ΠΜΣ «Οικονομική Επιστήμη»
ΕΚΠΑ Τμήμα Οικονομικών Επιστημών Microeconomic Theory – Producer Theory January 2020
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Problem Set 1 – Key

1. *Consider a firm whose production function $f(z)$ exhibits constant returns to scale. Show that its cost function can be expressed as $c(w,q) = q \cdot c(w,1)$, i.e., the cost per unit times the number of units produced.*

By the definition of constant returns to scale, a change in input usage of z where $a > 0$, entails a change in output of aq, and viceversa. As a consequence, changing output by q entails that the cost function satisfies $c(w, aq) = a \cdot c(w,q)$, i.e., initial costs change by exactly a. If $a = \frac{1}{q}$, we obtain that

$$c\left(w, \frac{1}{q}q\right) = \frac{1}{q} \cdot c(w,q) \Leftrightarrow c(w,1) = \frac{1}{q} \cdot c(w,q) \Leftrightarrow q \cdot c(w,1) = c(w,q)$$

where $c(w,1)$ should be the average cost, i.e., the cost per unit times the number of units produced.

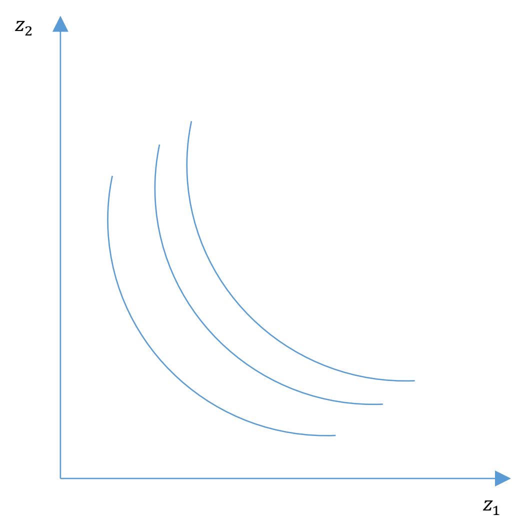
2. Suppose a firm's production function has the Cobb-Douglas form

$$q = z_1^{a_1} z_2^{a_2}$$

where z_1 and z_2 are inputs, q is output and a_1, a_2 are positive parameters.

(a) Draw the isoquants. Do they touch the axes?

The isoquants do not touch the axes. As seen on the figure below



(b) *What is the elasticity of substitution in this case?*

The elasticity of substitution is defined as:

$$\sigma_{ij} = -\frac{\vartheta \log(z_j/z_i)}{\vartheta \log(\phi_j(\boldsymbol{z})/\phi_i(\boldsymbol{z}))}$$

Since we have two inputs, we get

$$\sigma = -\frac{\vartheta \log(z_1/z_2)}{\vartheta \log(\phi_1(\boldsymbol{z})/\phi_2(\boldsymbol{z}))}$$

Calculating first the $\phi_1(\boldsymbol{z})/\phi_2(\boldsymbol{z})$ part,

$$\phi_1(\boldsymbol{z})/\phi_2(\boldsymbol{z}) = \frac{\alpha_1}{\alpha_2}/\frac{z_1}{z_2}$$

Taking the logarithms we have,

$$\log\left(\frac{z_1}{z_2}\right) = \log\left(\frac{\alpha_1}{\alpha_2}\right) - \log\left(\frac{\phi_1(\boldsymbol{z})}{\phi_2(\boldsymbol{z})}\right) or\ u = \log\left(\frac{\alpha_1}{\alpha_2}\right) - v$$

where $u := \log(z_1/z_2)$ and $v := \log(\phi_1(\boldsymbol{z})/\phi_2(\boldsymbol{z}))$

Differentiating u with respect to v we have,

$$\sigma = -\frac{\vartheta u}{\vartheta v} = -1$$

(c) *Using the Lagrangean method find the cost-minimising values of the inputs and the cost function.*

First we set up the Lagrangean,

$$\mathcal{L}(\boldsymbol{z}, \lambda) = w_1 z_1 + w_2 z_2 + \lambda\left[q - z_1^{\alpha_1} z_2^{\alpha_2}\right]$$

Which after taking the FOC and solving with some rearrangement we yield:

$$\left.\begin{aligned} z_1^* &= \frac{\alpha_1}{w_1}\lambda q \\ z_2^* &= \frac{\alpha_2}{w_2}\lambda q \end{aligned}\right\} (1)$$

which when substituted in the given production function we have,

$$\left[\frac{\alpha_1}{w_1}\lambda q\right]^{\alpha_1}\left[\frac{\alpha_2}{w_2}\lambda q\right]^{\alpha_2} = q\ or\ \lambda q = \left[q\left[\frac{w_1}{a_1}\right]^{a_1}\left[\frac{w_2}{a_2}\right]^{a_2}\right]^{\frac{1}{a_1+a_2}} (2)$$

Using (1) and (2) we derive the cost function,

$$C(\boldsymbol{w}, q) = w_1 z_1^* + w_2 z_2^* + [\alpha_1 + \alpha_2]\left[q\left[\frac{w_1}{a_1}\right]^{a_1}\left[\frac{w_2}{a_2}\right]^{a_2}\right]^{\frac{1}{a_1+a_2}}$$

(d) *Under what circumstances will the production function exhibit (a) decreasing (b) constant (c) increasing returns to scale? Explain this using first the production function and then the cost function.*

Using the production functions we have, for any $t > 0$,

$$\phi(t\boldsymbol{z}) = [tz_1]^{a_1}[tz_2]^{a_2} = t^{a_1+a_2}\phi(\boldsymbol{z})$$

Therefore DRTS/CRTS/IRTS is according to $a_1 + a_2 \gtreqless 1$. If we look at average cost as a function of q we find that AC is increasing/constant/decreasing in q according as $a_1 + a_2 \gtreqless 1$.

