, in turn, implies that the optimal consumption path is unique.

Consider an interior solution for $x \ \forall \ t \in [0,T]$. Then first-order condition (3.1.4) implies

$$x = u_x^{-1}(e^{(\rho - r)t}\lambda p)\,, \tag{3.1.5}$$

where $u_x^{-1}(\cdot)$ is the n-vector inverse of $u_x(\cdot)$. The strict monotonicity of $u(\cdot)$ combined with strictly positive prices p requires that $\lambda(t) > 0 \ \forall \ t \in [0,T]$. Because the Hamiltonian does not depend on current wealth, $M(t)$, equation (3.1.3) implies $\lambda(t)$ is constant over the entire planning horizon, $\lambda(t) \equiv \lambda_0 \ \forall \ t \in [0,T]$. Therefore, multiplying by $e^{-rt}p(t)$ and integrating with respect to t produces a defining relationship for the wealth shadow price, $\lambda_0(M_0 - M_T, \rho, r, T)$,

$$M_0 - M_T \equiv \int_0^T e^{-rt} p(t)' u_x^{-1}\left(e^{(\rho - r)t}\lambda_0(M_0 - M_T, \rho, r, T) \cdot p(t)\right) dt\,. \tag{3.1.6}$$

The integral form of (3.1.6) implies that the optimal solution for λ_0 depends on all prices at all points in time, but that for given t and any finite change in $p(t)$, with prices remaining unchanged at all other times, $\partial\lambda_0 / \partial p(t) \equiv 0$.[5]

Substituting $\lambda_0(M_0 - M_T, \rho, r, T)$ into (3.1.5) gives the optimal demands at time t,

$$x^*(t) \equiv u_x^{-1}\left(e^{(\rho - r)t}\lambda_0(M_0 - M_T, \rho, r, T) \cdot p(t)\right). \tag{3.1.7}$$

In contrast to the static model of the previous section, the neoclassical dynamic model with perfect foresight has a matrix of instantaneous uncompensated price slopes that is symmetric and negative definite,

$$\partial x^* / \partial p' = e^{(\rho - r)t}\lambda_0 u_{xx}^{-1}\,. \tag{3.1.8}$$

This difference in the symmetry properties of static and dynamic consumer choice models is the result of the intertemporal allocation of expenditure and is not due to perfect foresight or income smoothing, *per se*. The difference is due to the integral form of the budget constraint on total household wealth in the dynamic framework. The additive structure of intertemporal preferences implies that the flow of utility in any given instant is perfectly substitutable for utility flows at every other instant. Consequently, a change in market prices at a single point in time generates substitution effects which are perceptible at the given instant but are imperceptibly spread

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[5] More generally, $\lambda(\cdot)$ does not vary with any absolutely bounded changes in prices on any subset of [0, T] that has Lebesgue measure zero.

across the consumption bundles in all other times. Even in the simplest of dynamic contexts, therefore, the ubiquitously applied and tested Slutsky symmetry and negativity conditions of static consumer choice theory do not transcend to models in which wealth, rather than current income, is the constraint on consumption choices. Continuing this line of inquiry, the marginal wealth effects on the demands at each $t \in [0, T]$ satisfy

(3.1.9) $$\partial x * / \partial M_0 = e^{(\rho - r)t} u_{xx}^{-1} p \cdot \partial \lambda_0 / \partial M_0 .$$

By differentiating both sides of (3.1.6) with respect to M_0, combining the result obtained on the right-hand-side with (3.1.8), regrouping, canceling common terms, and distributing the integral, we have

(3.1.10) $$\partial \lambda_0 \,/\, \partial M_0 = 1 \Big/ \int_0^T e^{(\rho - 2r)t} p' u_{xx}^{-1} p dt < 0 \,,$$

where the inequality on the far right follows from the (strong) concavity of $u(\cdot)$ and $p(t) \gg \mathbf{0} \;\forall\; t \in [0, T]$. The maximal level of cumulative discounted utility is defined by

(3.1.11) $$V(M_0 - M_T, \rho, r, T) \equiv \int_0^T u(x^*) e^{-\rho t} dt \,.$$

Differentiating V with respect to M_0 gives

(3.1.12) $$\begin{aligned} \frac{\partial V}{\partial M_0} &\equiv \int_0^T \frac{\partial u}{\partial x'} \frac{\partial x^*}{\partial M_0} e^{-\rho t} dt \\ &\equiv \int_0^T \overbrace{e^{-rt} \lambda_0 p'}^{e^{-\rho t} \partial u / \partial x'} \overbrace{u_{xx}^{-1} p \cdot \partial \lambda_0 \,/\, \partial M_0}^{\partial x^* / \partial M_0} dt \\ &\equiv \lambda_0 \int_0^T e^{-rt} p' u_{xx}^{-1} p dt \Big/ \int_0^T e^{-rt} p' u_{xx}^{-1} p dt \equiv \lambda_0 > 0 \,, \end{aligned}$$

which is a direct intertemporal analogue to the envelope theorem (LaFrance and Barney (1991)). Following the same steps, but applied to M_T implies

(3.1.13) $$\partial \lambda_0 \,/\, \partial M_T = -1 \Big/ \int_0^T e^{(\rho - 2r)t} p' u_{xx}^{-1} p dt > 0 \,,$$

and

(3.1.14) $$\frac{\partial V}{\partial M_T} \equiv -\lambda_0 < 0 .$$

As a consequence, in the absence of any bequeath motive, the optimal terminal wealth vanishes. This is the intertemporal analogue to the static budget identity when preferences are nonsatiable. Note, however, that $V(\cdot)$ is (strongly) concave in the household's initial wealth as a direct consequence of the (strong) concavity of $u(\cdot)$ in x. This contrasts with the static model where the marginal utility of money may be constant, increasing or decreasing due to the ordinality of preferences.

To relate the intertemporal model more closely to the static framework, define total consumption expenditures at time t by $m(t) \equiv p(t)'x(t)$ and consider the static optimization problem of maximizing $u(x)$ subject to $x \geq \mathbf{0}$ and $p'x \leq m$. Let $\tilde{\lambda}$ denote the shadow price for the static budget constraint. The first-order conditions for an interior solution are $u_x = \tilde{\lambda} p$ and $p'x = m$, which produce the static neoclassical demand functions, $x = h(p, m)$. In the static problem, we take m as given, and calculate the comparative statics for x and $\tilde{\lambda}$ from

(3.1.15) $$\begin{bmatrix} \partial x/\partial p' & \partial x/\partial m \\ \partial \tilde{\lambda}/\partial p' & \partial \tilde{\lambda}/\partial m \end{bmatrix} = \begin{bmatrix} \tilde{\lambda}[u_{xx}^{-1} - (p'u_{xx}^{-1}p)^{-1}u_{xx}^{-1}pp'u_{xx}^{-1}] - (p'u_{xx}^{-1}p)^{-1}u_{xx}^{-1}px' & (p'u_{xx}^{-1}p)^{-1}u_{xx}^{-1}p \\ -(p'u_{xx}^{-1}p)^{-1}(\tilde{\lambda}p'u_{xx}^{-1} + x') & (p'u_{xx}^{-1}p)^{-1} \end{bmatrix}.$$

These in turn give the static Slutsky matrix,

(3.1.16) $$S = \partial x/\partial p' + (\partial x/\partial m)x' = \tilde{\lambda}[u_{xx}^{-1} - (p'u_{xx}^{-1}p)^{-1}u_{xx}^{-1}pp'u_{xx}^{-1}],$$

a symmetric, negative semidefinite, rank n-1 matrix.

Let $v(p,m) \equiv u(h(p,m))$ be the indirect utility function for the static problem, which defines the instantaneous flow of maximal utility subject to the static instantaneous budget constraint. Then, because of the additively separable structure of intertemporal preferences, the dynamic consumption problem can be represented equivalently as

(3.1.17) $$\text{maximize } \int_0^T v(p,m)e^{-\rho t}dt \quad \text{subject to} \quad \int_0^T me^{-rt}dt \leq M_0,\ m \geq 0\ \forall\ t \in [0,T].$$

The first-order conditions for an optimal (interior) solution are

(3.1.18) $$\partial v / \partial m \equiv \tilde{\lambda} = e^{(\rho - r)t}\lambda,$$

(3.1.19) $$\dot{\lambda} = 0,$$

(3.1.20) $$\int_0^T me^{-rt}dt = M_0,$$

where the identity in the center of (3.1.18) is due to the (static) envelope theorem. Total expenditure therefore is not predetermined in each period, but rather is jointly determined with quantities and is smoothed over time to equate the present value of the marginal utility of money across all points in time,

(3.1.21) $$\partial v(p,m^*) / \partial m \equiv e^{(\rho - r)t}\lambda_0,$$

where $m^*(t)$ denotes the optimal level of total consumption expenditures at each time $t \in [0, T]$.[6] Applying Roy's identity to the static problem then gives

$$x^*(t) \equiv h(p(t), m^*(t)) \equiv -\frac{\partial v(p(t), m^*(t))/\partial p}{\partial v(p(t), m^*(t))/\partial m} \quad \forall\ t \in [0, T]. \tag{3.1.22}$$

Taking the vector product of (3.1.22) with p, multiplying by e^{-rt} and integrating with respect to t, and utilizing (3.1.18) and (3.1.19) generates two alternative defining relationships for λ_0,

$$\lambda_0 \equiv -\int_0^T e^{-\rho t} p(t)' v_p(p(t), m^*(t)) dt \Big/ M_0 \equiv \int_0^T e^{-\rho t} m^*(t) v_m(p(t), m^*(t)) dt \Big/ M_0 > 0, \tag{3.1.23}$$

where the second identity follows from zero degree homogeneity of $v(\cdot)$ in (p, m) and the inequality follows from the fact that $v(\cdot)$ is strictly increasing in m. As before, we conclude that λ_0 is invariant to all absolutely bounded changes in prices on subsets of $[0, T]$ with Lebesgue measure zero. Hence, differentiating both sides of (3.1.21) with respect to p implies

$$\partial m^*/\partial p = -\frac{\partial^2 v(p, m^*)/\partial m \partial p}{\partial^2 v(p, m^*)/\partial m^2}. \tag{3.1.24}$$

The symmetry result in (3.1.8) is then obtained by differentiating (3.1.22) with respect to p, substituting (3.1.24) into the result, and canceling terms that vanish due to (3.1.21) and (3.1.22),

$$\begin{aligned} \frac{\partial x^*}{\partial p'} &= -\frac{\partial^2 v/\partial p \partial p'}{\partial v/\partial m} + \frac{\partial^2 v/\partial m \partial p}{\partial v/\partial m}\left(\frac{\partial m^*}{\partial p'}\right) + \frac{\partial v/\partial p}{[\partial v/\partial m]^2}\overbrace{\left[\frac{\partial^2 v}{\partial m \partial p} + \frac{\partial^2 v}{\partial m^2}\cdot\frac{\partial m^*}{\partial p}\right]'}^{\equiv 0} \\ &= \frac{-1}{\partial v/\partial m}\left[\frac{\partial^2 v}{\partial p \partial p'} - \left(\frac{1}{\partial v/\partial m}\right)^2 \frac{\partial^2 v}{\partial m \partial p}\frac{\partial^2 v}{\partial m \partial p'}\right]. \end{aligned} \tag{3.1.25}$$

Both matrices in square brackets on the last line of (3.1.25) are symmetric; hence $\partial x^*/\partial p'$ is symmetric.

[6] In other words, the optimal flow of consumption expenditures generally depends on the parameters of the utility function and market prices at time t, as well as initial wealth, individual and market discount rates, and the optimal value of the shadow price for the wealth constraint. Thus, except for models with myopic expectations, which are discussed in the next subsection, and with $\rho = r$, total consumption expenditures can not be treated as exogenous (Engel, Hendry, and Richard (1983)) in empirical models of intertemporal consumer choice.

3.2 Myopic Expectations

The opposite of full information regarding all future economic values on the part of consuming households is myopic expectations. In this case, the household is modeled as if it expects no change in relative prices of goods or services throughout its planning horizon, i.e., $p(t) = p_0 \; \forall \, t \geq 0$. This assumption plays an important part in many contemporary dynamic economic models (e.g., Cooper and McLaren (1980); McLaren and Cooper (1980); Epstein (1981, 1982); and Epstein and Denny (1983)). One drawback is the apparent contradiction between the level of sophistication that individuals are presumed to use to formulate their economic plans versus the manner in which they formulate and update their expectations about future events. As pointed out by Epstein and Denny (1983, pp. 649-650), "Current prices are … expected to persist indefinitely. As the base period changes and new prices … are observed, the [decision maker] revises its expectations and its previous plans. Thus only the $t = 0$ portion of the plan … is carried out in general." Thus, an unfortunate implication of the myopic expectations hypothesis is that economic decision makers are infinitely forward looking when they design their optimal consumption plans, but are totally myopic when they formulate their expectations about the future economic environment.