(e) *Find the conditional demand curve for input 1.*

Using the above solutions from *(c),* we yield:

$$H^1 = (\boldsymbol{w}, q) = \left[q\left[\frac{a_1 w_2}{a_2 w_1}\right]^{a_2}\right]^{\frac{1}{a_1+a_2}}$$

$$H^2 = (\boldsymbol{w}, q) = \left[q\left[\frac{a_2 w_1}{a_1 w_2}\right]^{a_1}\right]^{\frac{1}{a_1+a_2}}$$

3. Consider the following profit function that has been obtained from a technology that uses a single input, z:

$$\pi(p, w) = p^2 w^a$$

where p *is the output price,* w *is the input price and* a *is a parameter value.*

(a) *Check if the profit function satisfies homogeneity of degree one jointly in both* p *and* w*. In particular, determine for which values of this property is satisfied.*

The profit function is homogeneous of degree one if

$$\pi(\theta p, \theta w) = \theta\pi(p, w)$$

In this case we have that the left-hand term becomes

$$\pi(\theta p, \theta w) = (\theta p)^2(\theta w)^\alpha = \theta^{2+\alpha}p^2 w^\alpha$$

and, on the other hand, the right-hand term is

$$\theta\pi(p, w) = \theta p^2 w^\alpha$$

which implies that $2 + \alpha = 1 \Leftrightarrow \alpha = -1$

in this way the given profit function is obtained as

$$\pi(p, w) = p^2 w^{-1} \; or \; \pi(p, w) = \frac{p^2}{w}$$

(b) *Assuming the value of* a *for which the profit function satisfies homogeneity of degree one, check if the profit function* $\pi(p, w)$ *satisfies the following properties: (1) non-decreasing in output price* p*, (2) non-increasing in input prices* w*, and (3) convex in prices* p *and* w*.*

(1) Increasing output prices yields a weakly higher profit level since

$$\frac{\vartheta\pi(p, w)}{\vartheta p} = \frac{2p}{w} \geq 0$$

(2) Increasing all input prices weakly reduces profits since

$$\frac{\vartheta\pi(p,w)}{\vartheta w} = -\frac{p^2}{w^2} \leq 0$$

(3) In order to check the convexity in prices, we should derive the Hessian matrix and check how it is defined. Thus we yield,

$$\begin{vmatrix} \frac{\vartheta\pi^2(p,w)}{\vartheta p} & \frac{\vartheta\pi^2(p,w)}{\vartheta p\vartheta w} \\ \frac{\vartheta\pi^2(p,w)}{\vartheta w\vartheta p} & \frac{\vartheta\pi^2(p,w)}{\vartheta w^2} \end{vmatrix} = \begin{vmatrix} \frac{2}{w} & -\frac{2p}{w^2} \\ -\frac{2p}{w^2} & \frac{2p^2}{w^3} \end{vmatrix}$$

In particular, the Hessian is a positive semi-definite matrix, since

$$\frac{2}{w}\frac{2p^2}{w^3} - \left(-\frac{2p}{w^2}\right)\left(-\frac{2p}{w^2}\right) = \frac{4p^2}{w^4} - \frac{4p^2}{w^4} = 0$$

implying that the profit function $\pi(p,w)$ is convex.

(c) *Calculate the supply function of the firm,* $q(p,w)$*, and its demand for inputs,* $z(p,w)$

Using Hotelling's Lemma we can find the supply function, by differentiating the profit function with respect to p, as

$$q(p,w) = \frac{\vartheta\pi(p,w)}{\vartheta p} = \frac{2p}{w}$$

similarly w and the conditional factor demand correspondence can also be found by differentiating the profit function with respect to w,

$$z(p,w) = -\frac{\vartheta\pi(p,w)}{\vartheta w} = \frac{p^2}{w^2}$$

4. For the production function

$$q = z_1^{1/4} z_2^{1/4}$$

(a) *Find the conditional demand functions for z1 and z2.*
(b) *Find the cost function.*
(c) *Find the supply function.*
(d) *Find the input demand (Marshallian) function for* $z1$*. Briefly explain other ways of deriving the demand function.*
(e) *Find the short run supply function when* $\overline{z_2} = 16$ *(*$z_2^{1/4} = 2$*). Will this firm always supply at a positive price? Explain.*

**The key for this exercise is on this directory: https://eclass.uoa.gr/modules/document/index.php?course=ECON258&openDir=/54ce0fc5DNfu