Nevertheless, there are some good reasons to analyze the neoclassical intertemporal consumption model with myopic expectations. The appearance of the discount rate r in the consumer's budget constraint implies that any changes in relative prices must be real, rather than nominal. Absent general inflation in a competitive economy with stable production and consumption technologies, there is no *a priori* reason to anticipate future relative price changes. Focusing on constant relative prices, therefore, separates the primary economic forces associated with the structure of consumer preferences from those associated with technological change or other adjustments in the general economy when it is out of long-run equilibrium. In addition, a solid understanding of the dual structure of the intertemporal consumer choice problem under myopic expectations provides a foundation for analyzing more general frameworks, including adaptive and rational expectations, intertemporal models of household production, and naïve and rational habit formation in consumption. Therefore, in this subsection we develop and discuss the dynamic consumer choice model with myopic expectations at some length.

Under myopic expectations, the model and solution approach of the previous section continues to apply, with the caveat that $p(t)$ is replaced by p_0 at all points in time, which alters some of our previous conclusions. With this change, the shadow price for the budget constraint now satisfies the defining condition

$$M_0 \equiv \int_0^T e^{-rt} p_0' u_x^{-1}\left(e^{(\rho-r)t}\lambda_0(p_0, M_0, \rho, r, T)p_0\right)dt, \tag{3.2.1}$$

while, since p_0 is presumed to be constant over the entire planning horizon, (3.1.8) now has the form

$$\frac{\partial x^*(t)}{\partial p_0'} = e^{(\rho-r)t} u_{xx}^{-1}\left[\lambda_0 I + p_0 \frac{\partial \lambda_0}{\partial p_0'}\right]. \tag{3.2.2}$$

Differentiating the intertemporal budget identity, $\int_0^T e^{-rt} p_0' x^*(t)dt \equiv M_0$, with respect to p_0 implies that

$$\int_0^T e^{-rt} \frac{\partial x^*(t)'}{\partial p_0} p_0 dt \equiv -\int_0^T e^{-rt} x^*(t)dt. \tag{3.2.3}$$

Therefore, post-multiplying (3.2.2) by p_0, integrating over t, combining the results with (3.2.3), and solving for $\partial\lambda_0/\partial p_0$ gives

$$\frac{\partial \lambda_0}{\partial p_0} \equiv -\frac{\int_0^T e^{-rt}(x^* + u_{xx}^{-1}u_x)dt}{p_0'\left(\int_0^T e^{(\rho-2r)t}{}_0 u_{xx}^{-1}dt\right)p_0}. \tag{3.2.4}$$

The analogue to (3.1.9) when expectations are myopic replaces $p(t)$ with p_0,

$$\frac{\partial x^*(t)}{\partial M_0} \equiv \frac{e^{(\rho-r)t}u_{xx}^{-1}p_0}{p_0'\left(\int_0^T e^{(\rho-2r)\tau}u_{xx}^{-1}d\tau\right)p_0}. \tag{3.2.5}$$

Combining (3.2.4) with (3.2.3) gives the matrix of instantaneous uncompensated price derivatives as

$$\frac{\partial x^*(t)}{\partial p_0'} \equiv e^{(\rho-r)t}\left\{\lambda_0\left[u_{xx}^{-1} - \frac{u_{xx}^{-1}p_0 p_0' \int_0^T e^{(\rho-2r)\tau}\, u_{xx}^{-1}d\tau}{p_0'\left(\int_0^T e^{(\rho-2r)\tau}u_{xx}^{-1}d\tau\right)p_0}\right] - \frac{u_{xx}^{-1}p_0 \int_0^T e^{-r\tau}x^*(\tau)'d\tau}{p_0'\left(\int_0^T e^{(\rho-2r)\tau}u_{xx}^{-1}d\tau\right)p_0}\right\}. \tag{3.2.6}$$

Simple inspection or direct calculations show that the instantaneous uncompensated matrix of cross-price derivatives, the instantaneous "wealth-compensated" substitution matrix,

(3.2.7)
$$\frac{\partial x^*(t)}{\partial p_0'}+\frac{\partial x^*(t)}{\partial M_0}x^*(t)' \equiv$$
$$e^{(\rho-r)t}\left\{\lambda_0\left[u_{xx}^{-1}-\frac{u_{xx}^{-1}p_0p_0'\int_0^T e^{(\rho-2r)\tau}u_{xx}^{-1}d\tau}{p_0'\left(\int_0^T e^{(\rho-2r)\tau}u_{xx}^{-1}d\tau\right)p_0}\right]+\frac{u_{xx}^{-1}p_0\left(x^*(t)'-\int_0^T e^{-r\tau}x^*(\tau)'d\tau\right)}{p_0'\left(\int_0^T e^{(\rho-2r)\tau}u_{xx}^{-1}d\tau\right)p_0}\right\},$$

and, using the identity $x^*(t)\equiv h(p_0,m^*(t))$, the instantaneous "income-compensated" substitution matrix

(3.2.8)
$$\frac{\partial x^*(t)}{\partial p_0'}+\frac{\partial h(p_0,m^*(t))}{\partial m}x^*(t)' \equiv$$
$$e^{(\rho-r)t}\left\{\lambda_0\left[u_{xx}^{-1}-\frac{u_{xx}^{-1}p_0p_0'\int_0^T e^{(\rho-2r)\tau}u_{xx}^{-1}d\tau}{p_0'\left(\int_0^T e^{(\rho-2r)\tau}u_{xx}^{-1}d\tau\right)p_0}\right]-\frac{u_{xx}^{-1}p_0\int_0^T e^{-r\tau}x^*(\tau)'d\tau}{p_0'\left(\int_0^T e^{(\rho-2r)\tau}u_{xx}^{-1}d\tau\right)p_0}\right\}+\frac{u_{xx}^{-1}p_0x^*(t)'}{p_0'u_{xx}^{-1}p_0},$$

all are generally asymmetric. Thus, the primary mainstay of static consumer choice theory — the Slutsky symmetry and negativity condition — does not have ***any*** short-run (instantaneous) counterpart in intertemporal contexts. As we shall see in the next subsection, this result carries over to models of intertemporal consumer choice in which individuals formulate expectations for future values of relevant economic factors using smooth functions of the current values of those variables.

Prior to moving on to this more general formulation, however, first it is useful to develop a unifying framework to identify the economic structure and duality of the intertemporal consumer choice problem with myopic expectations. Many of the properties carry over to the more general situations considered later.

We begin heuristically and constructively by multiplying both sides of equations (3.2.5) and (3.2.6) by e^{-rt} and integrating over [0, T] to obtain

(3.2.9)
$$\int_0^T e^{-rt}\frac{\partial x^*(t)}{\partial M_0}dt \equiv \left(\int_0^T e^{(\rho-2r)t}u_{xx}^{-1}dt\right)p_0\Big/p_0'\left(\int_0^T e^{(\rho-2r)t}u_{xx}^{-1}dt\right)p_0,$$

(3.2.10)
$$\int_0^T e^{-rt}\frac{\partial x^*(t)}{\partial p_0'}dt \equiv \lambda_0\left[\int_0^T e^{(\rho-2r)t}u_{xx}^{-1}dt-\frac{\left(\int_0^T e^{(\rho-2r)t}u_{xx}^{-1}dt\right)p_0p_0'\left(\int_0^T e^{(\rho-2r)t}u_{xx}^{-1}dt\right)}{p_0'\left(\int_0^T e^{(\rho-2r)t}u_{xx}^{-1}dt\right)p_0}\right]$$
$$-\frac{\left(\int_0^T e^{(\rho-2r)t}u_{xx}^{-1}dt\right)p_0\left(\int_0^T e^{-rt}x^*(t)dt\right)'}{p_0'\left(\int_0^T e^{(\rho-2r)t}u_{xx}^{-1}dt\right)p_0}.$$

Then multiplying (3.2.9) by $[\int_0^T e^{-rt}x^*(t)dt]'$ and adding the result to (3.2.10) gives

$$(3.2.11)\qquad \int_0^T e^{-rt}\frac{\partial x^*(t)}{\partial p_0'}dt + \left(\int_0^T e^{-rt}\frac{\partial x^*(t)}{\partial M_0}dt\right)\left(\int_0^T e^{-rt}x^*(t)'dt\right) \equiv$$

$$\lambda_0\left[\int_0^T e^{(\rho-2r)t}u_{xx}^{-1}dt - \frac{\left(\int_0^T e^{(\rho-2r)t}u_{xx}^{-1}dt\right)p_0 p_0'\left(\int_0^T e^{(\rho-2r)t}\,u_{xx}^{-1}dt\right)}{p_0'\left(\int_0^T e^{(\rho-2r)t}u_{xx}^{-1}dt\right)p_0}\right].$$

The $n\times n$ matrix on the right-hand-side is symmetric, negative semidefinite, and has rank n-1. It turns out that this matrix of wealth-compensated substitution terms is precisely the intertemporal analogue to the static Slutsky symmetry condition, a fact which we will verify in the course of the developments below.[7]

We define the maximal level of discounted utility flows, subject to the wealth constraint, by

$$(3.2.12)\qquad V(p_0,M_0)\equiv \sup_{\{x(t)\}}\left\{\int_0^T e^{-\rho t}u(x)dt\colon\ \int_0^T e^{-rt}p_0'xdt = M_0\right\},$$

where the equality constraint is justified by the strict monotonicity of $u(\cdot)$. We call $V(p_0, M_0)$ the *dynamic indirect utility function.*[8] Under myopic expectations, the dynamic indirect utility function has properties that are intertemporal analogues to those of the static indirect utility function. That is, $V(p_0, M_0)$ is:

(DV.1) *twice continuously differentiable in* (p_0, M_0);

(DV.2) *decreasing and quasiconvex in* p_0;

(DV.3) *strictly increasing and strongly concave in* M_0; *and*

(DV.4) $0°$ *homogeneous in* (p_0, M_0); *and*

(DV.5) *satisfies the Dynamic Envelope Theorem,*

---

[7] A simple, heuristic argument for the validity of (3.2.11) as the dynamic Slutsky substitution matrix is the following. Let $\boldsymbol{U}_{xx}^{-1} \equiv \int_0^T e^{(r-2r)t}u_{xx}^{-1}dt$ and note that this $n\times n$ matrix is negative definite and defines, in a sense, the "inverse Hessian" matrix that determines how changes in consumption choices due to changes in initial prices are allocated over the life cycle. Direct substitution into (3.2.11) gives

$$\boldsymbol{S} \equiv \lambda_0\left[\boldsymbol{U}_{xx}^{-1} - (p_0'\boldsymbol{U}_{xx}^{-1}p_0)^{-1}\boldsymbol{U}_{xx}^{-1}p_0p_0'\boldsymbol{U}_{xx}^{-1}\right],$$

which has exactly the form of the static neoclassical Slutsky substitution matrix.

[8] The function $V(\cdot)$ also depends upon the discount rates, ρ and r, and the length of the planning horizon, T. Since these parameters are not the central focus of our discussion, they have been suppressed as arguments to reduce the notational burden.

$$\partial V(p_0, M_0)/\partial p_0 \equiv -\lambda_0(p_0, M_0)\int_0^T e^{-rt} h(p_0, M_0, t)dt \ll \mathbf{0},$$

$$\partial V(p_0, M_0)/\partial M_0 \equiv \lambda_0(p_0, M_0) > 0,$$

and the Dynamic Roy's Identity,

$$-\left(\frac{\partial V(p_0, M_0)/\partial p_0}{\partial V(p_0, M_0)/\partial M_0}\right) \equiv \int_0^T e^{-rt} h(p_0, M_0, t)dt,$$

where $x^*(t) \equiv h(p_0, M_0, t)$ is the vector of dynamic ordinary Marshallian demands at time t.

Twice continuous differentiability of $V(\cdot)$ follows from strict monotonicity and twice continuous differentiability of $u(\cdot)$. Strict monotonicity, and strong concavity in M_0 follow from the adaptation, without change, of (3.1.12) and (3.1.10) to the present situation. Monotonicity in p_0 also follows from the monotonicity of $u(\cdot)$ and the fact that the intertemporal budget set contracts as prices increase. Quasiconvexity is demonstrated in precisely the same manner as for a static problem. Homogeneity follows from the fact that the wealth constraint, $p_0'\int_0^T e^{-rt} x(t)dt = M_0$, is invariant to proportional changes in all prices and initial wealth.

In a very general context, including both equality and inequality constraints and multiple switch points over the planning horizon, the dynamic envelope theorem is demonstrated by LaFrance and Barney (1991). Their argument is complex and involved and will not be reproduced here. However, it is pedagogically useful to verify DV.5 by direct calculation to lend heuristic support for the complex dynamic envelope theorem results that are presented below. This is accomplished by simply differentiating

$$V(p_0, M_0) \equiv \int_0^T e^{-\rho t} u(h(p_0, M_0, t))dt$$

with respect to p_0 and M_0, substituting the first-order conditions into the resulting expressions, grouping terms, and integrating over the planning horizon, to obtain

(3.2.13) $$\frac{\partial V(p_0, M_0)}{\partial p_0} \equiv \int_0^T e^{-\rho t}\frac{\partial h'}{\partial p_0} u_x dt$$

$$\equiv \int_0^T e^{-\rho t}\left(e^{(\rho-r)t}\left\{\lambda_0 u_{xx}^{-1} - \frac{\lambda_0 u_{xx}^{-1} p_0 p_0'\left(\int_0^T e^{(\rho-2r)t} u_{xx}^{-1} dt\right) + \left(\int_0^T e^{-rt} h dt\right) p_0' u_{xx}^{-1}}{p_0'\left(\int_0^T e^{(\rho-2r)t} u_{xx}^{-1} dt\right) p_0}\right\} e^{(\rho-r)t}\lambda_0 p_0\right)dt$$

$$\equiv \lambda_0^2\left(\int_0^T e^{(\rho-2r)t} u_{xx}^{-1} dt\right) p_0$$

$$-\frac{\left[\lambda_0^2\left(\int_0^t e^{(\rho-2r)t}u_{xx}^{-1}dt\right)p_0p_0'\left(\int_0^T e^{(\rho-2r)t}u_{xx}^{-1}dt\right)p_0+\lambda_0\left(\int_0^T e^{-rt}hdt\right)p_0'\left(\int_0^t e^{(\rho-2r)t}u_{xx}^{-1}dt\right)p_0\right]}{p_0'\left(\int_0^T e^{(\rho-2r)t}u_{xx}^{-1}dt\right)p_0}$$

$$\equiv -\lambda_0\int_0^T e^{-rt}h(p_0,M_0,t)dt\,,$$

and

$$\frac{\partial V(p_0,M_0)}{\partial M_0}\equiv\int_0^T e^{-\rho t}u_x'\frac{\partial h}{\partial M_0}dt \tag{3.2.14}$$

$$\equiv\int_0^T e^{-\rho t}\left(e^{(\rho-r)t}\lambda_0p_0'\right)\left[\frac{e^{(\rho-r)t}u_{xx}^{-1}p_0}{p_0'\left(\int_0^T e^{(\rho-2r)\tau}u_{xx}^{-1}d\tau\right)p_0}\right]dt\equiv\lambda_0\,. \qquad \blacksquare$$

Dual to the intertemporal utility maximization problem, we define the minimum discounted present value of consumption expenditures subject to the constraint that the discounted cumulative flow of utility is no lower than a given value, U_0,

$$E(p_0,U_0)\equiv\inf_{\{x(t)\}}\left\{\int_0^T e^{-rt}p_0'xdt:\int_0^T e^{-\rho t}u(x)dt\ge U_0\right\}. \tag{3.2.15}$$

We call this the *dynamic expenditure function*. The dynamic expenditure function is:

(DE.1) *twice continuously differentiable, strictly increasing*, 1° *homogeneous, and concave in* p_0*;*

(DE.2) *twice continuously differentiable, strictly increasing and strongly convex in* U_0*; and*

(DE.3) *satisfies the Dynamic Envelope Theorem,*

$$\frac{\partial E(p_0,U_0)}{\partial p_0}\equiv\int_0^T e^{-rt}g(p_0,U_0,t)dt$$

(*the Dynamic Hotelling's/Shephard's Lemma*) *and*

$$\frac{\partial E(p_0,U_0)}{\partial U_0}\equiv\mu_0(p_0,U_0)>0\,,$$

where $x^*(t)=g(p_0,U_0,t)$ is the vector of wealth-compensated dynamic Hicksian demands at time t and $\mu_0(p_0,U_0)$ is the shadow price for the intertemporal utility constraint.

To lay the groundwork for the analysis of the more complicated and general models in subsequent sections, we fully develop these properties and the intertemporal duality between the dynamic indirect utility and expenditure functions. Let $U(0)=U_0$, $dU(t)/dt=-e^{-\rho t}u(x)\ \forall\ t\in[0,T]$, and introduce the discounted utility flow constraint as an inequality restriction, $U(T) = U(0) - \int_0^T e^{-\rho t} u(x)dt \geq 0$.[9] Then the Hamiltonian for the expenditure minimization problem is

(3.2.16) $$H = e^{-rt} p_0' x - \mu e^{-\rho t} u(x),$$

and the first-order necessary and sufficient conditions for an interior optimal path are:

(3.2.17) $$\partial H/\partial x = e^{-rt} p_0 - \mu e^{-\rho t}\ \partial u/\partial x = 0;$$

(3.2.18) $$\partial H/\partial U = 0 = -\dot{\mu};$$

(3.2.19) $$\partial H/\partial \mu = -e^{-\rho t} u = \dot{U},\ U(T) \equiv U_0 - \int_0^T e^{-\rho t} u dt \geq 0.$$

It is easy to see that strict monotonicity of $u(\cdot)$ implies that $U(T) = 0$ since otherwise the discounted present value of expenditures could be lowered without violating the inequality constraint on the present value of discounted utility flows. It also follows from the assumed properties of $u(\cdot)$ that the optimal path is unique. As in the case of dynamic utility maximization, condition (3.2.18) implies that the shadow price is constant throughout the planning horizon, $\mu(t) = \mu_0\ \forall\ t \in [0, T]$.

Let $x^*(t) \equiv g(p_0, U_0, t)$ denote the optimal dynamic Hicksian demands at time t and let $\mu_0(p_0, U_0) > 0$ denote the optimal shadow price for the intertemporal utility constraint. We verify DE.1 - DE.6 by direct calculation. We begin by first differentiating (3.2.17) with respect to p_0, and solving for $\partial g/\partial p_0'$,

(3.2.20) $$\frac{\partial g}{\partial p_0'} = \mu_0^{-1} u_{xx}^{-1}\left[e^{(\rho - r)t} I - u_x \frac{\partial \mu_0}{\partial p_0'}\right].$$

We differentiate $\int_0^T e^{-\rho t} u(g(p_0, U_0, t))dt \equiv U_0$ with respect to p_0 to get $\int_0^T e^{-\rho t}(\partial g'/\partial p_0) u_x dt \equiv \mathbf{0}$, transpose both sides of (3.2.20), post-multiply by $e^{-\rho t} u_x$, integrate over the planning horizon, and solve for $\partial \mu_0/\partial p_0$,

(3.2.21) $$\frac{\partial \mu_0}{\partial p_0} \equiv \frac{\int_0^T e^{-rt} u_{xx}^{-1} u_x dt}{\int_0^T e^{-\rho t} u_x' u_{xx}^{-1} u_x dt} \equiv \mu_0 \frac{\left(\int_0^T e^{(\rho - 2r)t} u_{xx}^{-1} dt\right) p_0}{p_0' \left(\int_0^T e^{(\rho - 2r)t} u_{xx}^{-1} dt\right) p_0}.$$

---

[9] This transformation converts the consumer's intertemporal expenditure minimization problem from an isoperimetric calculus of variations problem into a standard optimal control problem. The latter form is convenient for generating comparative dynamics results and the properties of the optimal solution path. The former, to which we will return momentarily, is useful for analyzing dynamic duality.

Substituting the far right-hand-side of (3.2.21) into (3.2.20) gives the instantaneous wealth-compensated matrix of cross-price substitution effects as

$$\text{(3.2.22)} \qquad \frac{\partial g}{\partial p_0'} \equiv e^{(\rho-r)t}\mu_0^{-1}\left[u_{xx}^{-1} - \frac{u_{xx}^{-1}p_0 p_0'\left(\int_0^T e^{(\rho-2r)\tau}u_{xx}^{-1}d\tau\right)}{p_0'\left(\int_0^T e^{(\rho-2r)\tau}u_{xx}^{-1}d\tau\right)p_0}\right].$$

We now can verify the dynamic analogue to Hotelling's/Shephard's Lemma. By definition of the dynamic expenditure function, $E(p_0,U_0) \equiv \int_0^T e^{-\rho t} p_0' g(p_0,U_0,t)dt$, we have

$$\text{(3.2.23)} \qquad \begin{aligned} \frac{\partial E(p_0,U_0)}{\partial p_0} &\equiv \int_0^T e^{-rt}\left(\frac{\partial g'}{\partial p_0}p_0 + g\right)dt \\ &\equiv \int_0^T e^{-rt}\left[\frac{\partial g'}{\partial p_0}\left(e^{(r-\rho)t}\mu_0 u_x\right) + g\right]dt \\ &\equiv \mu_0\int_0^T e^{-\rho t}\frac{\partial g'}{\partial p_0}u_x dt + \int_0^T e^{-rt}gdt \\ &\equiv \int_0^T e^{-rt}gdt \gg \mathbf{0}. \end{aligned}$$

By the converse to Euler's theorem, the dynamic expenditure function is linearly homogeneous in p_0. Since the right-hand-side of (3.2.22) is continuous, $E(p_0,U_0)$ is twice continuously differentiable in p_0. Although concavity in p_0 can be demonstrated with the arguments used for the static neoclassical model, it is useful to verify it directly. Differentiating (3.2.23) with respect to p_0, using (3.2.22) for the right-hand-side integrand, we have

$$\text{(3.2.24)} \qquad \begin{aligned} \frac{\partial E(p_0,U_0)}{\partial p_0 \partial p_0'} &\equiv \int_0^T e^{-rt}\frac{\partial g}{\partial p_0'}dt \\ &\equiv \int_0^T e^{(\rho-2r)t}\mu_0^{-1}\left[u_{xx}^{-1} - \frac{u_{xx}^{-1}p_0 p_0'\left(\int_0^T e^{(\rho-2r)\tau}u_{xx}^{-1}d\tau\right)}{p_0'\left(\int_0^T e^{(\rho-2r)\tau}u_{xx}^{-1}d\tau\right)p_0}\right]dt \\ &\equiv \mu_0^{-1}\left[\int_0^T e^{(\rho-2r)t}u_{xx}^{-1}dt - \frac{\left(\int_0^T e^{(\rho-2r)t}u_{xx}^{-1}dt\right)p_0 p_0'\left(\int_0^T e^{(\rho-2r)t}u_{xx}^{-1}dt\right)}{p_0'\left(\int_0^T e^{(\rho-2r)t}u_{xx}^{-1}dt\right)p_0}\right]. \end{aligned}$$

Since $\mu_0 > 0$ and $\boldsymbol{U}_{xx}^{-1} \equiv \int_0^T e^{(r-2r)t}u_{xx}^{-1}dt$ is symmetric, negative definite, the Hessian matrix for $E(p_0,U_0)$ is negative semidefinite with rank n-1. This completes the verification of DE.1 and the first half of DE.3. We return to (3.2.24) momentarily to verify that it is the symmetric, negative semidefinite, rank n-1 wealth-compensated Slutsky matrix given in (3.2.11) above.

The steps required to verify DE.2 are similar. Differentiating (3.2.17) with respect to U_0 implies

$$(3.2.25) \qquad \frac{\partial g}{\partial U_0} \equiv -\mu_0^{-1} u_{xx}^{-1} u_x \frac{\partial \mu_0}{\partial U_0}.$$

Differentiating $\int_0^T e^{-\rho t} u(g(p_0,U_0,t))dt \equiv U_0$ with respect to U_0 gives $\int_0^T e^{-\rho t} u_x' \, \partial g/\partial U_0 \, dt \equiv 1$. Therefore, premultiplying (3.2.25) by $e^{-\rho t} u_x'$, integrating over t, and using first-order condition (3.2.17) to replace u_x, we obtain

$$(3.2.26) \qquad \frac{\partial \mu_0}{\partial U_0} \equiv \frac{-\mu_0}{\int_0^T e^{-\rho t} u_x' u_{xx}^{-1} u_x dt} \equiv \frac{-\mu_0^3}{p_0'\left(\int_0^T e^{(\rho-2r)t} u_{xx}^{-1} dt\right) p_0} > 0,$$

where the inequality on the far right follows from strong concavity of $u(\cdot)$ and $p_0 \neq \mathbf{0}$. Substituting (3.2.26) into (3.2.25) gives

$$(3.2.27) \qquad \frac{\partial g}{\partial U_0} \equiv \frac{u_{xx}^{-1} u_x}{\int_0^T e^{-\rho t} u_x' u_{xx}^{-1} u_x dt} \equiv \frac{e^{(\rho-r)t} \mu_0 u_{xx}^{-1} p_0}{p_0'\left(\int_0^T e^{(\rho-2r)t} u_{xx}^{-1} dt\right) p_0}.$$

Differentiating the dynamic expenditure function with respect to U_0 then gives

$$(3.2.28) \qquad \frac{\partial E(p_0,U_0)}{\partial U_0} \equiv \int_0^T e^{-rt} p_0' \frac{\partial g}{\partial U_0} dt \equiv \int_0^T e^{-rt} p_0' \frac{e^{(\rho-r)t} \mu_0 u_{xx}^{-1} p_0}{p_0'\left(\int_0^T e^{(\rho-2r)\tau} u_{xx}^{-1} d\tau\right) p_0} dt \equiv \mu_0 > 0.$$

Inspection of (3.2.28) and then (3.2.26) shows us that $\partial^2 E(p_0,U_0)/\partial U_0^2 > 0$, thus completing the validation of DE.2 and the second half of DE.3.

The duality between the dynamic indirect utility function and the dynamic expenditure function can be established most directly by viewing them as problems in the classical theory of the calculus of variations (e.g., Clegg (1968), pp. 117-121). Recalling the strict monotonicity of $u(\cdot)$ and noting that $p_0'x$ is strictly decreasing in at least one x_i if $p_0 \neq \mathbf{0}$, the utility maximization and expenditure minimization problems can be restated in the isoperimetric form

$$V(p_0, M_0) \equiv \sup_{\{x(t)\}} \left\{ \int_0^T e^{-\rho t} u(x(t))dt : \int_0^T e^{-rt} p_0' x(t) dt = M_0 \right\},$$

$$E(p_0,U_0) \equiv \inf_{\{x(t)\}} \left\{ \int_0^T e^{-rt} p_0' x(t) dt : \int_0^T e^{-\rho t} u(x(t))dt = U_0 \right\}.$$

A well-known result in the theory of the calculus of variations is that, for isoperimetric control problems, the solutions to the two problems are equivalent throughout the entire optimal path if $M_0 = E(p_0,U_0)$, or equivalently, if $U_0 = V(p_0, M_0)$. This equivalence is analogous to the duality in static models of consumer choice,

$M_0 \equiv E(p_0, V(p_0, M_0))$ and $U_0 \equiv V(p_0, E(p_0, U_0))$, except that now all definitions are in terms of the discounted present values of consumption expenditures and utility flows. Several conclusions follow directly from this fact, each generating the dynamic analogue to a corresponding duality property in the static theory:

(3.2.29) $$\lambda_0(p_0, E(p_0, U_0)) \equiv \mu_0(p_0, U_0)^{-1};$$

(3.2.30) $$\mu_0(p_0, V(p_0, M_0)) \equiv \lambda_0(p_0, M_0)^{-1};$$

(3.2.31) $$g(p_0, U_0, t) \equiv h(p_0, E(p_0, U_0), t);$$

(3.2.32) $$h(p_0, M_0, t) \equiv g(p_0, V(p_0, M_0), t);$$

(3.2.33) $$\frac{\partial E(p_0, U_0)}{\partial p} \equiv \int_0^T e^{-rt} g(p_0, U_0, t) dt$$

$$\equiv \int_0^T e^{-rt} h(p_0, E(p_0, U_0), t) dt \equiv -\frac{\partial V(p_0, E(p_0, U_0))/\partial p}{\partial V(p_0, E(p_0, U_0))/\partial M};$$

(3.2.34) $$\frac{\partial g(p_0, U_0, t)}{\partial p'} \equiv \frac{\partial h(p_0, E(p_0, U_0), t)}{\partial p'} + \frac{\partial h(p_0, E(p_0, U_0), t)}{\partial M} \left(\int_0^T e^{-rt} h(p_0, E(p_0, U_0), t) dt \right)';$$

(3.2.35) $$\frac{\partial^2 E(p_0, U_0)}{\partial p \partial p'} \equiv \int_0^T e^{-rt} \frac{\partial g(p_0, U_0, t)}{\partial p'} dt$$

$$\equiv \int_0^T e^{-rt} \frac{\partial h(p_0, E(p_0, U_0), t)}{\partial p'} dt + \int_0^T e^{-rt} \frac{\partial h(p_0, E(p_0, U_0), t)}{\partial M} dt \left(\int_0^T e^{-rt} h(p_0, E(p_0, U_0), t) dt \right)'.$$

Equation (3.2.34) defines the *instantaneous Slutsky substitution matrix*. The first matrix on the right-hand-side denotes the instantaneous price effects on the ordinary demands at each point in time and the second right-hand-side matrix denotes the wealth effects. However, it is (3.2.35) and not (3.2.34) that is symmetric and negative semidefinite. Even in this simplest of possible dynamic contexts, therefore, caution is advisable when interpreting hypothesis tests for "Slutsky symmetry and negativity" and other strictures of the static theory. Moreover, if consumers plan ahead when designing their goods purchases, the manner in which they form expectations for the future economic environment also is a critical determinant of observable behavior. This is the topic of the next subsection.

3.3 Other Forecasting Rules

In the neoclassical model of competition, market prices are invariant to the purchasing and consumption choices of the individual. However, this does not imply that consumers are incapable of learning about market price mechanisms or of forming expectations about their future economic environment. Perfect foresight and myopic expectations are but two possibilities among an uncountable number of alternative forecasting rules that may be reasonable hypotheses in a model of consumption behavior. In this subsection, therefore, we analyze models in which consuming households employ rules for forecasting future economic conditions when they formulate their dynamic consumption plans. Important members of the class of rules we consider are myopic, adaptive, rational, and quasirational expectations. Notwithstanding the previous subsection's detailed analysis of myopic expectations, rather than treat each of these special cases separately, we attempt to embed all of these hypotheses as special cases within a general, unifying framework.

While it is clear that future incomes, rates of return on assets, and market rates of interest at which the individual can borrow or lend are important economic variables affecting future opportunity sets, the basic questions, arguments, and conclusions arising from expectations processes are most clear and simplest to present when we focus on forecasting market prices. Therefore, let the system of ordinary differential equations,

$$\dot{p}(t) = \psi(p(t),t), \quad p(0) = p_0 \,, \tag{3.3.1}$$

be the rule that a consumer is presumed to use to form expectations for future prices, where the "$\cdot$" over a variable (or vector of variables) denotes the ordinary time derivative. We assume throughout the discussion that $\psi : \mathbb{R}^n_+ \times \mathbb{R}_+ \to \mathbb{R}^n$ is twice continuously differentiable and $|\partial\psi/\partial p'| \neq 0$ throughout its domain. This implies the existence of a unique, twice continuously differentiable solution to the differential equation system which defines all future price forecasts as a function of the initial price vector, p_0, and current time, t,

$$p(t) \equiv \varphi(p_0,t) \equiv p_0 + \int_0^t \psi(\varphi(p_0,\tau),\tau)d\tau \,. \tag{3.3.2}$$

In addition to the above properties for $\psi(\cdot)$, we assume that the solution to the forecasting rule generates strictly positive price forecasts, $\varphi\colon \mathbb{R}^n_{++}\times\mathbb{R}_+\to\mathbb{R}^n_{++}$.

It is a well-accepted stylized empirical fact that observed market prices tend to have common trends. In the present context, the most general statement of such a property is that the forecasting solution, $\varphi(\cdot, t)$, is linearly homogeneous in p_0. It turns out that this property is necessary and sufficient for the dynamic expenditure function to be linearly homogeneous in current prices in this model. As the following lemma shows, this property can be stated equivalently in terms of the condition that $\psi(\cdot, t)$ is homogeneous of degree one in $p(t)$ $\forall$ $t \in [0, T]$.

> ***Lemma 1.*** *$\varphi(\cdot,t)$ is homogeneous of degree one in p_0 $\forall$ $(p_0,t)\in\mathbb{R}^n_{++}\times\mathbb{R}_+$, if and only if $\psi(\cdot,t)$ is homogeneous of degree one in $p\,\forall$ $(p,t)\in\mathbb{R}^n_{++}\times\mathbb{R}_+$.*

PROOF: Suppose that $\varphi(p,t)\equiv\dfrac{\partial\varphi(p,t)'}{\partial p}p$ $\forall$ $(p,t)\in\mathbb{R}^n_{++}\times\mathbb{R}_+$. Then

$$\frac{\partial\varphi(p_0,t)}{\partial p_0'}\equiv I+\int_0^t\frac{\partial\psi(\varphi(p_0,\tau),\tau)}{\partial p'}\times\frac{\partial\varphi(p_0,\tau)}{\partial p_0'}d\tau$$

$$\Rightarrow\ \varphi(p_0,t)\equiv\frac{\partial\varphi(p_0,t)}{\partial p_0'}p_0\equiv p_0+\int_0^t\frac{\partial\psi(\varphi(p_0,\tau),\tau)}{\partial p'}\varphi(p_0,\tau)d\tau\equiv p_0+\int_0^t\psi(\varphi(p_0,\tau),\tau)d\tau\ ,$$

where the far right-hand-side is the definition of the far left-hand-side, while the middle identity follows from the linear homogeneity of $\varphi(\cdot, t)$ in p_0. Subtracting p_0 from the last two expressions implies

$$\int_0^t\frac{\partial\psi(p(\tau),\tau)}{\partial p'}p(\tau)d\tau\equiv\int_0^t\psi(p(\tau),\tau)d\tau\ .$$

By the fundamental theorem of calculus, differentiating both sides with respect to t gives

$$\frac{\partial\psi(p(t),t)}{\partial p'}p(t)\equiv\psi(p(t),t)\ ,$$

hence $\psi(\cdot, t)$ is homogeneous of degree one in p by the converse to Euler's theorem. This proves necessity.

We verify sufficiency by employing the method of successive approximations to solve the ordinary differential equation system (3.3.1). Each iteration begins with an approximate solution that is linearly homogeneous. We then show that this property is inherited by the subsequent iteration's approximate solution. The proof is concluded by induction, and an appeal to the contraction mapping theorem to verify that the sequence of iterations converges to the unique solution to the ordinary differential equation system.

Let $\varphi^{(0)}(p_0,t) \equiv p_0$, which is trivially one degree homogeneous in p_0, and define

$$\varphi^{(1)}(p_0,t) \equiv p_0 + \int_0^t \psi(p_0,\tau)d\tau \equiv p_0 + \int_0^t \psi(\varphi^{(0)}(p_0,\tau),\tau)d\tau \ ,$$

so that

$$\frac{\partial \varphi^{(1)}(p_0,t)'}{\partial p_0} p_0 \equiv p_0 + \int_0^t \frac{\partial \psi(p_0,\tau)'}{\partial p_0} p_0 d\tau \equiv p_0 + \int_0^t \psi(p_0,\tau)d\tau \equiv \varphi^{(1)}(p_0,t) \ ,$$

which therefore also is one degree homogeneous in p_0. Proceeding by induction, if for any $i \geq 2$, we have

$$\frac{\partial \varphi^{(i-1)}(p_0,t)'}{\partial p_0} p_0 \equiv \varphi^{(i-1)}(p_0,t)$$

and we define

$$\varphi^{(i)}(p_0,t) \equiv p_0 + \int_0^t \psi(\varphi^{(i-1)}(p_0,\tau),\tau)d\tau \ ,$$

then

$$\frac{\partial \varphi^{(i)}(p_0,t)'}{\partial p_0} p_0 \equiv p_0 + \int_0^t \frac{\partial \psi(\varphi^{(i-1)}(p_0,\tau),\tau)'}{\partial p} \cdot \frac{\partial \varphi^{(i-1)}(p_0,\tau)'}{\partial p} p_0 d\tau$$

$$\equiv p_0 + \int_0^t \frac{\partial \psi(\varphi^{(i-1)}(p_0,\tau),\tau)'}{\partial p} \cdot \varphi^{(i-1)}(p_0,\tau)d\tau \equiv p_0 + \int_0^t \psi(\varphi^{(i-1)}(p_0,\tau),\tau)d\tau \equiv \varphi^{(i)}(p_0,t) \ ,$$

and $\varphi^{(i)}(p_0,t)$ is one degree homogeneous in p_0 $\forall$ $i \geq 1$. It follows that the unique solution to the ordinary differential equations,

$$\varphi(p_0,t) \equiv \lim_{i\to\infty} \varphi^{(i)}(p_0,t) \ ,$$

also must be linearly homogeneous in p_0. ■

For the remainder of the chapter, therefore, we assume that the forecasting rule, $\varphi(p_0,t)$, is twice continuously differentiable in (p_0,t), and increasing, positively linearly homogeneous, and concave in p_0. As we shall see in the course of the discussion that follows, the last condition is an essential ingredient for concavity of the dynamic expenditure function.

When relative prices change over time and consumers form expectations for future price levels according to some rule that is consistent with (3.3.2), the defining equation for the wealth constraint's shadow price takes the form

(3.3.3) $$\int_0^T e^{-rt} u_x^{-1}\left(e^{-rt}\lambda_0(p_0,M_0)\varphi(p_0,t)\right)' \varphi(p_0,t)dt \equiv M_0 \ .$$

Consequently, whenever consumers form price expectations in devising their consumption plans, market prices can not be exogenous in the empirical model.[10] This can be seen most clearly in the case of rational expectations, where $\varphi(p_0,t)$ equals the conditional mean of the price vector at time t given information available at time 0, so that the parameters of the marginal distribution for prices enter the conditional distribution for quantities given prices. The implication is that if consumers form expectations about their future economic environment as they develop consumption plans, then the expectation process must be modeled jointly with demand behavior to obtain consistent and efficient empirical results.

Following the same logic as in the previous subsections, we obtain the instantaneous price and wealth effects on demands to be

$$\frac{\partial h(p_0,M_0,t)}{\partial M_0} \equiv e^{(\rho-r)t}u_{xx}^{-1}\varphi(p_0,t)\frac{\partial \lambda_0(p_0,M_0)}{\partial M_0}, \tag{3.3.4}$$

$$\frac{\partial h(p_0,M_0,t)}{\partial p_0'} \equiv e^{(\rho-r)t}u_{xx}^{-1}\left[\lambda(p_0,M_0)\frac{\partial \varphi(p_0,t)}{\partial p_0'} + \varphi(p_0,t)\frac{\partial \lambda_0(p_0,M_0)}{\partial p_0'}\right], \tag{3.3.5}$$

while the impacts of a change in initial prices and wealth on the marginal utility of wealth are

$$\frac{\partial \lambda_0(p_0,M_0)}{\partial p_0} \equiv -\frac{\left(\int_0^T e^{(\rho-2r)t}(\partial\varphi'/\partial p_0)u_{xx}^{-1}\varphi dt + \int_0^T e^{-rt}(\partial\varphi'/\partial p_0)h dt\right)}{\int_0^T e^{(\rho-2r)t}\varphi' u_{xx}^{-1}\varphi dt} \tag{3.3.6}$$

$$\frac{\partial \lambda_0(p_0,M_0)}{\partial M_0} \equiv \frac{1}{\int_0^T e^{(\rho-2r)t}\varphi' u_{xx}^{-1}\varphi dt} < 0. \tag{3.3.7}$$

It follows from the last two equations that λ_0 is homogeneous of degree minus one in (p_0, M_0) if and only if φ is homogeneous of degree one in p_0. This implies that linear homogeneity of the price forecasting rule in current prices is necessary and sufficient for zero degree homogeneity of the ordinary demand functions in current prices and wealth. In turn, this latter property is necessary and sufficient for linear homogeneity of the dynamic expenditure function in current prices, p_0.

In the price forecasting model, the intertemporal Slutsky matrix has the form

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[10] Hendry (1995), especially chapter 5, contains a deep and exhaustive treatment of exogeneity in time series econometric models.

(3.3.8)
$$\boldsymbol{S} \equiv \int_0^T e^{-rt} \frac{\partial \varphi'}{\partial p_0} \frac{\partial h}{\partial p_0'} dt + \left(\int_0^T e^{-rt} \frac{\partial \varphi'}{\partial p_0} \frac{\partial h}{\partial M_0} dt \right) \times \left(\int_0^T e^{-rt} \frac{\partial \varphi'}{\partial p_0} h dt \right)'$$
$$\equiv \lambda_0 \left[\int_0^T e^{(\rho-2r)t} \frac{\partial \varphi'}{\partial p_0} u_{xx}^{-1} \frac{\partial \varphi}{\partial p_0'} dt - \frac{\left(\int_0^T e^{(\rho-2r)t} (\partial \varphi'/\partial p_0) u_{xx}^{-1} \varphi dt \right) \left(\int_0^T e^{(\rho-2r)t} (\partial \varphi'/\partial p_0) u_{xx}^{-1} \varphi dt \right)'}{\int_0^T e^{(\rho-2r)t} \varphi' u_{xx}^{-1} \varphi dt} \right],$$

a symmetric, negative semidefinite matrix with rank no greater than n-1.[11] Notwithstanding the effects of the additional terms $\partial\varphi'/\partial p_0$, the relationship between the intertemporal Marshallian and wealth-compensated (Hicksian) demands remains the same as in the previous subsection. However, the dynamic Slutsky matrix no longer is the Hessian matrix for the dynamic expenditure function. In particular, the dynamic envelope theorem now implies

(3.3.9)
$$\frac{\partial E(p_0,U_0)}{\partial p_0} \equiv \int_0^T e^{-rt} \frac{\partial \varphi(p_0,t)'}{\partial p_0} g(p_0,U_0,t) dt,$$

where $g(p_0,U_0,t)$ is the time t vector of Hicksian demands which solve the dynamic expenditure minimization problem

$$E(p_0,U_0) \equiv \inf_{\{x(t)\}} \left\{ \int_0^T e^{-rt} \varphi(p_0,t)' x(t) dt : \int_0^T e^{-\rho t} u(x(t)) dt = U_0 \right\}.$$

Differentiating (3.3.9) with respect to p_0 therefore implies

(3.3.10)
$$\frac{\partial^2 E(p_0,U_0)}{\partial p_0 \partial p_0'} \equiv \int_0^T e^{-rt} \frac{\partial \varphi(p_0,t)'}{\partial p_0} \frac{\partial g(p_0,U_0,t)}{\partial p_0'} dt + \sum_{i=1}^n \int_0^T e^{-rt} g_i(p_0,U_0,t) \frac{\partial^2 \varphi_i(p_0,t)}{\partial p_0 \partial p_0'} dt$$
$$\equiv \boldsymbol{S} + \sum_{i=1}^n \int_0^T e^{-rt} g_i(p_0,U_0,t) \frac{\partial^2 \varphi_i(p_0,t)}{\partial p_0 \partial p_0'} dt,$$

where $\lambda_0(p_0, E(p_0,U_0)) \equiv 1/\mu_0(p_0,U_0)$ has been used on the far right-hand-side. It follows that, in general, the dynamic expenditure function will be concave in p_0 only if all of the components of the price expectation rule are jointly concave in the initial price vector.

Introducing a general class of forecasting rules results in only minor changes to the duality between the dynamic indirect utility and expenditure functions. Writing the

[11] In fact, if the rank of $\partial\varphi/\partial p'$ is constant $\forall\ t \in [0, T]$, then

$$\operatorname{rank}(\boldsymbol{S}) = \min\{n-1, \operatorname{rank}(\partial\varphi/\partial p')\}.$$

utility maximization and expenditure minimization problems in their isoperimetric forms for the present case,

$$V(p_0, M_0) \equiv \sup_{\{x(t)\}} \left\{ \int_0^T e^{-\rho t} u(x(t))dt : \int_0^T e^{-rt} \varphi(p_0,t)' x(t)dt = M_0 \right\}, \tag{3.3.11}$$

$$E(p_0,U_0) \equiv \inf_{\{x(t)\}} \left\{ \int_0^T e^{-rt} \varphi(p_0,t)' x(t)dt : \int_0^T e^{-\rho t} u(x(t))dt = U_0 \right\}, \tag{3.3.12}$$

it follows that $M_0 \equiv E(p_0, V(p_0, M_0))$ and $U_0 \equiv V(p_0, E(p_0,U_0))$. Consequently, (3.2.39) — (3.2.32) remain unchanged, (3.2.35) becomes (3.3.10), and (3.2.33) and (3.2.34), respectively, become:

$$\begin{aligned} \frac{\partial E(p_0,U_0)}{\partial p_0} &\equiv \int_0^T e^{-rt} \frac{\partial \varphi(p_0,t)'}{\partial p_0} g(p_0,U_0,t)dt \\ &\equiv \int_0^T e^{-rt} \frac{\partial \varphi(p_0,t)'}{\partial p_0} h(p_0, E(p_0,U_0),t)dt \equiv -\frac{\partial V(p_0, E(p_0,U_0))/\partial p_0}{\partial V(p_0, E(p_0,U_0))/\partial M_0} \text{; and} \end{aligned} \tag{3.3.13}$$

$$\begin{aligned} &\frac{\partial g(p_0,U_0,t)}{\partial p_0'} \\ &\equiv \frac{\partial h(p_0, E(p_0,U_0),t)}{\partial p_0'} + \frac{\partial h(p_0, E(p_0,U_0),t)}{\partial M_0} \left(\int_0^T e^{-r\tau} \frac{\partial \varphi(p_0,\tau)'}{\partial p_0} h(p_0, E(p_0,U_0),\tau)d\tau \right)'. \end{aligned} \tag{3.3.14}$$

Similarly, the dynamic envelope theorem for the indirect utility function previously given in (3.2.13) above now takes the form

$$\frac{\partial V(p_0, M_0)}{\partial p_0} \equiv -\lambda_0 \int_0^T e^{-rt} \frac{\partial \varphi(p_0,t)'}{\partial p_0} h(p_0, M_0,t)dt\,, \tag{3.3.15}$$

while (3.2.14) continues to be $\partial V(p_0, M_0)/\partial M_0 \equiv \lambda_0(p_0, M_0)$. Note the effect of initial prices on future price expectations, which plays a significant and ubiquitous role in each of the above results, determining when the dynamic expenditure function is 1$^{\text{o}}$ homogeneous and concave in prices (equivalently, when the indirect utility function is 0$^{\text{o}}$ homogeneous in prices and wealth and quasiconvex in prices), as well as the functional expressions for the dynamic envelope theorem and the static (instantaneous) and dynamic Slutsky equations.

4. Dynamic Household Production Theory

This section merges household production theory with the theory of consumer choice over time. In this context, it is natural to incorporate durable goods into the household's production process. The basic model structure and variable definitions are analogous to previous sections, with $x(t)$ an n-vector flow of consumable market goods used at time t, $z(t)$ an m-vector flow of nonmarket commodities produced by the household and which generate utility directly, and $k(t)$ an ℓ-vector stock of household durables, some of which may be interpreted as consumption habits. We continue to take the household's objective to be to maximize the present value of discounted lifetime utility flows, but the flow of produced nonmarket commodities is now presumed to generate the flow of consumer satisfaction,

$$U = \int_0^T e^{-\rho t} u(z(t)) dt . \tag{4.1}$$

The efficient boundary of the household production possibility set for each point in time is defined by the joint consumables/durables/commodities transformation function

$$G(x(t), k(t), z(t), \beta, t) \leq 0 , \tag{4.2}$$

where β is an s-vector of quality characteristics of both the consumables and durables and the index t tacitly implies that the feasible household production possibilities set may vary over time.[12] The rates of change in the household's holdings of durable stocks are defined by the differential equations,

$$\dot{k}(t) = f(x(t), k(t), \gamma, t), \quad k(0) = k_0 \text{ , given ,} \tag{4.3}$$

where γ is a vector of durable goods' characteristics that affect the rates of accumulation and/or decay. The household's life cycle budget constraint is defined by

$$M_0 = \int_0^T e^{-rt} p(t)' x(t) dt . \tag{4.4}$$

We begin with a straightforward extension of theorem 2 to show that the derived instantaneous utility function defined over consumables, durables, qualities, and time,

$$u^*(x, k, \beta, t) \equiv \sup_{z \geq \mathbf{0}} \{ u(z) : G(x, k, z, \beta, t) \leq 0 \} \tag{4.5}$$

[12] One possibility is that the characteristics of market goods vary over time with consumer expectations for these changes modeled similarly as for price expectations in section 3.3 above. This would imply that β in equation (4.2) tacitly represents goods characteristics at the initial date in the planning horizon, while the structure of $f(\cdot)$ reflects the consuming household's expectations for both future household production technology and goods qualities.

is jointly strongly concave in (x, k) and increasing (decreasing) in x_i or k_j if and only if $f(\cdot)$ is increasing (decreasing) in the corresponding x_i or k_j. This in turn implies that the *instantaneous myopic indirect utility function*,

(4.6) $$\upsilon(p,m,k,\beta,t) \equiv \sup_{x\geq \mathbf{0}}\left\{ u^*(x,k,\beta,t) \colon p'x \leq m \right\},$$

is neoclassical in (p, m), i.e., $\upsilon(\cdot, k, \beta, t)$ is continuous and zero degree homogeneous in (p, m), increasing in m, and decreasing and quasiconvex in p. The corresponding myopic ordinary demands, $x = \tilde{h}(p,m,k,\beta,t)$, therefore also possess all of the neoclassical properties, while reflecting the structure commonly known as *naïve habit formation*.

Continuity of $f(\cdot)$ and $\tilde{h}(\cdot)$ implies that there is a unique solution for the time path of household durables holdings defined by

(4.7) $$\hat{k}(k_0,\beta,t) \equiv k_0 + \int_0^t f\left(\tilde{h}(p(\tau),m(\tau),\hat{k}(k_0,\beta,\tau),\beta,\tau),\hat{k}(k_0,\beta,\tau),\beta,\tau\right) d\tau .$$

Note that $\hat{k}(k_0,\beta,t)$ depends upon all past prices and consumption expenditures. This implies the following for consumption models under naïve habit formation:

Current stocks of durables are not weakly exogenous.

Preferences are intertemporally inconsistent, i.e., current preferences depend on the entire history of past consumption choices.

Consumers understand the effects of changes in household durables on the solution to their instantaneous utility maximization problem, but ignore this when planning for future consumption.

If consumers are assumed to be naïve regarding the influence of current consumption on future preferences and consumption possibilities, only myopic price expectations avoids a logical contradiction regarding household planning and foresight.

This essentially summarizes the current state of the art in empirical demand analysis. With a few notable exceptions, nearly all empirical demand analyses incorporate naïve habit formation and myopic price expectations. There are many reasons for this. Perhaps foremost is the fact that household holdings of durable stocks, including real capital items (as opposed to the somewhat ethereal concept of "consumption habits"), often are not observable, particularly with aggregate time series data. As a result, lagged quantities demanded of the consumable (i.e., nondurable) goods typically are used to proxy these and other unobservable trends in the data. Even with the enormous simplification that results when lagged quantities are used to proxy habits and other missing consumption trends, however, incorporating rational habit persistence in demand is difficult and complicated (Browning). *Rational habits* also suggest *rational expectations*, or at least expectations formation processes other than myopic, for future economic conditions. This complicates the required econometric analysis even further.

In this context, it is informative to analyze the economic structure of a dynamic model of household production that incorporates future expectations and a rational dynamic accounting of the effects of changes in the level of household stocks, whether they are interpreted as consumption habits or durable goods, on future (derived) preferences and feasible choice sets. With regard to future expectations, we will maintain our focus on prices and smooth expectation rules, although the analysis could be extended readily to include other economic variables as deemed appropriate. Our focus continues to be establishing the duality theory of the dynamic household production model.

When the household production technology is time dependent, e.g., as a result of technological change, the derived instantaneous utility function over consumable market goods and household stocks is a function of time, t. Hence, to reduce the notational burden, define $\tilde{u}(x,k,\beta,t) \equiv e^{-\rho t}u^*(x,k,\beta,t)$. The consumer's decision problem now is to solve

(4.8)
$$V(p_0, M_0, k_0, \beta, \gamma) \equiv \sup\left\{\int_0^T \tilde{u}(x,k,\beta,t)dt : \right.$$
$$\left. M_0 = \int_0^T e^{-rt}\varphi' x dt, \dot{k} = f(x,k,\gamma,t), k(0) = k_0, k(T) \geq 0\right\}.$$

Let ω be the ℓ-vector of co-state variables (i.e., shadow prices) for the equations of motion for household durables and let λ be the co-state variable for the equation of motion for the present value of wealth. Then the Hamiltonian can be written as

(4.9)
$$H = \tilde{u}(x,k,\beta,t) + \omega' f(x,k,\gamma,t) - \lambda e^{-rt}\varphi' x .$$

To simplify the discussion, assume that the Hamiltonian is jointly concave in (x, k), that for each $t \in [0,T)$, $\tilde{u} + \omega' g$ is strictly increasing in x throughout an open $n+\ell+1$-dimensional open tube in the neighborhood of the optimal path, and that the optimal path satisfies $(x^*(t), k^*(t)) \gg (\mathbf{0},\mathbf{0})\ \forall\, t \in [0,T)$. Then the necessary and sufficient first-order conditions plus the transversality conditions for the optimal path are:

$$\frac{\partial H}{\partial x} = \frac{\partial \tilde{u}}{\partial x} + \frac{\partial f'}{\partial x}\omega - \lambda e^{-rt} p = \mathbf{0}\,; \tag{4.10}$$

$$\frac{\partial H}{\partial k} = \frac{\partial \tilde{u}}{\partial k} + \frac{\partial f'}{\partial k}\omega = -\dot{\omega}\,; \tag{4.11}$$

$$\frac{\partial H}{\partial M} = 0 = -\dot{\lambda}\,; \tag{4.12}$$

$$\frac{\partial H}{\partial \omega} = f = \dot{k}\,,\ k(0) = k_0,\ k(T) \geq \mathbf{0}\,; \tag{4.13}$$

$$\frac{\partial H}{\partial \lambda} = -e^{-rt}\varphi' x = \dot{M}\,,\ M(0) = M_0\,,\ M(T) = 0\,; \text{ and} \tag{4.14}$$

$$\omega_j(T) k_j(T) = 0\ \ \forall\ \ j = 1,\ldots,\ell\,. \tag{4.15}$$

At time t, the optimal Marshallian demands, stocks of household durable goods, and shadow prices are

$$x^*(t) \equiv h(p_0, M_0, k_0, \beta, \gamma, t)\,, \tag{4.16}$$

$$k^*(t) \equiv \kappa(p_0, M_0, k_0, \beta, \gamma, t)\,, \tag{4.17}$$

$$\omega^*(t) \equiv \omega(p_0, M_0, k_0, \beta, \gamma, t)\,, \text{ and} \tag{4.18}$$

$$\lambda^*(t) \equiv \lambda_0(p_0, M_0, k_0, \beta, \gamma)\,, \tag{4.19}$$

respectively. Differentiating the *intertemporal budget identity*, $\int_0^T e^{-rt}\varphi' h\,dt \equiv M_0$, with respect to p_0 and M_0 generates the *intertemporal Cournot aggregation* and the *intertemporal Engel aggregation*, respectively,

$$\int_0^T e^{-rt}\left[\frac{\partial \varphi'}{\partial p_0} h + \frac{\partial h'}{\partial p_0}\varphi\right] dt \equiv \mathbf{0}\,, \text{ and} \tag{4.20}$$

$$\int_0^T e^{-rt}\varphi' \frac{\partial h}{\partial M_0}\, dt \equiv 1\,. \tag{4.21}$$

Define $H_{xx} \equiv \tilde{u}_{xx} + \sum_{i=1}^{\ell} \omega_i f_{xx}^i$, $H_{xk} \equiv \tilde{u}_{xk} + \sum_{i=1}^{\ell} \omega_i f_{xk}^i$, and $H_{kk} \equiv \tilde{u}_{kk} + \sum_{i=1}^{\ell} \omega_i f_{kk}^i$. Then, following the same steps as in the previous section, we obtain

$$\frac{\partial h}{\partial p_0'} = H_{xx}^{-1}\left[e^{-rt}\left(\lambda_0 \frac{\partial \varphi}{\partial p_0'} + \varphi \frac{\partial \lambda_0}{\partial p_0'}\right) - H_{xk}\frac{\partial \kappa}{\partial p_0'} - \frac{\partial f'}{\partial x}\frac{\partial \omega}{\partial p_0'}\right], \tag{4.22}$$

$$\frac{\partial \lambda_0}{\partial p_0} = \frac{-\displaystyle\int_0^T e^{-rt}\left\{\frac{\partial \varphi'}{\partial p_0} h + \left[e^{-rt}\lambda_0 \frac{\partial \varphi'}{\partial p_0} - \frac{\partial \kappa'}{\partial p_0} H_{kx} - \frac{\partial \omega'}{\partial p_0}\frac{\partial f}{\partial x'}\right] H_{xx}^{-1}\varphi\right\} dt}{\int_0^T e^{-2rt}\varphi' H_{xx}^{-1}\varphi\, dt}\,, \tag{4.23}$$

$$\frac{\partial h}{\partial M_0} = H_{xx}^{-1}\left[e^{-rt}\varphi \frac{\partial \lambda_0}{\partial M_0} - H_{xk}\frac{\partial \kappa}{\partial M_0} - \frac{\partial f'}{\partial x}\frac{\partial \omega}{\partial M_0}\right], \tag{4.24}$$

$$\frac{\partial \lambda_0}{\partial M_0} = \frac{1 + \int_0^T e^{-rt}\varphi' H_{xx}^{-1}\left(H_{xk}\frac{\partial \kappa}{\partial M_0} + \frac{\partial f'}{\partial x}\frac{\partial \omega}{\partial M_0}\right) dt}{\int_0^T e^{-2rt}\varphi' H_{xx}^{-1}\varphi dt}, \text{ and} \tag{4.25}$$

$$\int_0^T e^{-rt}\frac{\partial \varphi'}{\partial p_0}\frac{\partial h}{\partial p_0'} dt + \left(\int_0^T e^{-rt}\frac{\partial \varphi'}{\partial p_0}\frac{\partial h}{\partial M_0} dt\right)\left(\int_0^T e^{-rt}\frac{\partial \varphi'}{\partial p_0} h\, dt\right)' = \tag{4.26}$$

$$\lambda_0\left[\int_0^T e^{-2rt}\frac{\partial \varphi'}{\partial p_0}H_{xx}^{-1}\frac{\partial \varphi}{\partial p_0'} dt - \frac{\left(\int_0^T e^{-2rt}\frac{\partial \varphi'}{\partial p_0}H_{xx}^{-1}\varphi dt\right)\left(\int_0^T e^{-2rt}\frac{\partial \varphi'}{\partial p_0}H_{xx}^{-1}\varphi dt\right)'}{\int_0^T e^{-2rt}\varphi' H_{xx}^{-1}\varphi dt}\right]$$

$$+ \frac{\left(\int_0^T e^{-2rt}\frac{\partial \varphi'}{\partial p_0}H_{xx}^{-1}\varphi dt\right)}{\int_0^T e^{-2rt}\varphi' H_{xx}^{-1}\varphi dt}\left\{\int_0^T e^{-rt}\varphi' H_{xx}^{-1}\left(H_{xk}\frac{\partial \kappa}{\partial p_0'} + \frac{\partial f'}{\partial x}\frac{\partial \omega}{\partial p_0'}\right) dt\right.$$

$$\left. + \left[\int_0^T e^{-rt}\varphi' H_{xx}^{-1}\left(H_{xk}\frac{\partial \kappa}{\partial M_0} + \frac{\partial f'}{\partial x}\frac{\partial \omega}{\partial M_0}\right) dt\right]\left(\int_0^T e^{-rt}\frac{\partial \varphi'}{\partial p_0} h dt\right)'\right\}$$

$$- \left[\int_0^T e^{-rt}\frac{\partial \varphi'}{\partial p_0}H_{xx}^{-1}\left(H_{xk}\frac{\partial \kappa}{\partial M_0} + \frac{\partial f'}{\partial x}\frac{\partial \omega}{\partial M_0}\right) dt\right]\left(\int_0^T e^{-rt}\frac{\partial \varphi'}{\partial p_0} h dt\right)'.$$

Note, in particular, the added complexity of the intertemporal wealth-compensated price effects represented by the additional terms in the last three lines of (4.26). Even symmetry, much less negativity or homogeneity, of all but the first matrix on the right-hand-side is very difficult to prove using the direct methods of the previous sections. Therefore, another approach to the intertemporal duality of the dynamic household production problem given in equation (4.8) is needed.

Fortunately, such an approach is available, and this alternative way of looking at problems of this type has several advantages. The approach is simple and both heuristically and pedagogically appealing. It establishes a connection between the duality of static models and both discrete and continuous dynamic models, including the envelope theorem, adding up, homogeneity, symmetry and negativity, and the relationships between utility maximization and expenditure minimization. Finally, and at least as important as the simple and clear derivations, the arguments are valid for a very large class of problems — essentially all of optimal control theory.[13]

Define the "Lagrangean" function for (4.8) by

$$\mathcal{L}_1 \equiv \int_0^T \tilde{u}\,dt + \lambda\left(M_0 - \int_0^T e^{-rt}\varphi' x\,dt \right) + \int_0^T \omega'(f - \dot{k})\,dt \tag{4.27}$$

$$\equiv \int_0^T \left(\tilde{u} - \lambda e^{-rt}\varphi' x + \omega' f + \dot{\omega}' k \right) dt + \lambda M_0 + \omega(0)' k_0 - \omega(T)' k(T)\,,$$

where the second line follows from integrating the term $-\omega'\dot{k}$ by parts. Finding the pointwise maximum with respect to x of either the first or second lines of (4.27) reproduces first-order condition (4.10). Similarly, minimizing either expression for $\mathcal{L}_1$ with respect to λ reproduces (4.14) and also motivates the constant marginal utility of money condition given in (4.12). On the other hand, pointwise minimization of the first line of (4.27) with respect to ω gives (4.13), while pointwise maximization of the second line with respect to k generates (4.11). Also note that when the first-order conditions are satisfied $\forall\ t \in [0, T]$, the integrals of the constraints multiplied by their associated shadow prices vanish. This, in turn, implies that $\mathcal{L}_1^*(p_0, M_0, k_0, \beta, \gamma) \equiv V(p_0, M_0, k_0, \beta, \gamma)$. From the second line of (4.27), this simple observation immediately generates the pair of dynamic envelope theorem results:

$$\frac{\partial V}{\partial M_0} \equiv \lambda_0 > 0\text{; and} \tag{4.28}$$

$$\frac{\partial V}{\partial k_0} \equiv \omega(p_0, M_0, k_0, \beta, \gamma, 0)\,. \tag{4.29}$$

Several other dynamic envelope theorem results, as well as symmetry, curvature, and homogeneity properties also can be derived from the Lagrangean in (4.27). We do so in detail here for $\partial V/\partial p_0$ to illustrate the basic logic. We follow this with a statement of the properties of the dynamic indirect utility function for this problem.

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[13] In other words, once the necessary and sufficient conditions for an optimal solution path have been identified, the arguments of this section remain valid for: (a) absolutely bounded Lebesgue measurable controls; (b) nondifferental inequality and equality constraints; and (c) a countable number of switch points along the optimal path. The interested reader is referred to LaFrance and Barney (1991) for a discussion of one set of sufficient conditions and detailed derivations for the special case in which (c) is tightened to a finite number of switch points.

We then proceed with a brief development of the properties of the dynamic expenditure function. We conclude this section with a statement of the intertemporal duality for this problem.

We first proceed by substituting (4.16) — (4.19) into (4.27) to generate $\mathcal{L}_1^*(p_0, M_0, k_0, \beta, \gamma)$. Then we differentiate the resulting expression term-by-term with respect to p_0, which gives

$$\begin{aligned}(4.30)\qquad \frac{\partial \mathcal{L}_1^*}{\partial p_0} &\equiv \int_0^T \frac{\partial h'}{\partial p_0}\overbrace{\left(\frac{\partial \tilde{u}}{\partial x} - \lambda e^{-rt}\varphi + \frac{\partial f'}{\partial x}\omega\right)}^{\equiv 0} dt \\ &+ \int_0^T \frac{\partial \kappa'}{\partial p_0}\overbrace{\left(\frac{\partial \tilde{u}}{\partial k} + \frac{\partial f'}{\partial k}\omega + \dot{\omega}\right)}^{\equiv 0} dt + \frac{\partial \lambda_0}{\partial p_0}\overbrace{\left(M_0 - \int_0^T e^{-rt}\varphi' h dt\right)}^{\equiv 0} \\ &- \lambda_0 \int_0^T e^{-rt}\frac{\partial \varphi'}{\partial p_0} h dt + \int_0^T \frac{\partial \omega'}{\partial p_0} f dt + \int_0^T \frac{\partial^2 \omega'}{\partial t \partial p_0}\kappa dt + \frac{\partial \omega(\cdot,0)'}{\partial p_0}k_0 \\ &- \frac{\partial \omega(\cdot,T)'}{\partial p_0}\kappa(\cdot,T) - \overbrace{\frac{\partial \kappa(\cdot,T)'}{\partial p_0}\omega(\cdot,T)}^{\equiv 0},\end{aligned}$$

where the first three terms vanish by the first-order conditions (4.10) — (4.14) and the final term vanishes by the transversality conditions (4.15). Given the properties hypothesized for $\tilde{u}(\cdot)$, $f(\cdot)$, and $\varphi(\cdot)$, the Marshallian demands and the marginal utility of money are continuously differentiable, while the household durables and their shadow prices are twice continuously differentiable. Hence, by Young's theorem, we can integrate the terms $(\partial^2 \omega / \partial t \partial p_0)\kappa \equiv (\partial^2 \omega / \partial p_0 \partial t)\kappa$ by parts, which gives

$$(4.31)\qquad \int_0^T \frac{\partial^2 \omega}{\partial p_0 \partial t}\kappa dt \equiv \frac{\partial \omega(\cdot,T)'}{\partial p_0}\kappa(\cdot,T) - \frac{\partial \omega(\cdot,0)'}{\partial p_0}k_0 - \int_0^T \frac{\partial \omega}{\partial p_0}\frac{\partial \kappa}{\partial t}dt\,.$$

Canceling the terms that vanish on the right-hand-side of (4.30) and substituting the right-hand-side of (4.31) into (4.30) gives

$$(4.32)\qquad \frac{\partial \mathcal{L}_1}{\partial p_0} \equiv -\lambda_0 \int_0^T e^{-rt}\frac{\partial \varphi'}{\partial p_0} h dt + \int_0^T \frac{\partial \omega'}{\partial p_0}\overbrace{\left(f - \frac{\partial \kappa}{\partial t}\right)}^{\equiv 0} dt \equiv -\lambda_0 \int_0^T e^{-rt}\frac{\partial \varphi'}{\partial p_0} h dt\,.$$

Applying the identity $\mathcal{L}_1^*(p_0, M_0, k_0, \beta, \gamma) \equiv V(p_0, M_0, k_0, \beta, \gamma)$ we therefore can state the dynamic envelope theorem with respect to the initial price vector as

$$(4.33)\qquad \frac{\partial V}{\partial p_0} \equiv \int_0^T \left(\frac{\partial h'}{\partial p_0}\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \kappa'}{\partial p_0}\frac{\partial \tilde{u}}{\partial k}\right) dt \equiv -\lambda_0 \int_0^T e^{-rt}\frac{\partial \varphi'}{\partial p_0} h dt\,.$$

We follow essentially the same steps for the other parameters to obtain the following list of properties for the dynamic indirect utility function.

Theorem 6. *The dynamic indirect utility function in* (4.8) *is twice continuously differentiable in* $(p_0, M_0, k_0, \beta, \gamma)$ *and satisfies*

(6.a) $$\frac{\partial V}{\partial p_0} \equiv \int_0^T \left(\frac{\partial h'}{\partial p_0}\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \kappa'}{\partial p_0}\frac{\partial \tilde{u}}{\partial k} \right) dt \equiv -\lambda_0 \int_0^T e^{-rt} \frac{\partial \varphi'}{\partial p_0} h dt\,;$$

(6.b) $$\frac{\partial V}{\partial M_0} \equiv \int_0^T \left(\frac{\partial h'}{\partial M_0}\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \kappa'}{\partial M_0}\frac{\partial \tilde{u}}{\partial k} \right) dt \equiv \lambda_0 > 0\,;$$

(6.c) $$\frac{\partial V}{\partial k_0} \equiv \int_0^T \left(\frac{\partial h'}{\partial k_0}\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \kappa'}{\partial k_0}\frac{\partial \tilde{u}}{\partial k} \right) dt \equiv \omega(\cdot,0)\,;$$

(6.d) $$\frac{\partial V}{\partial \beta} \equiv \int_0^T \left(\frac{\partial h'}{\partial \beta}\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \kappa'}{\partial \beta}\frac{\partial \tilde{u}}{\partial k} + \frac{\partial \tilde{u}}{\partial \beta} \right) dt \equiv \int_0^T \frac{\partial \tilde{u}}{\partial \beta} dt \text{ ; and}$$

(6.e) $$\frac{\partial V}{\partial \gamma} \equiv \int_0^T \left(\frac{\partial h'}{\partial \gamma}\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \kappa'}{\partial \gamma}\frac{\partial \tilde{u}}{\partial k} \right) dt \equiv \int_0^T \frac{\partial f'}{\partial \gamma} \omega dt\,.$$

(6.f) *If* $\varphi(\cdot,t)$ *is* 1° *homogeneous in* p_0, *then* $V(\cdot,k_0,\beta,\gamma)$ *is* 0° *homogeneous in* (p_0, M_0).

(6.g) *If* $\varphi(\cdot,t)$ *is increasing and concave in* p_0, *then* $V(\cdot,k_0,\beta,\gamma)$ *is decreasing and quasiconvex in* p_0.

In addition, the dynamic Marshallian demand functions satisfy the intertemporal budget identity, Cournot aggregation, Engle aggregation, and Roy's identity,

(6.h) $\int_0^T e^{-rt}\varphi' h dt \equiv M_0$,

(6.i) $$\int_0^T e^{-rt}\left[\frac{\partial \varphi'}{\partial p_0} h + \frac{\partial h'}{\partial p_0}\varphi \right] dt \equiv \mathbf{0}\,,$$

(6.j) $$\int_0^T e^{-rt}\varphi' \frac{\partial h}{\partial M_0} dt \equiv 1\,, \text{ and}$$

(6.k) $$-\frac{\partial V/\partial p_0}{\partial V/\partial M_0} \equiv \int_0^T e^{-rt} \frac{\partial \varphi'}{\partial p_0} h dt\,, \text{ respectively.}$$

The properties of the dynamic expenditure function,

(4.35) $$E(p_0, U_0, k_0, \beta, \gamma) \equiv \inf \left\{ \int_0^T e^{-rt}\varphi' x dt : u_0 = \int_0^T \tilde{u}(x,k,\beta,t)dt, \dot{k} = f(x,k,\gamma,t), k(0) = k_0, k(T) \geq 0 \right\}.$$

are derived in a similar way. We first define the Lagrangean function for the consumer's intertemporal cost minimization problem as

$$(4.36)\qquad \mathcal{L}_2 \equiv \int_0^T e^{-rt}\varphi' x dt + \mu\left(U_0 - \int_0^T \tilde{u}\,dt\right) + \int_0^T \psi'\left(\dot{k} - f\right)dt$$
$$\equiv \int_0^T \left(e^{-rt}\varphi' x dt - \mu\tilde{u} - \psi' f - \dot{\psi}' k\right)dt + \mu U_0 + \omega(T)' k(T) - \omega(0)' k_0 ,$$

where μ is the shadow price on the discounted utility constraint, ψ is the vector of shadow prices for the equations of motion for household durable goods, and as before, the second line of (4.36) is obtained by integrating the terms $\psi'\dot{k}$ by parts. We continue to assume that $(x^*(t), k^*(t)) \gg (\mathbf{0},\mathbf{0})\ \forall t \in [0,T)$, as well as the previous regularity conditions for $\tilde{u}(\cdot)$ and $f(\cdot)$. The necessary and sufficient first-order conditions for the unique optimal path now are:

$$(4.37)\qquad e^{-rt}\varphi - \mu\frac{\partial \tilde{u}}{\partial x} - \frac{\partial f'}{\partial x}\psi = \mathbf{0}\ ;$$

$$(4.38)\qquad \mu\frac{\partial \tilde{u}}{\partial k} + \frac{\partial f'}{\partial k}\psi + \dot{\psi} = \mathbf{0}\ ;$$

$$(4.39)\qquad \dot{\mu} = 0\ ;$$

$$(4.40)\qquad f = \dot{k},\ k(0) = k_0,\ k(T) \geq \mathbf{0}\ ;\ \text{and}$$

$$(4.41)\qquad U_0 = \int_0^T \tilde{u}\,dt\ ;$$

together with the transversality conditions

$$(4.42)\qquad \psi_j(T)k_j(T) = 0\ \ \forall\ \ j = 1,\ldots,\ell\,.$$

At time t, the optimal Hicksian demands, stocks of household durable goods, and shadow prices are

$$(4.43)\qquad x^*(t) \equiv g(p_0, U_0, k_0, \beta, \gamma, t)\,,$$

$$(4.44)\qquad k^*(t) \equiv \xi(p_0, U_0, k_0, \beta, \gamma, t)\,,$$

$$(4.45)\qquad \psi^*(t) \equiv \psi(p_0, U_0, k_0, \beta, \gamma, t)\,,\ \text{and}$$

$$(4.46)\qquad \mu^*(t) \equiv \mu_0(p_0, U_0, k_0, \beta, \gamma)\,,$$

respectively. The first-order conditions imply that the optimal Lagrangean function and the dynamic expenditure function satisfy

$$(4.47)\qquad \mathcal{L}_2^*(p_0, U_0, k_0, \beta, \gamma) \equiv E(p_0, U_0, k_0, \beta, \gamma) \equiv \int_0^T e^{-rt}\varphi(p_0, t)' g(p_0, U_0, k_0, \beta, \gamma, t)dt\,.$$

This, in turn, when combined with the discounted utility constraint, $U_0 \equiv \int_0^T \tilde{u}(\cdot)dt$, implies the following set of properties for $E(p_0, U_0, k_0, \beta, \gamma)$:

> ***Theorem 7.*** *The dynamic expenditure function in (4.36) is twice continuously differentiable in* $(p_0, U_0, k_0, \beta, \gamma)$ *and satisfies*

(6.a) $$\frac{\partial E}{\partial p_0} \equiv \int_0^T e^{-rt} \frac{\partial \varphi'}{\partial p_0} g dt \;;$$

(6.b) $$\frac{\partial E}{\partial U_0} \equiv \mu_0 > 0 \;;$$

(6.c) $$\frac{\partial E}{\partial k_0} \equiv -\psi(\cdot,0) \;;$$

(6.d) $$\frac{\partial E}{\partial \beta} \equiv \int_0^T e^{-rt} \frac{\partial g'}{\partial \beta} \varphi dt \equiv -\mu_0 \int_0^T \frac{\partial \tilde{u}}{\partial \beta} dt \;; \text{ and}$$

(6.e) $$\frac{\partial E}{\partial \gamma} \equiv \int_0^T e^{-rt} \frac{\partial g'}{\partial \gamma} \varphi dt \equiv -\int_0^T \frac{\partial f'}{\partial \gamma} \psi dt \;.$$

(6.f) *$E(\cdot,U_0,k_0,\beta,\gamma)$ is 1^0 homogeneous in p_0 if and only if $\varphi(\cdot,t)$ is 1^0 homogeneous in p_0.*

(6.g) *If $\varphi(\cdot,t)$ is increasing and concave in p_0, then $E(\cdot,U_0,k_0,\beta,\gamma)$ is increasing and concave in p_0, with Hessian matrix defined by*

$$\frac{\partial^2 E}{\partial p_0 \partial p_0'} \equiv \int_0^T e^{-rt} \left(\sum_{i=1}^{n_x} g^i \frac{\partial^2 \varphi_i'}{\partial p_0 \partial p_0'} + \frac{\partial \varphi'}{\partial p_0} \frac{\partial g}{\partial p_0'} \right) dt \;.$$

In addition, the dynamic Hicksian demand functions and the expenditure minimizing demands for household durables satisfy,

(6.h) $$\textstyle\int_0^T \tilde{u}(g,\xi,\beta,t) dt \equiv U_0 \,,$$

(6.i) $$\int_0^T \left(\frac{\partial g'}{\partial p_0} \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \xi'}{\partial p_0} \frac{\partial \tilde{u}}{\partial k} \right) dt \equiv \mathbf{0} \,,$$

(6.j) $$\int_0^T \left(\frac{\partial g'}{\partial U_0} \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \xi'}{\partial U_0} \frac{\partial \tilde{u}}{\partial k} \right) dt \equiv 1 \,,$$

(6.k) $$\int_0^T \left(\frac{\partial g'}{\partial \beta} \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \xi'}{\partial \beta} \frac{\partial \tilde{u}}{\partial k} + \frac{\partial \tilde{u}}{\partial \beta} \right) dt \equiv \mathbf{0} \,, \text{ and}$$

(6.l) $$\int_0^T \left(\frac{\partial g'}{\partial \gamma} \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \xi'}{\partial \gamma} \frac{\partial \tilde{u}}{\partial k} \right) dt \equiv \mathbf{0} \,.$$

The final piece of the puzzle is to establish the dual relationship between the dynamic expenditure and indirect utility functions as inverses to each another with respect to their $n+1^{\text{st}}$ arguments. In other words, we now will show that if $U_0 = V(p_0, M_0, k_0, \beta, \gamma)$, then $E(p_0, V(p_0, M_0, k_0, \beta, \gamma), k_0, \beta, \gamma) \equiv M_0$; equivalently, if $M_0 = E(p_0, U_0, k_0, \beta, \gamma)$, then $V(p_0, E(p_0, U_0, k_0, \beta, \gamma), k_0, \beta, \gamma) \equiv U_0$. Intuitively, this seems obvious —the minimum present value of discounted consumption expenditures necessary to obtain the maximum present value of discounted utility flows that can be afforded with initial wealth M_0 must be M_0. In fact, utilizing the Lagrangeans in (4.27) and (4.36) makes the proofs of these facts nearly as obvious.

The intuition is simple. From theorem 4, recall that in the static case, $\mu(p,u,b) \equiv 1/\lambda(p,e(p,u,b),b)$, and similarly, $\lambda(p,m,b) \equiv 1/\mu(p,v(p,m),b)$, where λ is the marginal utility of money and μ is the marginal cost of utility, and that the first-order conditions for expenditure minimization and utility maximization are identical when income is set equal to expenditure. In the dynamic case, we will show that analogous properties hold, although the argument is slightly more involved.

Let $M_0 = E(p_0,U_0,k_0,\beta,\gamma)$, and note that

(4.48) $$\mathcal{L}_1^*(p_0,E(p_0,U_0,k_0,\beta,\gamma),k_0,\beta,\gamma) \equiv$$
$$\int_0^T \tilde{u}\big(h(p_0,E(p_0,U_0,k_0,\beta,\gamma,t),k_0,\beta,\gamma,t),\kappa(p_0,E(p_0,U_0,k_0,\beta,\gamma,t),\beta,\gamma,t),\beta,t\big)dt \equiv$$
$$V(p_0,E(p_0,U_0,k_0,\beta,\gamma),k_0,\beta,\gamma) \geq U_0,$$

where the inequality follows from maximization and the fact that U_0 is affordable. Writing out $\mathcal{L}_1^*$ explicitly, we have

(4.49) $$0 \leq \mathcal{L}_1^* - U_0 = \int_0^T \tilde{u}\,dt + \lambda_0\Big(M_0 - \int_0^T e^{-rt}\varphi' h\,dt\Big) + \int_0^T \omega'(f-\dot{\kappa})dt - U_0$$
$$= \lambda_0\left\{M_0 - \left[\int_0^T e^{-rt}\varphi' h\,dt + (1/\lambda_0)\Big(U_0 - \int_0^T \tilde{u}\,dt\Big) + \int_0^T (\omega/\lambda_0)'(\dot{\kappa}-f)dt\right]\right\}$$
$$= \lambda_0\Big(M_0 - \hat{\mathcal{L}}_2\Big) \leq 0,$$

where $\hat{\mathcal{L}}_2$ is the Lagrangean for the expenditure minimization problem evaluated along the utility maximizing path for x and k with $\mu_0 = 1/\lambda_0$ and $\xi \equiv \omega/\lambda_0$. The second inequality follows from the fact that this path is feasible, so that

$$M_0 = \int_0^T e^{-rt}\varphi' g\,dt \equiv \mathcal{L}_2^*(p_0,U_0,k_0,\beta,\gamma) \leq \hat{\mathcal{L}}_2 \equiv \int_0^T e^{-rt}\varphi' h\,dt.$$

It follows immediately from this that

$$V(p_0,E(p_0,U_0,k_0,\beta,\gamma),k_0,\beta,\gamma) \equiv U_0.$$

The argument for

$$E(p_0,V(p_0,M_0,k_0,\beta,\gamma),k_0,\beta,\gamma) \equiv M_0$$

is identical, with the roles of the dynamic expenditure and indirect utility functions interchanged. We therefore have the following rather remarkable result. Only parts h and i of the next theorem are not immediately obvious from the previous developments. However, this pair of conclusions follows from: (a) the uniqueness of the optimal paths for the two problems; (b) the above relationships among the shadow prices; and (c) the fact that the first-order conditions for the two problems are equivalent $\forall\ t \in [0, T]$. Hence, no further proof is necessary to establish the following.

Theorem 8. *The dynamic indirect utility and expenditure functions for the intertemporal consumer choice problem with household production and non-static price expectations satisfy*

(8.a) $V(p_0, E(p_0, U_0, k_0, \beta, \gamma), k_0, \beta, \gamma) \equiv U_0$ *;*

(8.b) $E(p_0, V(p_0, M_0, k_0, \beta, \gamma), k_0, \beta, \gamma) \equiv M_0$ *;*

(8.c) $$\frac{\partial V(p_0, E(p_0, U_0, k_0, \beta, \gamma), k_0, \beta, \gamma)}{\partial M_0} \equiv \lambda_0(p_0, E(p_0, U_0, k_0, \beta, \gamma), k_0, \beta, \gamma)$$

$$\equiv \frac{1}{\mu_0(p_0, U_0, k_0, \beta, \gamma)} \equiv \frac{1}{\partial E(p_0, U_0, k_0, \beta, \gamma)/\partial U_0}\ ;$$

(8.d) $$\frac{\partial E(p_0, V(p_0, M_0, k_0, \beta, \gamma), k_0, \beta, \gamma)}{\partial U_0} \equiv \mu_0(p_0, V(p_0, M_0, k_0, \beta, \gamma), k_0, \beta, \gamma)$$

$$\equiv \frac{1}{\lambda_0(p_0, M_0, k_0, \beta, \gamma)} \equiv \frac{1}{\partial V(p_0, M_0, k_0, \beta, \gamma)/\partial M_0}\ ;$$

(8.e) $$\frac{\partial V(p_0, E(p_0, U_0, k_0, \beta, \gamma), k_0, \beta, \gamma)}{\partial k_0} \equiv \omega(p_0, E(p_0, U_0, k_0, \beta, \gamma), k_0, \beta, \gamma, 0)$$

$$\equiv \frac{\psi(p_0, U_0, k_0, \beta, \gamma, 0)}{\mu_0(p_0, U_0, k_0, \beta, \gamma)} \equiv -\frac{\partial E(p_0, U_0, k_0, \beta, \gamma)/\partial k_0}{\partial E(p_0, U_0, k_0, \beta, \gamma)/\partial U_0}\ ;$$

(8.f) $$\frac{\partial V(p_0, E(p_0, U_0, k_0, \beta, \gamma), k_0, \beta, \gamma)}{\partial \beta} \equiv -\frac{\partial E(p_0, U_0, k_0, \beta, \gamma)/\partial \beta}{\partial E(p_0, U_0, k_0, \beta, \gamma)/\partial U_0}\ ;$$

(8.g) $$\frac{\partial V(p_0, E(p_0, U_0, k_0, \beta, \gamma), k_0, \beta, \gamma)}{\partial \gamma} \equiv -\frac{\partial E(p_0, U_0, k_0, \beta, \gamma)/\partial \gamma}{\partial E(p_0, U_0, k_0, \beta, \gamma)/\partial U_0}\ ;$$

(8.h) $$-\frac{\partial V(p_0, E(p_0, U_0, k_0, \beta, \gamma), k_0, \beta, \gamma)/\partial p_0}{\partial V(p_0, E(p_0, U_0, k_0, \beta, \gamma), k_0, \beta, \gamma)/\partial M_0}$$

$$\equiv \int_0^T e^{-rt} \frac{\partial \varphi(p_0, t)'}{\partial p_0} h(p_0, E(p_0, U_0, k_0, \beta, \gamma), k_0, \beta, \gamma, t)\,dt$$

$$\equiv \int_0^T e^{-rt} \frac{\partial \varphi(p_0, t)'}{\partial p_0} g(p_0, U_0, k_0, \beta, \gamma, t)\,dt \equiv \frac{\partial E(p_0, U_0, k_0, \beta, \gamma)}{\partial p_0}\ ;$$

(8.h) $h(p_0, E(p_0, U_0, k_0, \beta, \gamma), k_0, \beta, \gamma, t) \equiv g(p_0, U_0, k_0, \beta, \gamma, t)\ \forall\ t \in [0, T]$ *;*

(8.i) $g(p_0, V(p_0, M_0, k_0, \beta, \gamma), k_0, \beta, \gamma, t) \equiv h(p_0, M_0, k_0, \beta, \gamma, t)\ \forall\ t \in [0, T]$ *; and*

$$\text{(8.h)}\qquad \frac{\partial^2 E(p_0,U_0,k_0,\beta,\gamma)}{\partial p_0 \partial p_0'} \equiv \int_0^T e^{-rt}\sum_{i=1}^n g^i(p_0,U_0,k_0,\beta,\gamma,t)\frac{\partial^2 \varphi_i(p_0,t)}{\partial p_0\partial p_0'}dt$$

$$+\int_0^T e^{-rt}\frac{\partial \varphi(p_0,t)'}{\partial p_0}\frac{\partial g(p_0,U_0,k_0,\beta,\gamma,t)}{\partial p_0'}dt$$

$$\equiv \int_0^T e^{-rt}\sum_{i=1}^n h^i(p_0,E(p_0,U_0,k_0,\beta,\gamma),k_0,\beta,\gamma,t)\frac{\partial^2 \varphi_i(p_0,t)}{\partial p_0\partial p_0'}dt$$

$$+\int_0^T e^{-rt}\frac{\partial \varphi(p_0,t)'}{\partial p_0}\frac{\partial h(p_0,E(p_0,U_0,k_0,\beta,\gamma),k_0,\beta,\gamma,t)}{\partial p_0'}dt$$

$$+\left(\int_0^T e^{-rt}\frac{\partial \varphi(p_0,t)'}{\partial p_0}\frac{\partial h(p_0,E(p_0,U_0,k_0,\beta,\gamma),k_0,\beta,\gamma,t)}{\partial M_0}dt\right)\times$$

$$\left(\int_0^T e^{-rt}\frac{\partial \varphi(p_0,t)'}{\partial p_0}h(p_0,E(p_0,U_0,k_0,\beta,\gamma),k_0,\beta,\gamma,t)dt\right)'$$

is a symmetric $n\times n$ *matrix with rank no greater than* $n-1$ *and is negative semidefinite if* $\varphi(\ \ ,t)$ *is (weakly) concave in* p_0.

4. Discussion

The static neoclassical model provides a solid foundation for the entire host of generalizations to consumer choice theory considered in this chapter. The core duality theory of the neoclassical model transcends the theory of household production, characteristics theory, and intertemporal consumer choice, models of consumer expectations for future values of important economic variables, durable goods, consumption habits, and changing household production technologies or quality characteristics. This illustrates a robust theoretical framework. But the way that the duality theory is manifested varies substantially across specifications. When intertemporal considerations are added, there no longer is a static, short-run, or instantaneous, counterpart to the neoclassical model's Slutsky symmetry and negativity conditions. Once the proper concept of substitution has been taken into account in a dynamic setting, however, the precise nature of the symmetry condition becomes self evident. In addition, the standard homogeneity and curvature conditions of the static model are not necessarily satisfied in a dynamic framework. Again, however, once the influences that expectations for the future economic environment have on consumers' optimal plans have been identified, the conditions in which homogeneity and curvature are satisfied become apparent.

The above analysis also shows that the naïve way that consumption habits and durable goods are typically treated in empirical demand analysis has some problems, particularly when anything other than myopic expectations is assumed. This gives rise to questions that could open several avenues for future research in empirical demand analysis. Can "rational" consumption habits be estimated econometrically? How does one distinguish between changes in households' holdings of durable assets and changes in their stocks of consumption habits? Does habit formation even exist? Can future expectations be modeled successfully in empirical demand studies? In dynamic settings with fully "rational" consumers, how do we measure the economic consequences on consumer of policy changes? My hope is this chapter stimulates some fresh thoughts and new answers to these and other important questions in the economics of consumer choice.

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